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ON FINITE DIFFUSING SPEED FOR UNIFORMLY DEGENERATE QUASILINEAR PARABOLIC EQUATIONS

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Abstract

In this paper, the author studies the quasilinear parabolic equation

$$u_t=(a_{ij}(x,\ t,\ u)u_{x_j})_{x_i}+b_i(x,\ t,\ u)u_{x_i}+c(x,\ t,\ u)$$

in $Q^T=\{(x,t)\,|\,x\in\mathbf{R}^N,\ 0<t\leqslant T\}$, which is unifommly degenerate wherever $u=0$. Under some conditions of the coefficients of the equation and the presupposition $0\leqslant u(x,\ t)\leqslant M$, the author proves that non-negative weak solutions, $u(x,\ t)$, to the equation satisfy the estimation that, for any $(x^0,\ t^0)\in Q^T$,

$$\frac{1}{C}\min\{\inf_{|x-x^0|<b\sqrt{t^0}}u(x,0),1\}\leqslant u(x^0,\ t^0)\leqslant C\sup_{|x-x^0|<b\sqrt{t^0}}u(x,\ 0),$$

where the constants b and C depend only upon M, T and the coefficients. It is a more exact description on the finite diffusing speed for the equation which has not been obtained even for one-dimensional porous medium eq uations.

§ 1.

In the paper [1] we studied existence and uniqueness of the weak solutions to the Cauchy problem and the first boundary-value problem for the quasilinear parabolic equation

$$u_t=(a_{ij}(x,\ t,\ u)u_{x_j})_{x_i}+b_i(x,\ t,\ u)u_{x_i}+c(x,\ t,\ u), \tag{1.1}$$

where the matrix $(a_{ij})_{N\times N}$ is uniformly degenerate wherever $u=0$. Now we are going to discuss some properties of the weak solutions.

B. F. Knerr[3], L. A. Caffarelli and A. Friedman[4] made systematic studies on the properties of the weak solutions for one-dimensional porous medium problems. Afterwards, L. A. Caffareli and A. Friedman[5] studied the regularity of the free boundary for N-dimensional porous medium equation

$$u_t=\Delta(u^m)\quad(m>1).$$

It is difficult to generalize their method to the equation (1.1). In this paper we obtain some properties of the weak solutions to the equation (1.1), using the method

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given in the paper [2]. It is remarkable that we give a more exact description about the finite diffusing speed for the equation (1.1) which has not been obtained even for one dimensional problems.

Let $\mathbf{R}^N$ be N-dimensional Euclidean space and Ω a bounded open domain in $\mathbf{R}^N$ and $\partial\Omega$ the boundary of Ω. Let

$$Q_1^T=\{(x,t)\,|\,x\in\Omega,\ 0<t\leqslant T\},$$
$$Q^T=\{(x,t)\,|\,x\in\mathbf{R}^N,\ 0<t\leqslant T\},$$
$$\Gamma=\partial\Omega\times[0,\ T].$$

Assmme that the coefficients of the equation (1.1) satisfy the following conditions:

(EA) For any $(x,t)\in Q_1^T, 0<u<\infty, \xi\in\mathbf{R}^n$,

$$\nu(|u|)|\xi|^2\leqslant a_{ij}(x,t,u)\xi_i\xi_j\leqslant \Lambda\nu(|u|)|\xi|^2, \tag{1.2}$$

where Λ is a constant and $\nu(s)$ has the following properties:

(i) $\nu(s)\in C[0,\infty]$, $\nu(0)=0$ and $\nu(s)>0$ if $s>0$.

(ii) Let

$$w=\varphi(u)=\int_0^u\nu(s)ds \text{ and its inverse be } u=\Phi(w). \tag{1.3}$$

There exist $\delta>0$ and $m>1$ such that, for any w_1,w_2: $0<w_1<w_2\leqslant\delta$,

$$\frac{1}{m}\left(\frac{w_1}{w_2}\right)^{1-\frac{1}{m}}\leqslant\frac{\Phi'(w_2)}{\Phi'(w_1)}\leqslant\lambda\left(\frac{w_1}{w_2}\right), \tag{1.4}$$

where $\lambda(\tau)$ is non-decreasing and $\lambda(\tau)\to0$ as $\tau\to0^+$.

(EB) $a_{ij}(x,\ t,\ u)$, $b_i(x,\ t,\ u)b_{ix_i}(x,\ t,\ u)$, $c_u(x,\ t,\ u)$ $(i,\ j=1,\ 2,\ \cdots,\ N)$ are in $C(\overline{Q}_1^T\times[0,\ \infty))$ and satisfy that, for $(x,t)\in\overline{Q}_1^T$ and $0<u<\infty$,

$$\frac{1}{\nu(|u|)}\sum_{i=1}^N b_i^2+\sum_{i=1}^N|(b_i)_{x_i}|+|c_u|\leqslant\eta(|u|), \tag{1.5}$$

$$c(x,\ t,\ 0,)=0, \tag{1.6}$$

where $\eta(\tau)$ is a non-decreasing function.

When we consider the weak solutions to (1.1) in Q^T, the above conditions can be changed correspondingly.

Definition 1.1. *A function $u(x,\ t)$ defined in $\overline{Q}_1^T$ (or $\overline{Q}^T$) is called a non-negative weak solution to the equation (1.1) if it is bounded and there exists a sequence of positive classical solutions, $\{u_n(x,t)\}$, to the equation (1.1) such that the sequence $\{u_n(x,\ t)\}$ converges uniformly (or bounded and local uniformly) to $u(x,t)$ in $\overline{Q}_1^T$ (or $\overline{Q}^T$).*

In this paper, we always suppose that (EA), (EB) are satisfied and

$$0<u_n(x,t)\leqslant M \quad \text{in } \overline{Q}_1^T (\text{or } \overline{Q}^T). \tag{1.7}$$

Obviously, we can get from (1.4) that, for any w_1 and w_2,

$$0<w_1<w_2=\widetilde{M}=\varphi(Me^{\eta(M)T}),$$

$$\frac{1}{m}\left(\frac{w_1}{w_2}\right)^{1-\frac{1}{m}}\leqslant\frac{\Phi'(w_2)}{\Phi'(w_1)}\leqslant\lambda\left(\frac{w_1}{w_2}\right)\leqslant m,$$

if changing appropriately the constant m and the function $\lambda(\tau)$ (they depend upon δ and M as well at this time). It can be proved from (1.4)′ (cf. [1]) that for $0<w_1<w_2<\widetilde{M}$

$$\frac{1}{m}\left(\frac{w_1}{w_2}\right)\leqslant\frac{\Phi(w_1)}{\Phi(w_2)}\leqslant m\left(\frac{w_1}{w_2}\right)^{\frac{1}{m}} \tag{1.8}$$

and that for $0<u_1<u_2\leqslant M$

$$\frac{1}{m^m}\left(\frac{u_1}{u_2}\right)^m\leqslant\frac{\varphi(u_1)}{\varphi(u_2)}\leqslant m\frac{u_1}{u_2}, \tag{1.8$'$}$$

$$\frac{1}{m^m}\left(\frac{u_1}{u_2}\right)^{m-1}\leqslant\frac{\nu(u_1)}{\nu(u_2)}\leqslant m. \tag{1.9}$$

In what follows we shall describe the main results in this paper.

At first we review how to describe the finite diffusing speed for one-dimensional porous medium equation before (for example, cf. [3]).

Consider the Cauchy problem

$$u_t=[\varphi(u)]_{xx},\ -\infty<x<+\infty,\ t>0,$$
$$u|_{t=0}=u_0(x)\geqslant 0,\ -\infty<x<+\infty.$$

If $u_0(x)$ has a compact support and

$$\int_0^1\frac{\varphi'(s)}{s}\,ds<\infty,$$

then for any $t>0$ there exists X_t such that

$$u(x,t)=0\quad\text{if } |x|\geqslant X_t.$$

Now our description about this property is given in the following theorems.

In virtue of the properties of $\lambda(\tau)$, we can certainly find $q_0\in(0,1)$ such that

$$\lambda(q_0)\leqslant\frac{1}{16} \tag{1.10}$$

and then define

$$\beta=1-\sqrt[3]{1-q_0}. \tag{1.11}$$

Theorem 1.1. *Let $u(x,t)$ be a weak solution to the equation (1.1) in Q_1^T and (EA), (EB) and (1.7) be satisfied, Then for any $(x^0,t^0)\in Q_1^T$, there exists a constant b, which depends onl y upon N, m, δ, T, M, Λ, $\eta(s)$ and $\lambda(s)$, such that*

$$u(x^0,t^0)\leqslant\frac{me^{\eta(M)T}}{\beta}\max\Big\{\sup_{K(x^0,b\sqrt{t^0})\cap\Omega}u(x,0),\ \sup_{\substack{K(x^0,b\sqrt{t^0})\cap\partial\Omega\\0\leqslant t\leqslant t^0}}u(x,t)\Big\}, \tag{1.12}$$

where $K(x^0,\rho)$ is a ball in $\mathbf{R}^N$ with the centre at x^0 and radius ρ.

If $u(x,t)$ is a weak solution in Q^T, then (1.12) reduces to

$$u(x^0,t^0)\leqslant\frac{me^{\eta(MT)}}{\beta}\sup_{K(x^0,b\sqrt{t^0})}u(x,0). \tag{1.12$'$}$$

Corollary 1.2. *Let $u(x,t)$ be a weak solution to the equation (1.1) in Q^T and (EA), (EB) and (1.7) be satisfied. Suppose that $u(x,0)$ has a compact support. Then, for any t, $0\leqslant t\leqslant T$, there exists a constant R_t such that*

$$u(x,t)=0 \text{ if } |x|\geqslant R_t.$$

In fact, if the support of $u(x,0)$ is on the sphere $K(0,R_0)$, then it suffcies to take

$$R_t=R_0+b\sqrt{t},$$

where b is defined in Theorem 1.1.

Theorem1.3. *Let $u(x,t)$ be a weak solution to the equation (1.1) in Q^T and (EA), (EB) and (1.7) be satisfied. For any $(x^0,t^0)\in Q^T$, define*

$$\tilde{\mu}_\rho=\inf_{|x-x^0|<\rho} u(x,0).$$

Then, there exists a constant b, which depends upon the same quantities asthose in Theorem 1.1, such that

$$u(x^0,t^0)\geqslant\frac{e^{-\eta(M)t^0}}{4m}\min\{\tilde{\mu}_\rho,\Phi(\tilde{M}\lambda^{-1}(\rho^2/b^2t^0))\}, \tag{1.13}$$

where $\tilde{M}=\varphi(Me^{\eta(M)T})$ and $\lambda^{-1}(\tau)$ is the inversion of the function $\lambda(s)$ in (1.4)′.

Corollary 1.4. *Let $u(x,t)$ be a weak solution to the equation (1.1) in Q^T and (EA), (EB) and (1.7) be satisfied. Then, that $u(x^0, t^0)>0$ implies*

$$u(x^0, t)>0 \text{ for } t\geqslant t^0.$$

Theorem 1.5. *Let $u(x,t)$ be a weak solution to the equation (1.1) in Q^T and (EA), (EB) and (1.7) be satisfied. Then, for any $(x^0,t^0)\in Q^T$,*

$$C^{-1}\min\{\inf_{|x-x^0|\leqslant b\sqrt{t^0}} u(x,0),1\}\leqslant u(x^0, t^0)\leqslant C\max_{|x-x^0|\leqslant b\sqrt{t^0}} u(x, 0), \tag{1.14}$$

where b is defined in Theorem 1.1 and C depends only upon N, m δ, Λ, T, M, $\eta(s)$ and $\lambda(s)$.

This theorem is a direct inference from Theorems 1.1 and 1.3, and it gives us a description about finite diffusing speed.

Remak 1. 1. The description is very different from that for hyperbolic equations since the estimations (1.14) are obtained under the presupposition $0\leqslant u(x,t)\leqslant M$ and the constant b depends on M.

Remark 1.2. In general, β defined in (1.11) will depend on $M=\max\{u(x, t)\}$ since $\lambda(\tau)$ in (1.4)′ depends on M, and so Theorem 1.1 perhaps does not give any more information than that $u(x,t)\leqslant M$ if $u(x, t)$ is a classical positive solution. However, if $\nu(u)=u^{m-1}$ $(m>1)$, then β defined in (1.11) will be independent of M. If $c(x, t, u)\equiv0$, the factor $\exp\{\eta(M)T\}$ in (1.12)′ can be dropped. Thus the estimation (1.12)′ shows that classical positive solutions for some class of quasilinear parabolic equations have the property on finite diffusing speed as well.

In this paper, we also discuss how the free boundary expands under the influence of initial data. The result is well-known for one-dimensional problems[3] but our method is suitable for general equations.

Let $u(x,t)$ be a weak solution in Q^T. Set

$$D(t)=\{x\in\mathbf{R}^N\,|\,u(x, t)>0\},\ D=\{(x, t)\,|\,x\in D(t),\ 0<t\leqslant T\},$$
$$\Gamma=\{(x,t)\,|\,x\in\partial D(t),\ 0\leqslant t\leqslant T\}. \tag{1.15}$$

The set Γ is called the free boundary. By Corollary 1.4, it follows that

$$D(t_1)\subset D(t_2) \text{ if } t_1<t_2, \tag{1.16}$$

which means that the free boundary is expanding as t increases.

Theorem 1.6. *Let $x^0\in\partial D(0)$. If there exists $\delta>0$ such that, in the δ-neighborhood of x^0,*

$$|\nu(u(x, 0))|\leqslant K|x-x^0|^2,$$

then we can find $t^*>0$ such that

$$x^0\in\partial D(t) \text{ for } 0\leqslant t\leqslant t^*. \tag{1.17}$$

In § 2, we shall study further properties of class-$\mathscr{B}_2$ which provide the essential tools for our purpose. In § 3 and § 4, we shall prove Theorem 1.1, Theorems 1.3 and 1.6 respectively.

§ 2.

In the paper [2], we introduced the generalized class-$\mathscr{B}_2$ and studied some properties of it. Now we shall give some more estimates and further properties.

Let $x^0\in\Omega$. For $0\leqslant t\leqslant T$, define

$$\begin{aligned} A_{k,\rho}(t;w)&=\{x\in K(\rho)\cap\Omega\,|\,w(x,t)>k\},\\ B_{k,\rho}(t;w)&=\{x\in K(\rho)\cap\Omega\,|\,w(x,t)<k\},\end{aligned} \tag{2.1}$$

where $K(\rho)=K(x^0, \rho)$.

Set

$$\zeta(x;\rho_1,\rho_2)=\begin{cases}1, & |x-x^0|\leqslant\rho_2,\\ \dfrac{\rho_1-|x-x^0|}{\rho_1-\rho_2}, & \rho_2\leqslant|x-x^0|<\rho_1,\\ 0, & |x-x^0|\geqslant\rho_1.\end{cases} \tag{2.2}$$

Definition 2.1. *A function $w(x, t)$ defined in $\overline{Q}_1^T$ is said to belong the class super-$\mathscr{B}_2(\overline{Q}_1^T, \lambda_0, \gamma, M, \Phi)$ if it satisfies*

(i) $w(x,t)\in C^{2,1}_{x,t}(Q_1^T)\cap C(\overline{Q}_1^T)$ *and* $0<w(x, t)\leqslant M$ *on* $\overline{Q}_1^T$.

(ii) *For any* $\rho\in(0,1)$, $x^0\in\Omega$ *and* $t\in[0,T]$, if

$$k\geqslant\max_{x\in K(\rho)\cap\partial\Omega} w(x,t)\quad (K(\rho)=K(x^0,\rho)),$$

we have

$$\begin{aligned}&\frac{\partial}{\partial t}\left[e^{-\gamma t}\int_{A_{k,\rho}(t)}\zeta^2(x)\chi_k(w-k)\,dx\right]+\lambda_0e^{-\gamma t}\int_{A_{k,\rho}(t)}\zeta^2(x)\,|\nabla w|^2\,dx\\ &\leqslant\gamma e^{-\gamma t}\int_{A_{k,\rho}(t)}|\nabla\zeta|^2(w-k)^2\,dx,\end{aligned} \tag{2.3}$$

where ∇ is a gradient operator with respect to the variable x, $\zeta(x)=\zeta(x;\rho,(1-\sigma)\rho)$ $(\forall\sigma\in(0,1))$ $A_{k,\rho}(t)=A_{k,\rho}(t;w)$ and

$$\chi_k(s)=\int_0^s\Phi'(k+\tau)\tau\,d\tau. \tag{2.4}$$

Here $\Phi(s)$ has the following properties:

(i) $\Phi(s)\in C[0,M]\cap C^1[0,M]$, $\Phi(0)=0$ and $\Phi(s)>0$ if $s>0$,

(ii) $(1.4)'$ is satisfied for $0<w_1<w_2\leqslant M$.

Definition 2.2. *A function $w(x,t)$ defined in Q_1^T is said to belong to the class sub-$\mathscr{B}_2(Q_1^T,\lambda_0,\gamma,M,\Phi)$, if*

(i) $w(x,t)\in C^{2,1}_{x,t}(Q_1^T)\cap C(\bar{Q}_1^T)$ *and* $0<w(x,t)\leqslant M$ *on* $\bar{Q}_1^T$,

(ii) *For any* $\rho\in(0,1)$, $x^0\in\Omega$ *and* $t\in[0,T]$ *if*

$$0<k\leqslant\min_{x\in K(\rho)\cap\partial\Omega}w(x,t),$$

we have

$$\frac{\partial}{\partial t}\left[e^{-\gamma t}\int_{B_{k,\rho}(t)}\zeta^2(x)\tilde{\chi}_k(k-w)dx\right]+\lambda_0e^{-\gamma t}\int_{B_{k,\rho}(t)}\zeta^2(x)|\nabla w|^2dx$$
$$\leqslant\gamma e^{-\gamma t}\int_{B_{k,\rho}(t)}|\nabla\zeta|^2(w-k)^2dx, \tag{2.5}$$

where

$$\tilde{\chi}_k(s)=\int_0^s\Phi'(k-\tau)\tau\,d\tau. \tag{2.6}$$

Theorem 2.1. *Let $w(x,t)$ belong to the class super-$\mathscr{B}_2$. Then there exists a constant b, which depends only on the parameters of $\mathscr{B}_2$, such that, for any $(x^0,t^0)\in Q_1^T$,*

$$w(x^0,t^0)\leqslant\frac{1}{\beta}\max\{\sup_{K(x^0,b\sqrt{t^0})\cap\Omega}w(x,0),\ \sup_{\substack{x\in K(x^0,b\sqrt{t^0})\cap\partial\Omega\\0\leqslant t\leqslant t^0}}w(x,t)\}. \tag{2.7}$$

We first prove the following lemma:

Lemma 2.2. *Let $w(x,t)$ belong to the class super-$\mathscr{B}_2$. Then, for any $\beta\in(0,1)$, there exists some constant $a>0$, which depends only on β and the parameters of the class $\mathscr{B}_2$, such that if μ and k satisfy*

$$\mu\geqslant\max_{\substack{x\in K(\rho)\cap\Omega\\0<t\leqslant t^0}}w(x,t),\ t^0\leqslant a\Phi'(\mu)\rho^2,$$
$$k\geqslant\max\{\beta\mu,\ \max_{x\in K(\rho)\cap\Omega}w(x,0),\ \max_{\substack{x\in K(\rho)\cap\partial\Omega\\0\leqslant t\leqslant t^0}}w(x,t)\}, \tag{2.8}$$
$$H\triangleq\mu-k>0,$$

then

$$\text{mes}\,A_{\mu-(1-\beta)^2H,\frac{\rho}{4}}(t)=0\ \textit{for}\ t\in[0,t^0]. \tag{2.9}$$

Proof Integrating the inequality (2.3) from 0 to t with $\zeta(x)=\zeta\left(x;\rho,\frac{\rho}{2}\right)$ and noting that mes $\{A_{k,\rho}(0)\}=0$ by the hypothesis of the lemma, we have, for $t\leqslant t^0$,

$$e^{-\gamma t}\int_{A_{k,\rho}(t)}\zeta^2\chi_k(w-k)\,dx\leqslant\frac{4\gamma H^2t^0}{\rho^2}\varkappa_N\delta^N,$$

where $\varkappa_N=$ mes $\{K(1)\}$. It follows that, for $t\in[0,t^0]$,

$$\chi_k(\beta H)\ \text{mes}\,A_{k+\beta H,\frac{\rho}{2}}(t)\leqslant4a\gamma e^{\gamma T}H^2\Phi'(\mu)\varkappa_N\rho^N. \tag{2.10}$$

In virtue of $(1.4)'$, we can find

$$\frac{\chi_k(\beta H)}{\Phi'(\mu)}=\int_0^{\beta H}\frac{\Phi'(k+\tau)}{\Phi'(\mu)}\tau\,d\tau\geqslant\frac{1}{2m}(\beta H^2). \tag{2.11}$$

Combining (2.10) with (2.11), we obtain

$$\operatorname{mes} A_{k+\beta H,\rho/2}(t)\leqslant\frac{aC_1}{\beta^2}\rho^N \quad \text{for } t\in[0,t^0]. \tag{2.12}$$

Hereafter we use C_k ($k=1,2,\cdots$) to express constants depending only upon the parameters of the class $\mathscr{B}_2$.

Now set

$$\begin{aligned} &k_h=k+\beta(2-\beta)H-\frac{\beta(1-\beta)H}{2^h}, \qquad \rho_h=\frac{\rho}{4}+\frac{\rho}{2^{h+2}},\\ &\mu_h=\max_{0\leqslant t\leqslant t^0}\operatorname{mes}A_{k_h,\rho_h}(t), \qquad \zeta_h(x)=\zeta(x;\rho_h,\rho_{h+1}), \\ &I_h(t)=e^{-\gamma t}\int_{A_{k_h,\rho_h}(t)}\zeta_h^2(x)\chi_{k_h}(w-k_h)dx. \end{aligned} \tag{2.13}$$

The inequality (2.12) implies

$$\mu_0\leqslant\frac{aC_1}{\beta^2}\rho^N. \tag{2.14}$$

We are going to prove

$$\lim_{h\to\infty}\mu_h=0$$

if a is small enough.

By the condition (1.4)$'$ we have, for $w\geqslant k_h$,

$$\frac{\chi_{k_h}(w-k_h)}{\Phi'(\mu)}\leqslant m\int_0^{w-k_h}\left(\frac{\mu}{k_h+\tau}\right)^{1-\frac{1}{m}}\tau\,d\tau\leqslant\frac{m}{2\beta^{1-\frac{1}{m}}}(w-k_h)^2, \tag{2.15}$$

and so

$$I_h(t)\leqslant\frac{m}{2\beta^{1-\frac{1}{m}}}\Phi'(\mu)\int_{A_{k_h,\rho_h}(t)}\zeta_h^2(w-k_h)^2dx. \tag{2.16}$$

For any $\mu\in W_2^{(1)}(\Omega)$, by the Sobolev's lemma, we have

$$\int_{A_0}u^2dx\leqslant C(\operatorname{mes}A_0)^{2/N}\int_{A_0}|\nabla u|^2dx, \tag{2.17}$$

where $A_0=\{x\in\Omega\,|\,u(x)>0\}$ and $C=C(N)$. Applying (2.17) to the right hand side of (2.16), we get

$$I_h(t)\leqslant C_2\frac{1}{\beta^{1-\frac{1}{m}}}\Phi'(\mu)\,\mu_h^{\frac{2}{N}}\Big[\int_{A_{k_h,\rho_h}(t)}\zeta_h^2|\nabla w|^2dx+\frac{H^2}{(\rho_{h+1}-\rho_h)^2}\mu_h\Big]. \tag{2.18}$$

On the other hand, the inequality (2.3) implies

$$I_h'(t)+\lambda_0e^{-\gamma t}\int_{A_{k_h,\rho_h}(t)}\zeta_h^2|\nabla w|^2dx\leqslant\frac{\gamma H^2}{(\rho_h-\rho_{h+1})^2}\mu_h.$$

Define

$$\tau=\sup\{s\in[0,t]\,|\,I'(s)\geqslant0\}.$$

Since $I_h(0)=0$ and $I_h'(0)\geqslant0$, τ exists. It is clear that $I_h'(\tau)\geqslant0$ and $I_h(t)\leqslant I_h(\tau)$. Thus, we obtain from (2.18) and (2.19) that

$$I_h(t)\leqslant I_h(\tau)\leqslant\frac{C_2}{\beta^{1-\frac{1}{m}}}\Phi'(\mu)\left(\frac{\gamma}{\lambda_0}e^{\gamma T}+1\right)\frac{H^2\mu_h^{1+\frac{2}{N}}}{(\rho_h-\rho_{h+1})^2}. \tag{2.20}$$

Noting that, by the condition (1.4)',

$$I_h(t)\geq e^{-\gamma T}\chi_{k_h}(k_{h+1}-k_h)\text{mes }A_{k_{h+1},\rho_{h+1}}(t)$$
$$\geq\frac{1}{2m}e^{-\gamma T}\Phi'(\mu)(k_{h+1}-k_h)^2\text{mes}A_{k_{h+1},\rho_{h+1}}(t),$$

we can obtain from (2.20) that

$$(k_{h+1}-k_h)^2\mu_{h+1}\leq\frac{2mC_2e^{\gamma T}}{\beta^{1-\frac{1}{m}}}\left(\frac{\gamma}{\lambda_0}e^{\gamma T}+1\right)\frac{H^2\mu_h^{1+\frac{2}{N}}}{(\rho_h-\rho_{h+1})^2}.$$

Hence

$$\mu_{h+1}\leq\frac{C_3}{\beta^{3-\frac{1}{m}}}\frac{2^{4h}\mu^{1+\frac{2}{N}}}{\rho^2}. \tag{2.21}$$

Let $y_h=\mu_h/\rho^N$. Then (2.21) yields

$$y_{h+1}\leq\frac{C_3}{\beta^{3-\frac{1}{m}}}2^{4h}y_h^{1+\frac{2}{N}}. \tag{2.22}$$

The inequality (2.14) means

$$y_0\leq\frac{aC_1}{\beta^2}.$$

We shall argue by induction

$$y_h\leq\frac{aC_1}{\beta^2}2^{-2Nh}\quad(h=0,1,2,\cdots). \tag{2.23}$$

In fact

$$y_{h+1}\leq\frac{C_3}{\beta^{3-\frac{1}{m}}}2^{4h}y_h^{1+\frac{2}{N}}\leq\frac{C_3}{\beta^{3-\frac{1}{m}}}2^{4h}\left(\frac{aC_1}{\beta^2}2^{-2Nh}\right)^{1+\frac{2}{N}}$$
$$\leq\frac{C_3}{\beta^{3-\frac{1}{m}}}2^{-2Nh}\left(\frac{aC_1}{\beta^2}\right)^{1+\frac{2}{N}}.$$

In order to make (2.23) valid, it suffices to take

$$a\leq\frac{\beta^2}{C_1}\left(\frac{2^{-2N}\beta^{3-\frac{1}{m}}}{C_3}\right)^{\frac{N}{2}}. \tag{2.24}$$

Now, setting $h\to\infty$ in (2.23), we find what we want.

The proof of Theorem 2.1 Set

$$\begin{aligned}&\mu_0=M,\ \mu_l=q_0\mu_{l-1},\ k_l=\beta\mu_l\\&\rho_0=\sqrt{\frac{t^0}{a\Phi'(M)}},\ \rho_l=\rho_{l-1}/4,\end{aligned} \tag{2.25}$$

$$Q_l=\{(x,t)\in Q_1^T\,|\,|x-x^0|<\rho_l, 0<t\leq t^0\}\quad(l=1,2,\cdots),$$
$$M_0=\max\{\max_{K(\rho_0)\cap\Omega}w(x,0),\ \max_{\substack{x\in K(\rho_0)\cap\partial\Omega\\0\leq t\leq t^0}}w(x,t)\},$$
$$l^*=\sup\{l\,|\,k_l>M_0\}, \tag{2.26}$$

where a is determined in Lemma 2.2 and q and β are defined in (1.10) and (1.11) respectively.

We shall argue by induction that

$$\begin{cases}\max\limits_{Q_l}\{w(x,t)\}\leq\mu_l\\ t^0\leq a\Phi'(\mu_l)\rho_l^2\end{cases}\quad\text{for } l\leq l^*+1. \tag{2.27}$$

In view of the selection of μ_0 and ρ_0, (2.27) is valid for $l=0$. Now suppose that (2.27) is valid for $l\leqslant l^*$, we shall show that (2.27) is valid for $l+1$ as well.

In virtue of $l\leqslant l^*$, it follows that

$$k_l\geqslant M_0,$$

which implies

$$k_l\geqslant \max\{\beta\mu_l,\ M_0\}.$$

Applying Lemma 2.2 to the domain Q_l, we obtain

$$\operatorname{mes} A_{\mu_l-(1-\beta)^2H_l,\rho_l/4}(t)=0 \text{ for } 0\leqslant t\leqslant t^0,$$

where $H_l=\mu_l-k_l=(1-\beta)\mu_l$, which yields

$$\max_{Q_{l+1}}\{w(x,t)\}\leqslant \mu_l-(1-\beta)^2H_l=\mu_l-(1-\beta)^3\mu_l=q_0\mu_l=\mu_{l+1}.$$

Furthermore, by the condition (1.4)′ we have

$$a\Phi'(\mu_{l+1})\rho_{l+1}^2=\frac{\Phi'(\mu_{l+1})}{\Phi'(\mu_l)}\cdot\frac{1}{16}\cdot a\Phi'(\mu_l)\rho_l^2$$

$$\geqslant\frac{1}{\lambda\left(\dfrac{\mu_{l+1}}{\mu_l}\right)}\cdot\frac{1}{16}\,t^0=\frac{1}{\lambda(q_0)}\cdot\frac{1}{16}\,t^0\geqslant t^0.$$

Thus (2.27) is valid for $l\geqslant l^*+1$ by induction and, in particular, we have

$$\max_{Q_{l^*+1}}\{w(x,t)\}\leqslant\mu_{l^*+1}.$$

By the definition (2.26) of l^*, $k_{l^*+1}\leqslant M_0$, and so we have

$$w(x^0,t^0)\leqslant\max_{Q_{l^*+1}}\{w(x,t)\}\leqslant\mu_{l^*+1}=\frac{1}{\beta}\,k_{l^*+1}\leqslant\frac{1}{\beta}\,M_0$$

as claimed in (2.7) if taking $b=[a\Phi'(M)]^{-\frac{1}{2}}$.

Theorem 2.3. *Let $w(x,t)$ belong to the class sub-$\mathscr{B}_2(Q^T,\lambda_0,\gamma,M,\Phi)$. If*

$$\inf_{x\in K(x^0,\rho)} w(x,0)\geqslant k>0,$$

then there exists some constant a depending only upon the parameters of $\mathscr{B}_2$ such that

$$\inf_{x\in K\left(x^0,\frac{\rho}{4}\right)} w(x,t)\geqslant\frac{k}{4} \text{ for } 0\leqslant t\leqslant t^0=\min\{a\Phi'(k)\rho^2,T\}.$$

Proof It is similar to the proof of Lemma 2.2, if noting that

$$\frac{1}{2m}\,s^2\leqslant\frac{\tilde{\chi}_k(s)}{\Phi'(k)}\leqslant m^2s^2 \text{ for } k\geqslant s>0,$$

which was obtained in the paper [2] (Lemma 3.3).

§ 3

In this section, we shall give the proof of Theorem 1.1 and Theorem 1.3.

Lemma 3.1. *Let $u(x,t)$ be a classical positive solution to the equation* (1.1) *with $0<u(x,t)\leqslant M$. Then, $v(x,t)=\varphi(e^{\eta(M)t}u(x,t))$ and $w(x,t)=\varphi(e^{-\eta(M)t}u(x,t)$ belong to sub-$\mathscr{B}_2(Q_1^T,\lambda_0,\gamma,\widetilde{M},\Phi)$ and super-$\mathscr{B}_2(Q_1^T,\lambda_0,\gamma,\widetilde{M},\Phi)$ respectively, where*

$\widetilde{M}=\varphi(Me^{\eta(M)T})$, λ_0 *and* γ *are some constants depending only upon the specified data and* $\Phi(w)$ *is defined in* (1.3).

The proof refer to § 2 in the paper [2].

The Proof of Theorem 1.1.

Let $\{u\ (x,t)\}$ be the sequence in the Definition 1.1 of weak solutions satisfying (1.7) and $w_n(x,t)=\varphi\,(u_n(x,t)e^{-\eta(M)t})$. By Lemma 3.1, $w_n(x,t)$ belong to the class super-$\mathscr{B}_2$ with the parameters independent of n.

Using Theorem 2.1 to $w_n(x,t)$, we obtain that for any $(x^0,t^0)\in Q_1^T$, there exists a constant b independent of n such that

$$w_n(x^0,t^0)\leqslant\frac{1}{\beta}\max\{\max_{K(x^0,b\sqrt{t^0})\cap\overline{\Omega}}w_n(x,0),\ \max_{\substack{K(x^0,b\sqrt{t^0})\cap\partial\Omega\\ 0\leqslant t\leqslant t^0}}w_n(x,t)\}.$$

In virtue of (1.8)′, it follows that

$$u_n(x^0,\ t^0)\leqslant\frac{me^{\eta(M)T}}{\beta}\max\{\max_{K(x^0,c\sqrt{t^0})\cap\Omega}u_n(x,\ 0),\ \max_{K(x^0,b\sqrt{t^0})\cap\partial\Omega}u_n(x,t)\}.$$

Let $n\to\infty$, the proof is complete.

The proof of Theorem 1.3.

In view of Lemma 3.1, the functions $v_n(x,t)=\varphi(e^{\eta(M)t}u_n(x,t))$ belong to the class sub-$\mathscr{B}_2(Q^T,\ \lambda_0,\ \gamma,\ \widetilde{M},\ \Phi)$. For simplicity, we drop the subscript n.

If $\widetilde{\mu}_\rho=0$, Theorem 1.3 is trivial. Now suppose that $\widetilde{\mu}_\rho>0$. We shall take $k>0$ such that

$$a\Phi'(k)\rho^2\geqslant t^0.$$

By the con dition (1.4)′, it suffices to select

$$\lambda\Big(\frac{k}{\widetilde{M}}\Big)\leqslant\frac{a\rho^2\Phi'(\widetilde{M})}{t^0}\quad\text{i. e.}\quad k\leqslant\widetilde{M}\lambda^{-1}\Big(\frac{\rho^2}{b^2t^0}\Big),$$

where $b=(a\Phi'(\widetilde{M}))^{-\frac{1}{2}}$. By Theorem 2.3, if

$$k=\min\Big\{\ \varphi(\widetilde{\mu}_\rho),\ \widetilde{M}\lambda^{-1}\Big(\frac{\rho^2}{b^2t^0}\Big)\Big\},$$

then

$$v(x^0,t^0)\geqslant\frac{1}{4}\min\Big\{\varphi(\widetilde{\mu}_\rho),\ \widetilde{M}\lambda^{-1}\Big(\frac{\rho^2}{b^2t^0}\Big)\Big\}.$$

Using the inequality (1.8)′, we obtain

$$u(x^0,t^0)\geqslant\frac{e^{-\eta(M)t^0}}{4m}\min\Big\{\widetilde{\mu}_\rho,\ \Phi\Big(\widetilde{M}\lambda^{-1}\Big(\frac{\rho^2}{b^2t^0}\Big)\Big)\Big\}$$

as claimed.

§ 4.

The proof of Theorem 1.6.

Let $w_n(x,\ t)=\varphi(e^{-\eta(M)t}u_n(x,\ t))$. Then $w_n(x,t)$ $(n=1,\ 2,\ \cdots)$ belong to the class super-$\mathscr{B}_2$. We omit the subscript n as well.

At first we take $\delta>0$ so small that

$$\delta\leqslant[(m+1)K\Phi'(\beta\widetilde{M})]^{-\frac{1}{2}}, \tag{4.1}$$

where K is the constant in (1.17) and $\widetilde{M}=\varphi(Me^{\gamma(M)T})$.

Set

$$\rho_l=\delta/4^l,\quad (l=0,1,2\cdots) \tag{4.2}$$

$$\varepsilon_l=\inf\left\{s\,|\,\Phi'(s)\leqslant\frac{1}{(m+1)K\rho_l^2}\right\}.$$

By the condition (1.17) of Theorem 1.6, we shall have

$$\varepsilon_l>\max_{|x-x^0|\leqslant\rho_l} w(x,0). \tag{4.3}$$

In fact, the condition (1.17) implies

$$\inf_{|x-x^0|<\rho_l}\Phi'(w(x,0))\geqslant\frac{1}{K\rho_l^2}\geqslant(m+1)\Phi'(\varepsilon_l).$$

Combining it with (1.4)′, we get (4.3).

Now we define

$$\begin{aligned}&\mu_0=\max\{\widetilde{M},\beta^{-1}\varepsilon_0\},\\ &\mu_l=\max\{\beta^{-1}\varepsilon_l,\ q_0\mu_{l-1}\}(l=1,\ 2,\ \cdots),\quad k_l=\beta\mu_l(l=0,\ 1,\ 2,\ \cdots),\\ &Q_l=\{(x,t)\,|\,|x-x^0|<\rho_l,0<t<t^*=a\Phi'(k_0)\rho_0^2\},\end{aligned} \tag{4.4}$$

where β and q_0 are defined in (1.10) and (1.11) respectively, and a is determined in Lemma 2.2.

We shall have

$$t^*\leqslant\frac{a}{(m+1)K}. \tag{4.5}$$

Actually, if $\mu_0=\beta^{-1}\varepsilon_0$, then

$$t^*=a\Phi'(k_0)\rho_0^2=a\Phi'(\varepsilon_0)\rho_0^2=\frac{a\rho_0^2}{(m+1)K\rho_0^2}=\frac{a}{(m+1)K},$$

and if $\mu_0=\widetilde{M}$, then, by the selection of δ, it follows that

$$t^*=a\Phi'(\beta\widetilde{M})\rho_0^2\leqslant\frac{a}{(m+1)K}.$$

In what follows we shall argue by induction that

$$\left.\begin{aligned}&\max_{Q_l}\{w(x,t)\}\leqslant\mu_l,\\ &t^*\leqslant a\Phi'(k_l)\rho_l^2\end{aligned}\right.\quad(l=0,1,2,\cdots). \tag{4.6}$$

Obviously, (4.6) is valid for $l=0$. Now suppose that (4.6) is valid for l.

In view of (4.3) it follows that

$$k_l\geqslant\varepsilon_l\geqslant\max_{|x-x^0|\leqslant\rho_l} w(x,0)\ (l=0,\ 1,\ 2,\ \cdots). \tag{4.7}$$

Using Lemma 2.2 in Q_l, we obtain

$$\max_{Q_{l+1}}\{w(x,t)\}\leqslant\mu_l-(1-\beta)^2H_l\leqslant\mu_{l+1},$$

where $H_l=\mu_l-k_l$. As for the second inequality of (4.6), if $\mu_{l+1}=q_0\mu_l$, then

$$a\Phi'(k_{l+1})\rho_{l+1}^2 \geqslant \frac{\Phi'(k_{l+1})}{\Phi'(k_l)}\cdot\frac{1}{16}t^* \geqslant \frac{1}{\lambda(q_0)}\cdot\frac{1}{16}t^* \geqslant t^*$$

and if $\mu_{l+1}=\beta^{-1}\varepsilon_{l+1}$, then it follows from (4.2) and (4.5) that

$$a\Phi'(k_{l+1})\rho_{l+1}^2 = a\Phi'(\varepsilon_{l+1})\rho_{l+1}^2 = \frac{a}{(m+1)K\rho_{l+1}^2}\,\rho_{l+1}^2 \geqslant t^*.$$

Thus, (4.6) is valid for $l+1$ as well. By induction, (4.6) has been proved and, in particular, we have

$$\max_{0\leqslant t\leqslant t^*} w(x^0, t) \leqslant \mu_l. \tag{4.8}$$

At present we should prove that

$$\mu_l \to 0 \quad \text{as} \quad l\to\infty.$$

We consider two cases:

(1) If $q_0\mu_{l-1}\geqslant\beta^{-1}\varepsilon_l$ for any $l\geqslant 0$, then since

$$\mu_l = q_0\mu_{l-1} \quad (l=1,2,\cdots),$$

the fact (4.9) is obvious.

(2) If $q_0\mu_{l-1}\leqslant\beta^{-1}\varepsilon_l$ for some l_0, we shall show that

$$q_0\mu_{l_0}\leqslant\beta^{-1}\varepsilon_{l_0+1}. \tag{4.10}$$

In fact, by (4.2) and (1.4)', it follows that

$$\frac{\Phi'(\varepsilon_{l_0})}{\Phi'(\varepsilon_{l_0+1})} = \frac{1}{16} \leqslant \lambda\left(\frac{\varepsilon_{l_0+1}}{\varepsilon_{l_0}}\right).$$

In virtue of the definition (1.10) of q_0, we have

$$\frac{\varepsilon_{l_0+1}}{\varepsilon_{l_0}} \geqslant q_0 \quad \text{i. e.} \quad \varepsilon_{l_0+1} \geqslant q_{l_0}\varepsilon_{l_0} = q_0\beta\mu_{l_0}.$$

Thus, (4.10) has been proved. By induction, we have

$$\beta^{-1}\varepsilon_l \geqslant q_0\mu_{l-1} \quad (l=l_0,\ l_0+1,\ \cdots).$$

Obviously, by (4.2), the sequence $\{\varepsilon_l\}$ converges to zero as $l\to\infty$ and so does $\{\mu_l\}$.

Let $l\to\infty$ in (4.8) we have proved what we want.

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