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GLOBAL SMOOTH SOLUTIONS TO THE SYSTEM OF ONE-DIMENSIONAL THERMOELASTICITY WITH DISSIPATION BOUNDARY CONDITIONS

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Abstract

In this paper, the authors consider the initial boundary value problem for the a system of one-domensional thermoelasticity with dissipation conditions. By means of the delicate energy estimates and the continuation argument, the authors proved the global existence, uniqueness and the exponential decay of smooth solutions provided that the initial data are sufficiently small.

§ 1. Introduction

In the recent years, wide interest has been paid to the initial boundary value problem of the system of the thermoelasticity (see [1], [2]). In this paper, we consider the following initial boundary value problem with dissipation boundary conditions for the system of one-dimensional thermoelasticity (see [3]).

$$u_t - v_x = 0, \tag{1.1}$$

$$v_t + p(v, \theta)_x = 0, \tag{1.2}$$

$$\left(e(u, \theta) + \frac{v^2}{2} \right)_t + (p(u, \theta)v)_x = \theta_{xx}. \tag{1.3}$$

Here u is the deformation gradient, v is the velocity, p is the pressure, e-inner energy and θ-temperature.

The initial conditions and the boundary conditions are the following:

$$t=0: \; u=u^0(x), \; v=v^0(x), \; \theta=\theta^0(x), \tag{1.4}$$

$$x=0: \; -p(u, \theta) - \gamma v = 0, \tag{1.5}$$

$$\theta = \theta_0, \tag{1.6}$$

$$x=1: \; v=0, \tag{1.7}$$

$$\theta_x = 0, \tag{1.8}$$

where $\theta^0(x) > 0$, γ, θ_0 are positive constants, the boundary condition (1.5) represents

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that the string or rod described by (1.1)—(1.3)is connected with a damper at the end $x=0$. Similar to [4], we call this kind of boundary conditions the dissipation boundary condition

It is well known that (see [3])

$$p_u<0,\ e_\theta>0,\ e_u=\theta^2\left(\frac{p}{\theta}\right)_\theta. \tag{1.9}$$

The main results obtained in this paper can be described as follows. If the functions e, p are suitably smooth and the compatibility conditions on $x=0$, $t=0$ and $x=1$, $t=0$ are satisfied, then problem (1.1)—(1.9) admits a unique global solution provided that the initial data are sufficiently small.

(1.5) is a nonlinear boundary condition. For the convenience, we first make the following reduction for problem (1.1)—(1.9).

Set

$$u_1=-p(u,\theta_0),\ u_2=v,\ v=\theta-\theta_0. \tag{1.10}$$

By (1.9), u can be solved from the first equation of (1.10)

$$u=\sigma(u_1),\ \text{with}\ \sigma'(u_1)>0. \tag{1.11}$$

Also set

$$\tilde{p}(u_1,v)=p(\sigma(u_1),v+\theta_0),\ \tilde{p}_{u_1}<0. \tag{1.12}$$

Substituting (1.10)—(1.12) into the system (1.1)—(1.3), we arrive at

$$-\sigma'(u_1)\frac{\partial\tilde{p}}{\partial u_1}u_{1t}+\frac{\partial\tilde{p}}{\partial u_1}u_{2x}=0, \tag{1.13}$$

$$u_{2t}+\frac{\partial\tilde{p}}{\partial u_1}u_{1x}+\frac{\partial\tilde{p}}{\partial v}v_x=0, \tag{1.14}$$

$$\frac{e_v}{v+\theta_0}v_t-\left(\frac{1}{v+\theta_0}v_x\right)_x+\frac{\partial\tilde{p}}{\partial v}u_{2x}-\frac{1}{v+\theta_0}v_x^2=0, \tag{1.15}$$

$$t=0:\ u=u_1^0(x),\ u_2=u_2^0(x),\ v=v^0(x), \tag{1.16}$$

$$x=0:\ u_1=\gamma u_2, \tag{1.17}$$

$$v=0, \tag{1.18}$$

$$x=1:\ u_2=0, \tag{1.19}$$

$$v_x=0. \tag{1.20}$$

Instead of (1.13)—(1.15), in what follows we will study the following general system with the initial boundary value conditions (1.16)—(1.20):

$$\begin{cases}\begin{pmatrix}a(u_1,v) & 0\\ 0 & 1\end{pmatrix}\dfrac{\partial}{\partial t}\begin{pmatrix}u_1\\ u_2\end{pmatrix}-\begin{pmatrix}0 & b(u_1,v)\\ b(u_1,v) & 0\end{pmatrix}\dfrac{\partial}{\partial x}\begin{pmatrix}u_1\\ u_2\end{pmatrix}+\begin{pmatrix}0\\ \beta(u_1,v)v_x\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}, & (1.21)\\ c(u_1,v)v_t-(k(u_1,v)v_x)_x+\beta(u_1,v)u_{2x}+d(v,v_x)=0. & (1.22)\end{cases}$$

In view of (1.9), (1.12) and the definition of u_1 and $\tilde{p}$, we make the following assumptions on (1.21), (1.22), (1.16)—(1.20):

(1) a, b, β, $c\in C^3$, $k\in C^4$ and $d=d_1(v)v_x^2$, $d_1(v)\in C^3$.

(2) There exist positive constants R and a^0, a_0, b^0, b_0, c^0, c_0, k^0, k_0, β^0, β_0, k_β, k_c such that when $|u_1|$, $|v|\leqslant R$,

$$a^0 \geqslant a \geqslant a_0 > 0,\quad b^0 \geqslant b \geqslant b_0 > 0,\quad c^0 \geqslant c \geqslant c_0 > 0,\ k^0 \geqslant k \geqslant k_0 > 0,$$
$$\beta^0 \geqslant \beta \geqslant \beta_0 > 0,\ \frac{\beta^2}{k} \geqslant k_\beta > 0,\ \frac{\beta^c}{k} \geqslant k_c > 0. \tag{1.23}$$

Under the above assumptions, (1.21) is a quasilinear symmetric hyperbolic system with respect to $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and (1.22) is a quasilinear parabolic equation with respect to v. They couple each other.

(3) $u^0(x) = \begin{pmatrix} u_1^0(x) \\ u_2^0(x) \end{pmatrix} \in H^3$, $v^0(x) \in H^4$, the following compatibility conditions are satisfied:

$$u_1^0(0) = \gamma u_2^0(0),\ u_2^0(1) = 0,\ v^0(0) = 0,\ v_x^0(1) = 0,\ \frac{\partial u_1^0}{\partial x}(1) = 0. \tag{1.24}$$

Let

$$M_0 = (\|u^0\|_{H^3}^2 + \|v^0\|_{H^4}^2)^{\frac{1}{2}},\ Du = \{u_t,\ u_x\},$$
$$D^2u = \{u_{tt},\ u_{tx},\ u_{xx}\},\ |u|^2 = u_1^2 + u_2^2,\ |u|_1^2 = |u|^2 + |Du|^2, \tag{1.25}$$
$$|u|_2^2 = |u|^2 + |Du|^2 + |D^2u|^2.$$

Now our main theorem is the following

Theorem 1. *Under the assumptions* (1)—(3), *when* M_0 *is sufficiently small, the initial boundary value problem* (1.21), (1.22), (1.16)—(1.20) *admits a unique global smooth solution* (u, v).

$$u,\ Du,\ D^2u,\ v,\ Dv,\ D^2v,\ v_{xxx},\ v_{xxt} \in C([0, +\infty), L^2),$$
$$v_{xtt} \in L^2([0, +\infty), L^2). \tag{1.26}$$

Moreover, $\int_0^1 |u|_2^2 dx$, $\int_0^1 |v|_2^2 dx$, $\int_0^1 v_{xxx}^2\, dx$ and $\int_0^1 v_{xxt}^2\, dx$ decay exponentially to zero as $t \to +\infty$.

Especially, for the initial boundary value problem of one-dimensional thermoelasticity with dissipation boundary conditions, Theorem 1 implies the global existence and uniqueness of smooth solutions and the exponential decay of solutions.

§ 2. Existence and Uniqueness of Local Smooth Solutions

Let

$$M_1 = \|\{u,\ Du,\ D^2u\}|_{t=0}\|^2 + \|\{v,\ Dv,\ D^2v\}|_{t=0}\|^2, \tag{2.1}$$

where we denote by $\|\cdot\|$ the L^2 norm in the interval $[0, 1]$ and $Du|_{t=0}$, $D^2u|_{t=0}$, $Dv|_{t=0}$, $D^2v|_{t=0}$ are obtained from the initial conditions and system (1.21), (1.22). From the concrete expressions of (1.21), (1.22), it follows easily that M_1 tends tozero as M_0 tends tozero.

Let

$$Q_h=(0,1)\times(0,h),$$
$$H_2^2(t)=\max_{0\leqslant\tau\leqslant t}(\|u\|_2^2+\|v\|_2^2), \tag{2.2}$$
$$H_3^2(t)=\max_{0\leqslant\tau\leqslant t}(\|v_{xxx}(\tau)\|^2+\|v_{xxt}(\tau)\|^2)$$
$$+\int_0^t(\|v_{xxx}\|^2+\|v_{xxt}\|^2+\|v_{xtt}\|^2)d\tau, \tag{2.3}$$

where $\|u\|_2^2$ is the L^2 norm of $\{u, Du, D^2u\}$ in the interval $[0, 1]$. For any positive constants M_2, M_3, h, we difine the set of functions:

$$\Sigma^h(M_2, M_3)=\left\{(u,v)\left|\begin{array}{l}(u,v)\in C^\infty(\bar{Q}_h),\ H_2(t)\leqslant M_2,\ H_3(t)\leqslant M_3,\ \forall t\in[0,h],\\ u_1|_{x=0}=\gamma u_2|_{x=0},\ u_2|_{x=1}=0,\ v|_{x=0}=v_x|_{x=1}=0,\\ u_{1x}|_{x=1}=u_{2xx}|_{x=1}=0,\end{array}\right.\right\} \tag{2.4}$$

and $\overline{\Sigma^h}(M_2, M_3)$ is the closure of $\Sigma^h(M_2, M_3)$ with the corresponding norm $H_2(h)+H_3(h)$.

By the imbedding theorem, there is a constant R_1 such that if $H_2(t)\leqslant R_1$ $\forall t\in[0, h]$, then $|u|$, $|v|\leqslant R$.

Establishing the corresponding auxiliary linear problems and noting that the boundary condition (1.17) is admissible under the assumption (2), by means of the energy estimate method for the linear symmetric hyperbolic systems and the linear parabolic equations of second order and the contractive mapping theorem, we can prove the following local existence and uniqueness theorem.

Theorem 2. *Under the assumptions* (1)—(3) *in the previous section, there exist positive constants* C_2, C_3, $M_{10}(C_2M_{10}\leqslant R_1)$ *and* t_0 *depending only on* M_{10} *such that when* $M_1\leqslant M_{10}$, *problem* (1.21), (1.22), (1.16)—(1.20) *admits a unique smooth solution* (u, v) in $\bar{Q}_{t_0}=[0,1]\times[0,t_0]$. *Moveover*

$$(u,v)\in\overline{\Sigma^{t_0}}(C_1M_1, C_2M_1). \tag{2.5}$$

Since the proof of Theorem 2 is standard (see [5]), we omit the detail here.

Remark 1. It is easy to see that when the coefficients and the initial data have more regularity, the solution will have more regularity, too. Hence when the higher smoothness is required in the process of getting the uniform a priori estimates in the following section, we can apply the usual dense argument.

§ 3. Uniform a Priori Extimates of Solutions for Quasilinear Systems

In this section we are going to derive the uniform a priori estimates of solution $(u, v)\in\overline{\Sigma^T}$ for problem (1.21), (1.22), (1.16)—(1.20) in $Q_T=(0,1)\times(0,T)$, $\forall T>0$. This is the key step in proving the global existence of solutions. From now on, we denote by K the positive constant independent of u, v, T with no particular regard to distinguishing one from another and by ε the appropriately small positive constant. For simplicity, we omit the integral element $dxd\tau$ and the integral limit of the double integral.

Assume that $(u, v)\in\overline{\Sigma^T}$, and

$$|u|_1+|v|_1+|v_{xx}|+|v_{xt}|\leqslant\varepsilon. \tag{3.1}$$

The main theorem in this section is the following

Theorem 3. *For any $T>0$, there exists a positive constant ε_1 independent of T such that when $\varepsilon\leqslant\varepsilon_1$ and the solution (u, v) of (1.21), (1.22), (1.16)—(1.20) satisfies (3.1), the following uniform à priori estimate holds:*

$$\int_0^1(|u|_2^2+|v|_2^2+v_{xxx}^2+v_{xxt}^2)dx+\int_0^t\int_0^1(|u|_2^2+|v|_2^2+v_{xxx}^2+v_{xxt}^2+v_{xtt}^2)\leqslant K_1M_1, \tag{3.2}$$

where M_1 is defined by (2.1), K_1 is a positive constant independent of u, v, t, T.

Proof For the initial boundary value problem with the dissipation boundary condions, the main difficulty in getting uniform a priori estimate consists in getting the estimate of boundary integral. To do this, we first prove the following

Lemma 1. (1) *If $f(x)$ is a smooth function in $[0, 1]$, $f(0)=0$ or $f(1)=0$, then*

$$f^2(x)=\int_0^1 f_x^2(x)dx, \tag{3.3}$$

$$\int_0^1 f^2(x)\,dx\leqslant\int_0^1 f_x^2dx. \tag{3.4}$$

(2) *If $f(x)$ is a smooth function in $[0, 1]$, then $\forall\delta>0$,*

$$f_x^2(x)\leqslant C_1\Big(\delta\int_0^1 f_{xx}^2\,dx+\frac{1}{\delta}\int_0^1 f^2\,dx\Big),\ \forall x\in[0, 1]. \tag{3.5}$$

Hereafter $C_i(i=1, 2, \cdots\cdots)$ are the constants independent of f and δ.

Proof (1) is trivial. For (2), by the Nirenberg inequality

$$\sup_{0\leqslant x\leqslant1}|f_x(x)|\leqslant C_2\Big(\int_0^1 f_{xx}^2\,dx\Big)^{\frac38}\Big(\int_0^1 f^2dx\Big)^{\frac18}+C_3\Big(\int_0^1 f^2\,dx\Big)^{\frac12}. \tag{3.6}$$

From the Young inequality

$$a\cdot b\leqslant\frac{(\delta_1a)^p}{p}+\frac{1}{q}\Big(\frac{b}{\delta_1}\Big)^q, \tag{3.7}$$

where $\frac{1}{p}+\frac{1}{q}=1$, δ_1 is an arbitrarily small positive constant, it follows that

$$\sup_{0\leqslant x\leqslant1}|f_x(x)|\leqslant C_4\Big(\delta_1\int_0^1 f_{xx}^2\,dx\Big)^{\frac12}+\frac{C_5}{\delta_1}\Big(\int_0^1 f^2\,dx\Big)^{\frac12}. \tag{3.8}$$

By squaring both sides, we get (3.5).

Lemma 2. *If the solution $(u, v)\in\overline{\Sigma^T}$ and satisfies (3.1), then $\forall 0\leqslant t\leqslant T$, $\delta>0$,*

$$u_{2x}^2|_{x=0}\leqslant K\int_0^1 u_{2tx}^2\,dx, \tag{3.9}$$

$$v_{xt}^2|_{x=0}\leqslant K\int_0^1 v_{xxt}^2\,dx, \tag{3.10}$$

$$v_x^2|_{x=0}\leqslant K\int_0^1 v_{xx}^2\,dx\leqslant K\int_0^1\left(v_{xxt}^2+\delta u_{2xx}^2+\frac{1}{\delta}u_{2x}^2\right)dx+K\varepsilon\int_0^1 v_{xx}^2\,dx, \tag{3.11}$$

$$v_{xx}^2|_{x=0}\leqslant K\int_0^1\left(\delta u_{2xx}^2+\frac{1}{\delta}u_{2x}^2\right)dx+K\varepsilon\int_0^1 v_{xx}^2\,dx. \tag{3.12}$$

Proof By noting the boundary conditions, (3.10) is the corollary of Lemma 1. From (1.21) and the boundary conditions, it follows that

$$u_{2x}^2|_{x=0}=\frac{a^2}{b^2}u_{1t}^2|_{x=0}=\frac{a^2\gamma^2}{b^2}u_{2t}^2|_{x=0}\leqslant K\int_0^1 u_{2tx}^2\,dx \tag{3.13}$$

and (3.11) (3.12) can be obtained from Lemma 1 and (1.21), (1.22), (1.16)—(1.20).

Lemma 3. (i) *If the solution* $(u, v)\in\overline{\Sigma}^T$, *then* $\forall 0\leqslant t\leqslant T$, $\delta>0$

$$\int_0^1 bu_{1x}u_{2x}|_{x=0}^{x=1}\,dt\leqslant K\delta\iint v_{xx}^2+\frac{K}{\delta}\iint v^2 \tag{3.14}$$

and

$$\int_0^1 bu_{1x}u_{2x}|_{x=0}^{x=1}\,dt\leqslant K\iint v_{xx}^2. \tag{3.15}$$

(ii) *If the solution* (u, v) *is suitably smooth and satisfies* (3.1), *then* $\forall\ 0\leqslant t\leqslant T$, $\delta>0$,

$$\int_0^1\left(bu_{1xx}u_{2xx}+\frac{\beta cb}{k}u_{1xx}v_x\right)\Big|_{x=0}^{x=1}dt\leqslant K\iint\left(v_{xxt}^2+\delta u_{2xx}^2+\frac{1}{\delta}u_{2x}^2\right)$$
$$+K\varepsilon\iint(|Du|_1^2+|Dv|_1^2). \tag{3.16}$$

$$\int_0^1 bu_{1xt}u_{2xt}|_{x=0}^{x=1}\,dt\leqslant K\iint\left(\delta v_{xxt}^2+\frac{1}{\delta}v_t^2\right)+K\varepsilon\iint(|Du|_1^2+|Dv|_1^2+v_{xxt}^2). \tag{3.17}$$

$$\int_0^1 bu_{1xt}u_{2xt}|_{x=0}^{x=1}\,dt\leqslant K\iint v_{xxt}^2+K\varepsilon\iint(v_{xxt}^2+|Du|_1^2+|Dv|_1^2). \tag{3.17$'$}$$

Proof (i) From (1.21)

$$u_{1x}|_{x=0}=\left(\frac{1}{b}(u_{2t}+\beta v_x)\right)|_{x=0}, \tag{3.18}$$

$$u_{2x}|_{x=0}=\left(\frac{a}{b}u_{1t}\right)|_{x=0} \tag{3.19}$$

and from (1.19), (1.20)

$$u_{1x}|_{x=1}=0. \tag{3.20}$$

Hence

$$\int_0^t bu_{1x}u_{2x}|_{x=0}^{x=1}\,dx=-\int_0^t\frac{1}{b}(u_{2t}+\beta v_x)a\gamma u_{2t}|_{x=0}\,dt$$
$$=-\int_0^t\frac{a\gamma}{b}u_{2t}^2|_{x=0}\,dt-\int_0\frac{a\gamma\beta}{b}u_{2t}v_x|_{x=0}dt. \tag{3.21}$$

By means of Lamma 1, we have

$$-\frac{a\gamma\beta}{b}u_{2t}v_x|_{x=0}\leqslant\frac{a\gamma\beta}{b}\frac{\eta}{2}u_{2t}^2|_{x=0}+\frac{a\gamma\beta}{b}\frac{1}{2\eta}v_x^2|_{x=0}$$

$$\leqslant\frac{a\gamma\beta}{b}\frac{\eta}{2}u_{2t}^2|_{x=0}+\frac{C_6\delta}{2\eta}\int_0^1 v_{xx}^2\,dx+\frac{C_6'}{2\eta\delta}\int_0^1 v^2\,dx,\qquad(3.22)$$

where η, δ are arbitrarily small positive constants. Substituting (3.22) into (3.21) and taking η sufficiently small, we get (3.14). It is easy to obtain (3.15) in a similar way.

(2) From (1.21), (1.22), (1.16)—(1.20) it follows that

$$u_{1xx}|_{x=0}=\frac{1}{b}\Big(\frac{a\gamma}{b}u_{2tt}+\beta v_{xx}+I_1\Big)|_{x=0},\qquad(3.23)$$

$$u_{2xx}|_{x=0}=\frac{a}{b^2}(u_{2tt}+\beta v_{xt}+I_2)|_{x=0},\qquad(3.24)$$

$$u_{2xx}|_{x=1}=0.\qquad(3.25)$$

Hereafter $I_i(i=1,2,\cdots\cdots)$ are the terms, such as $\left(\frac{a}{b}\right)_x u_{1t}$, including the product of first order derivative of solutions and coefficients.

Hence

$$\int_0^t\Big(bu_{1xx}u_{2xx}+\frac{\beta cb}{k}u_{1xx}v_x\Big|_{\substack{x=1\\x=0}}dt=-\int_0^t\Big(bu_{1xx}u_{2xx}+\frac{\beta cb}{k}u_{1xx}v_x\Big)\Big|_{x=0}dt$$

$$=-\int_0^t\Big[\frac{a^2\gamma}{b^3}u_{2tt}^2+(A_1v_{xt}+A_2v_{xx}+A_3v_x)u_{2tt}+A_4v_{xx}v_{xt}+A_5v_{xx}v_x+I_1u_{2tt}+I_1^2\Big]\Big|_{x=0}dt,\qquad(3.26)$$

where $A_i(i=1,2,\cdots\cdots)$ simply denote the given functions which consist of the coefficients.

By Lemma 2 we get

$$(A_1v_{xt}+A_2v_{xx}+A_3v_x)u_{2tt}|_{x=0}\geqslant-\frac{C_7\eta}{2}u_{2tt}^2\Big|_{x=0}-\frac{C_7}{2\eta}\int_0^1\Big(v_{xxt}^2+\delta u_{2xx}^2+\frac{1}{\delta}u_{2x}^2\Big)dx$$

$$-K\varepsilon\int_0^1 v_{xx}^2\,dx,\qquad(3.27)$$

$$(A_4v_{xx}v_{xt}+A_5v_{xx}v_x)|_{x=0}\geqslant-C_8\int_0^1\Big(v_{xxt}^2+\delta u_{2xx}^2+\frac{1}{\delta}u_{2x}^2\Big)dx-K\varepsilon\int_0^1 v_{xx}^2\,dx.\quad(3.28)$$

From (3.1) it follows that

$$I_1u_{2tt}|_{x=0}\geqslant-C_9\varepsilon\frac{\eta}{2}u_{2tt}^2\Big|_{x=0}-\frac{C_9\varepsilon}{2\eta}\int_0^1(|Du|^2+|Dv|^2)dx,\qquad(3.29)$$

$$I_1\cdot I_1|_{x=0}\geqslant-C_{10}\varepsilon\int_0^1(|Du|^2+|Dv|^2)dx.\qquad(3.30)$$

Combining (3.26) with (3.27)—(3.30) and taking η sufficiently small, we get (3.16).

It is easy to obtain (3.17), (3.17)′ in a similar way.

Now we are going to get the energy estimates.

Let

$$E_1(t)=\frac{1}{2}\int_0^1(au_1^2+u_2^2+cv^2)dx,$$
$$E_2(t)=\frac{1}{2}\int_0^1(au_{1t}^2+u_{2t}^2+cv_t^2)dx,$$
$$E_3(t)=\frac{1}{2}\int_0^1(au_{1x}^2+u_{2x}^2+cv_x^2)dx, \qquad (3.31)$$
$$E_4(t)=\frac{1}{2}\int_0^1(a\ u_{1tt}^2+u_{2tt}^2+cv_{tt}^2)dx,$$
$$E_5(t)=\frac{1}{2}\int_0^1(au_{1xt}^2+u_{2xt}^2+cv_{xt}^2)dx.$$

Thus we have

Lemma 4. *For the solution* $(u, v)\in\overline{\Sigma^T}$, *satisfing* (3.1), *when* ε *is appropriately small, for* $0\leqslant t\leqslant T$ *the following estimates hold.*

$$E_1(t)-E_1(0)+\frac{1}{2}\iint kv_x^2\leqslant K\varepsilon\iint|Du|^2, \qquad (3.32)$$
$$E_2(t)-E_2(0)+\frac{1}{2}\iint kv_{xt}^2\leqslant K\varepsilon\iint(|Du|^2+|Dv|^2), \qquad (3.33)$$
$$E_3(t)-E_3(0)+\frac{1}{2}\iint kv_{xx}^2\leqslant K\iint v_x^2+K\varepsilon\iint(|Du|^2+|Dv|^2), \qquad (3.34)$$
$$E_4(t)-E_4(0)+\frac{1}{2}\iint kv_{xtt}^2\leqslant K\varepsilon\iint(|Du|_1^2+|Dv|_1^2), \qquad (3.35)$$
$$E_5(t)-E_5(0)+\frac{1}{2}\iint kv_{xxt}^2\leqslant K\iint v_t^2+K\varepsilon\iint(|Du|_1^2+|Dv|_1^2). \qquad (3.36)$$

Proof Establishing the usual energy integral for (1.21), (1.22) and noteing that under the boundary condition (1.17), (1.19)

$$u_1^2(x, t)=\left(\int_0^x u_{1x}\,dx-u_1(0, t)\right)^2,$$
$$=\left(\int_0^x u_{1x}\,dx-\gamma u_2(0, t)\right)^2=\left(\int_0^x u_{1x}\,dx+\gamma\int_0^1 u_{2x}\,dx\right)^2$$
$$\leqslant C_{11}\int_0^1(u_{1x}^2+u_{2x}^2)dx, \qquad (3.37)$$

we have

$$\int_0^t\int_0^1 u_1^2\leqslant C_{11}\int_0^t\int_0^1|u_x|^2. \qquad (3.38)$$

So when ε is appropriatly small, we can obtain (3.32).

Differentiating (1.21), (1.22) with respect to t, using the energy estimate method, we can get (3.33). Similarly, (3.35) can be obtained.

Differentiating (1.21), (1.22) with respect to x, multiplying them by u_x and v_x, respectively, summing up and integrating with respect to x and t, we arrive at

$$E_3(t)-E_3(0)+\iint kv_{xx}^2=\iint\frac{\partial(bu_{1x}u_{2x})}{\partial x}-\frac{\partial}{\partial x}(\beta u_{2x}v_x-(kv_x)_x v_x)+I_2) \qquad (3.39)$$

and

$$\left|\iint I_2\right| \leqslant C_{12}\varepsilon\iint(|Du|^2+|Dv|^2+v_{xx}^2). \tag{3.40}$$

From (3.14)

$$\iint\frac{\partial(bu_{1x}u_{2x})}{\partial x}=\int_0^t bu_{1x}u_{2x}\big|_{x=0}^{x=1}dt \leqslant K\delta\iint v_{xx}^2+\frac{K}{\delta}\iint v^2 \leqslant K\delta\iint v_{xx}^2+\frac{K}{\delta}\iint v_x^2 \tag{3.41}$$

and from (1.22), (1.17)—(1.20)

$$-\iint\frac{\partial}{\partial x}\,(\beta u_{2x}v_x-(kv_x)_x v_x)=-\iint\frac{\partial}{\partial x}[(-cv_t-d)v_x]$$

$$=-\int_0^t dv_x\big|_{x=0}\,dt \leqslant C_{13}\varepsilon\int_0^t v_x^2\big|_{x=0}\,dt \leqslant C_{14}\varepsilon\iint v_{xx}^2. \tag{3.42}$$

Combining (3.39)–(3.42), taking δ and ε suitably small, we obtain (3.34).

When solution (u, v) is suitably smooth, differentiating (1.21), (1.22) with respect to x and t, using the usual energy method and noting (3.17), in a similar way to the proof of (3.34), we get (3.36).

Applying the usual dense argement we conclude (3.36) holds for $(u, v)\in\overline{\Sigma^T}$.

To estimate $\iint|Du_2|^2+|D^2u_2|^2$, we introduce the following auxiliary functions:

$$\begin{aligned}
F_1(t)&=\int_0^1 u_2u_{2t}\,dx,\\
F_2(t)&=\int_0^1 u_{2x}\,\frac{\beta c}{k}v\,dx,\\
F_3(t)&=\int_0^1 u_{2xx}\,\frac{\beta c}{k}\,v_x\,dx,\\
F_4(t)&=\int_0^1 u_{2xt}\,\frac{\beta c}{k}\,v_t\,dx,\\
F_5(t)&=\int_0^1 u_{2t}u_{2tt}\,dx
\end{aligned} \tag{3.43}$$

and

$$\begin{aligned}
D_2(t)&=\frac{1}{2}\int_0^1(au_{1x}^2+u_{2x}^2)\,dx,\\
D_3(t)&=\frac{1}{2}\int_0^1(au_{1xx}^2+u_{2xx}^2)\,dx,\\
D_4(t)&=\frac{1}{2}\int_0^1(au_{1xt}^2+u_{2xt}^2)\,dx.
\end{aligned} \tag{3.44}$$

Lemma 5. *When the solutoin $(u, v)\in\overline{\Sigma^T}$ satisfies (3.1) and ε is suitably small, $\forall 0\leqslant t\leqslant T$ the following estimates hold:*

$$F_1(t)-F_1(0)+\frac{1}{2}\,b\gamma u_2^2\big|_{x=0} \geqslant \iint u_{2t}^2-K\iint(u_{2x}^2+u_{1t}^2+v_t^2)-K\varepsilon\iint(u_2^2+v_x^2+u_{1x}^2)$$
$$+\frac{1}{2}\,b\gamma u_2^2\big|_{\substack{x=0\\t=0}}, \tag{3.45}$$

$$F_2(t)-F_2(0)+D_2(t)-D_2(0)+\iint\frac{\beta^2}{k}\,u_{2x}^2 \leqslant K\iint\left(\frac{1}{\delta}\,v_t^2+\delta u_{1x}^2+v_{xx}^2\right)$$
$$+K\varepsilon\iint(|Du|^2+|Dv|^2), \tag{3.46}$$

$$F_3(t)-F_3(0)+D_3(t)-D_3(0)+\frac{1}{2}\iint\frac{\beta^2}{k}u_{2xx}^2\leqslant K\iint(v_{xx}^2+u_{1xx}^2+u_{2x}^2+v_{xxt}^2)$$
$$+K\varepsilon\iint(|Du|_1^2+|Dv|_1^2), \tag{3.47}$$
$$F_4(t)-F_4(0)+D_4(t)-D_4(0)+\frac{1}{2}\iint\frac{\beta^2}{k}u_{2xt}^2\leqslant K\iint\left(\frac{1}{\delta}v_{xt}^2+\delta u_{1xt}^2+v_{xxt}^2\right)$$
$$+K\varepsilon\iint(v_t^2+v_{xx}^2+u_{xt}^2+u_{xx}^2), \tag{3.48}$$
$$F_5(t)-F_5(0)+\frac{1}{2}b\gamma u_{2t}^2\Big|_{x=0}\geqslant\frac{1}{2}b\gamma u_{2t}^2\Big|_{t=0}^{x=0}+\iint u_{2tt}^2-K\iint(u_{2tx}^2+u_{1tt}^2+v_{tt}^2)$$
$$-K\varepsilon\iint(|Du|^2+u_{1xt}^2+v_{xt}^2), \tag{3.49}$$

where δ is an arbitrarily positive constant.

Proof Noting that when (u, v) is smooth function

$$F_1(t)-F_1(0)=\int_0^t\frac{dF_1}{dt}dt=\iint u_{2t}^2+u_2u_{2tt}, \tag{3.50}$$
$$F_5(t)-F_5(0)=\int_0^t\frac{dF_5}{dt}dt=\iint u_{2tt}^2+u_{2t}u_{2ttt}. \tag{3.51}$$

Differentiating the second equation of (1.21) with respect to t, multiplying by u_2 in both sides, integrating with respect to x, t, by integration by parts and (1.17), we get (3.45). Similarly, differentiating twice the second equation of (1.21) with respect to t, multiplying by u_{2t} in both sides, we get (3.49). By the usual dense argument, (3.45) and (3.49) hold for $(u, v)\in\overline{\Sigma}^T$, too.

To prove (3.46)—(3.48), differentiating (1.21) with respect to x and multiplying by u_x in both sides, we get

$$\frac{1}{2}\frac{d}{dt}(au_{1x}^2+u_{2x}^2)-\frac{\partial(bu_{1x}u_{2x})}{\partial x}+\beta u_{2x}v_{xx}+I_3=0. \tag{3.52}$$

Differentiate the second equation with respect to x, multiply it by $\frac{\beta cv}{k}$, we have

$$\frac{\beta c}{k}vu_{2xt}-\frac{\beta c}{k}bvu_{1xx}+\frac{\beta^2c}{k}vv_{xx}+I_3=0. \tag{3.53}$$

Again multiplying (1.22) by $\frac{\beta}{k}u_{2x}$, adding it with (3.52), (3.53) together and integrating with respect to x, t, by (3.15) we get (3.46).

When (u, v) is suitably smooth, differentiating (1.21) twice with respect to x, multiplying it by u_{1xx}, u_{2xx}, respectively, we have

$$\frac{1}{2}\frac{d}{dt}(au_{1xx}^2+u_{2xx}^2)-\frac{\partial(bu_{1xx}u_{2xx})}{\partial x}+\beta v_{xxx}u_{2x}+I_4+J_1=0. \tag{3.54}$$

Hereafter $J_i(i=1, 2, \cdots)$ denote the terms, such as $\frac{\partial^2a}{\partial x^2}u_{1t}u_{1xx}$, including the second order derivatives of the coefficients.

Differentiating twice the second equation of (1.21) with respect to x, multiplying it by $\frac{\beta c}{k} v_x$, we have

$$\frac{\beta c}{k} v_x u_{2xxt} - \frac{\beta cb}{k} v_x u_{1xxx} + \frac{\beta^2 c}{k} v_x v_{xxx} + I_5 + J_2 = 0. \tag{3.55}$$

Differentiating (1.22) with respect to x, multiplying it by $\frac{\beta}{k} u_{2xx}$, we have

$$\frac{\beta c}{k} u_{2xx} v_{xt} - \frac{\beta}{k} u_{2xx} \frac{\partial}{\partial x} (k v_{xx}) + \frac{\beta^2}{k} u_{2xx}^2 + I_6 + J_3 = 0. \tag{3.56}$$

Adding (3.54)—(3.56) together, integrating with respect to x, t, we have

$$F_3(t) - F_3(0) + D_3(t) - D_3(0) + \iint \frac{\beta^2}{k} u_{2xx}^2 = \iint \frac{\partial (b u_{1xx} u_{2xx})}{\partial x}$$
$$+ \iint \frac{\beta cb}{k} u_{1xxx} v_x + \iint \frac{\beta^2 c}{k} v_{xxx} v_x + I_7 + J_4 = 0. \tag{3.57}$$

Obviously

$$\iint (|I_7| + |J_4|) \leqslant C_{15} \varepsilon \iint (|Du|_1^2 + |Dv|_1^2). \tag{3.58}$$

From (3.16)

$$\iint \frac{\partial (b u_{1xx} u_{2xx})}{\partial x} + \frac{\beta cb}{k} u_{1xxx} v_x \leqslant C_{16} \iint \left(v_{xxt}^2 + \delta u_{2xx}^2 + \frac{1}{\delta} u_{2x}^2 + u_{1xx}^2 + v_{xx}^2 \right)$$
$$+ C_{17} \varepsilon \iint (|Du|_1^2 + |Dv|_1^2) \tag{3.59}$$

and from (3.11), (3.12)

$$\iint \frac{\beta^2 c}{k} v_x v_{xxx} = -\int_0^t \frac{\beta^2 c}{k} v_x v_{xx} \Big|_{x=0} dt - \iint \frac{\partial}{\partial x} \left(\frac{\beta^2 c}{k} v_x \right) v_{xx}$$
$$\leqslant C_{18} \iint \left(v_{xx}^2 + \delta u_{2xx}^2 + \frac{1}{\delta} u_{2x}^2 \right) + C_{18} \varepsilon \iint v_{xx}^2. \tag{3.60}$$

Combining (3.57)—(3.60), taking δ sufficiently small, we get (3.47).

In a similar way, by (3.17) we can get (3.48).

It is easy to see from the equation (1.21) that we can estimate the derivatives of u_1 by the derivatives of u_2 and v.

Lemma 6. *For the solution $(u, v) \in \Sigma^T$ satisfing (3.1), $\forall 0 \leqslant t \leqslant T$ the following estimates hold:*

$$\begin{aligned} &\iint u_{1x}^2 \leqslant K \iint (u_{2t}^2 + v_x^2), \\ &\iint u_{1t}^2 \leqslant K \iint u_{2x}^2, \\ &\iint u_{1xt}^2 \leqslant K \iint (u_{2xt}^2 + v_{xt}^2) + K\varepsilon \iint (v_x^2 + u_{1x}^2), \\ &\iint u_{1xx}^2 \leqslant K \iint (u_{2xt}^2 + v_{xx}^2) + K\varepsilon \iint (v_x^2 + u_{1x}^2), \\ &\iint u_{1tt}^2 \leqslant K \iint u_{2xt}^2 + K\varepsilon \iint (u_{1t}^2 + u_{2x}^2). \end{aligned} \tag{3.61}$$

In what follows we are going to get the uniform à priori estimates of the solution for the quasilinear system.

Let

$$P(t)=\sum_{i=1}^{5}N_iE_i(t)-\eta_1\Big(F_1(t)+\frac{1}{2}\,b\gamma u_2^2|_{x=0}\Big)+\sum_{i=2}^{4}\eta_i(F_i(t)+D_i(t)$$
$$-\eta_5\Big(F_5(t)+\frac{1}{2}\,b\gamma u_{2t}^2|_{x=0}\Big),$$

where N_1 and $\eta_i(i=1\cdots\cdots5)$ are positive constants to be specified later. From Lemma 4 and Lemma 5, there exists a constant $\varepsilon_0>0$ such that when $\varepsilon<\varepsilon_0$ the following holds:

$$\begin{aligned}P(t)-P(0)\leqslant&-\frac{N_1}{2}\iint kv_x^2-\frac{N_2}{2}\iint kv_{xt}^2-\frac{N_3}{2}\iint kv_{xx}^2-\frac{N_4}{2}\iint kv_{xtt}^2\\&-\frac{N_5}{2}\iint kv_{xxt}^2+\tilde{K}\varepsilon\iint(|Du|_1^2+|Dv|_1^2)+K\iint N_5v_t^2+N_3v_x^2\\&-\eta_1\iint u_{2t}^2+K\eta_1\iint(u_{2x}^2+u_{1t}^2+v_t^2)-\eta_2\iint\frac{\beta^2}{k}\,u_{2x}^2\\&+K\eta_2\iint\Big(v_{xx}^2+\delta u_{1x}^2+\frac{1}{\delta}\,v_x^2\Big)-\eta_3\iint\frac{\beta^2}{2k}\,u_{2xx}^2+K\eta_3\iint(v_{xx}^2+u_{1xx}^2\\&+u_{2x}^2+v_{xxt}^2)-\eta_4\iint\frac{\beta^2}{2k}\,u_{2xt}^2+K\eta_4\iint\Big(v_{xxt}^2+\delta u_{1xt}^2+\frac{1}{\delta}\,v_{xt}^2\Big)\\&-\eta_5\iint u_{2tt}^2+K\eta_5\iint(u_{2tx}^2+u_{1tt}^2+v_{tt}^2).\end{aligned}\tag{3.62}$$

By Lemma 6 we arrive at

$$\begin{aligned}P(t)-P(0)\leqslant&-\iint\Big(\frac{N_1k}{2}-KN_3-K\eta_2\Big(\frac{1}{\delta}+K\delta\Big)\Big)v_x^2\\&-\iint\Big(\frac{N_2k}{2}-KN_5-K\eta_1-K\eta_4\Big(\frac{1}{\delta}+K\delta\Big)\Big)v_{xt}^2\\&-\iint\Big(\frac{N_3k}{2}-K\eta_2-K\eta_3(1+K)\Big)\,v_{xx}^2-\iint\Big(\frac{N_4k}{2}-K\eta_5\Big)v_{xtt}^2\\&-\iint\Big(\frac{N_5k}{2}-K\eta_3-K\eta_4\Big)v_{xxt}^2-\iint(\eta_1-K^2\eta_2\delta)u_{2t}^2\\&-\iint\Big(\frac{\beta}{k}\,\eta_2-K(1+K)\eta_1-K\eta_3\Big)\,u_{2x}^2-\iint\frac{\beta^2}{2k}\,\eta_3u_{2xx}^2\\&-\iint\Big(\frac{\beta^2}{2k}\,\eta_4-K^2\eta_3-K(1+K)\eta_5\Big)u_{2xt}^2-\iint(\eta_5-K^2\delta\eta_4)u_{2tt}^2\\&+\tilde{K}\varepsilon\iint(|Du|_1^2+|Dv|_1^2).\end{aligned}\tag{3.63}$$

Hereafter $\tilde{K}$ denotes the positive constant depending on K, N_i, η_i.

In what follows we will explain that by appropriately choosing η_i, N_i and δ, there exist positive constants σ_1, σ_2 such that the following inequalities hold simultaneously:

$$\begin{aligned}&P(t)+\sigma_1\iint(|Du_2|_1^2+v_x^2+v_{xt}^2+v_x^2+v_{xxt}^2+v_{xtt}^2)\\&\leqslant P(0)+\tilde{K}\varepsilon\iint(|Du|_1^2+|Dv|_1^2),\quad\forall 0\leqslant t\leqslant T\end{aligned}\tag{3.64}$$

and

$$P(t)\geqslant\sigma_2\int_0^1(|u|_2^2+|v|_1^2+v_{xt}^2+v_{tt}^2)dx. \tag{3.65}$$

Hereafter $\sigma_i(i=1, 2, \cdots)$ denote positive constants independent of u, v, T, t.

In fact, for (3.64) to be satisfied, it only need, by (3.63), to choose η_i, $N_i(i=1,\cdots, 5)$ and δ so that the coefficients of each terms (except the last term) are less than zero. For instance, by taking $\eta_2=\eta_4$ as positive constant, $\eta_3=\dfrac{K_\beta\eta_2}{4K^2}$, $\eta_1=\eta_5=\dfrac{K_\beta\eta_2}{8K(1+K)}$ and $\delta=\dfrac{\eta_1}{2K^2\eta_2}$, the coefficients of u_{2t}^2, u_{2x}^2, u_{2xx}^2, u_{2xt}^2, and u_{2tt}^2 are less than zero. Then we take N_3, N_4, N_5 so large that the coefficients of v_{xt}^2, v_{xtt}^2, v_{xxt}^2 are less than zero. Finally, taking N_1, N_2 large enough to ensure that the coefficients of the remaining two terms are less than zero, so (3.64) is satisfied. It is easy to see that when N_i are fixed, (3.64) still holds for η_i being reduced.

For (3.65) to be satisfied, by the definition of $P(t)$, it only need to take $\eta_i(i=1, \cdots, 5)$ small enough so that the non-square power terms, such as $u_{2x}v$, appearing in $F_i(t)(i=1, \cdots, 5)$, can be bounded by the square power terms appearing in $E_i(t)(i=1, \cdots, 5)$ and $D_i(t)(i=2, 3, 4)$. For instance, by taking $\eta_2=\dfrac{N_1C_0}{2k_C^2}$, $\eta_3=\dfrac{N_3C_0}{2k_C^2}$, $\eta_4=\dfrac{N_2C_0}{2k_C^2}$, we have

$$\begin{aligned}
&\eta_2(F_2(t)+D_2(t))+\frac{N_1}{2}\int_0^1 cv^2\geqslant\sigma_3\int_0^1(u_x^2+v^2),\\
&\eta_3(F_3(t)+D_3(t))+\frac{N_3}{2}\int_0^1 cv_x^2\geqslant\sigma_3\int_0^1(u_{xx}^2+v_x^2),\\
&\eta_4(F_4(t)+D_4(t))+\frac{N_2}{2}\int_0^1 cv_t^2\geqslant\sigma_3\int_0^1(u_{xt}^2+v_t^2).
\end{aligned} \tag{3.66}$$

Again take η_1, η_5 so small that by Lemma 1 the following inequality holds

$$\begin{aligned}
&\frac{1}{2}\int_0^1(N_1u_2^2+N_2u_{2t}^2+N_3u_{2x}^2+N_4u_{2tt}^2+N_5u_{2xt}^2)\\
&\qquad-\eta_1\Big(\int_0^1 u_2u_{2t}\,dx+\frac{1}{2}\,b\gamma u_2^2|_{x=0}\Big)-\eta_5\Big(\int_0^1 u_{2t}u_{2tt}\,dx+\frac{1}{2}\,b\gamma u_{2t}^2|_{x=0}\Big)\\
&\geqslant\sigma_4\int_0^1(u_2^2+u_{2t}^2+u_{2x}^2+u_{2tt}^2+u_{2xt}^2)dx.
\end{aligned} \tag{3.67}$$

In order to prove (3.2), by the system (1.21), (1.22) and the boundary conditions we can easily obtain

Lemma 7. *When the solution $(u, v)\in\overline{\Sigma^T}$ satisfies (3.1), $\forall 0\leqslant t\leqslant T$, the following hold:*

$$\begin{aligned}
&\int_0^1 v_{xx}^2\,dx\leqslant dx\leqslant K\int_0^1(|Dv|^2+u_{2x}^2)dx,\\
&\int_0^1 v_{xxx}^2\,dx\leqslant K\int_0^1(|Du|_1^2+|Dv|^2+v_{xt}^2)dx,\\
&\int_0^1 v_{xxt}^2dx\leqslant K\int_0^1(|Du|_1^2+|Dv|^2+v_{tt}^2+v_{xt}^2),
\end{aligned} \tag{3.68}$$

$$\iint v_{xxx}^2 \leqslant K \iint (v_{xt}^2 + u_{2xt}^2) + K\varepsilon \iint (v_{xx}^2 + |Du|^2 + |Dv|^2).$$

Moreover, from (3.38) *and Lemma* 1 *we can get*

$$\iint |u|^2 + v^2 \leqslant K \iint |Du|^2 + |Dv|^2, \tag{3.69}$$

$$\iint v_t^2 \leqslant K \iint v_{tx}^2, \tag{3.70}$$

$$\iint v_{tt}^2 \leqslant K \iint v_{ttx}^2. \tag{3.71}$$

So (3.64) and (3.65) can be rewritten as

$$P(t) + \sigma_4 \iint (|u|_2^2 + |v|_2^2 + v_{xxx}^2 + v_{xxt}^2 + v_{xtt}^2) \leqslant P(0) + \widetilde{K}\varepsilon \iint |Du|_1^2 + |Dv|_1^2 \tag{3.72}$$

and

$$P(t) \geqslant \sigma_5 \int_0^1 (|u|_2^2 + |v|_2^2 + v_{xxx}^2 + v_{xxt}^2)\, dx. \tag{3.73}$$

Substituting (3.73) into (3.72), taking $\varepsilon_1 = \min\left(\frac{1}{2\widetilde{K}}, \varepsilon_0\right)$ we arrive at (3.2). Thus the proof of Theorem 3 is completed.

§ 4. Global Existence and Decay of Solutions

Based on sections 2 and 3, in this section we are going to prove the main theorem, that is, Theorem 1. By Theorem 2, when $M_1 \leqslant M_{10}$, there exists a positive constant t_0 depending on M_{10} such that problem (1.21), (1.22), (1.16)—(1.20) admits a unique smooth solution $(u, v) \in \overline{\Sigma^{t_0}}(C_1 M_1, C_2 M_1)$. By the imbedding theorem, the following holds in $[0, 1] \times [0, t_0]$,

$$|u|_1 + |v|_1 + |v_{xx}| + |v_{xt}| \leqslant K_2 M_1, \tag{4.1}$$

where K_2 is a positive constant independent of u, v, x, t and t_0. Taking

$$M_1 \leqslant M_1^* = \min\left(\frac{\varepsilon_1}{k_2}, \frac{\varepsilon_1}{k_1 k_2}, M_{10}, \frac{M_{10}}{k_1}\right), \tag{4.2}$$

where ε_1, K_1 are the constants appearing in Theorem 3, and M_{10} in Theorem 2. We can get the global existence by the well known continuation argument. To get the decay of the solution, similarly to (3.72), we can get

$$P(t) - P(\tau) + \sigma_6 \int_\tau^t \int_0^1 (|u|_2^2 + |v|_2^2 + v_{xxt}^2 + v_{xxx}^2 + v_{xtt}^2)$$
$$\leqslant \widetilde{K}\varepsilon \int_\tau^t \int_0^1 (|Du|_1^2 + |Dv|_1^2). \tag{4.3}$$

By tne definition of $P(t)$, it is easy to see that

$$P(t) \leqslant K_3 \int_0^1 (|u|_2^2 + |v|_2^2 + v_{xxx}^2 + v_{xxt}^2)\, dx, \tag{4.4}$$

provided that ε (equivalently, the initial data) is sufficiently small. It follows from (4.3) and (4.4) that

$$P(t)-P(\tau)+\frac{\sigma_6}{2K_3}\int_\tau^t P(t')dt' \leqslant 0. \tag{4.5}$$

Therefore, when $P(t)\in C^1$, i. e., the solution is more regular, we have

$$\frac{dP(t)}{\partial t}+\frac{\sigma_6}{2K_3}P(t)\leqslant 0. \tag{4.6}$$

Hence

$$P(t)\leqslant e^{-\frac{\sigma_6}{2K_3}t}P(0). \tag{4.7}$$

Substituting (3.73) into (4.7), we get

$$\int_0^1 (|u|_2^2+|v|_2^2+v_{xxx}^2+v_{xxt}^2)\leqslant Ke^{-\frac{\sigma_6}{2K_3}t}. \tag{4.8}$$

By the usual dense argument and Banach-Saks theorem, the restriction $P(t)\in C^1$ can be droped. Thus the proof of Theorem 1 is completed.

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