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UNUNIQUENESS OF SOME HOLOMORPHIC FUNCTIONS*

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Abstract

In the present paper, we show that there exists a bounded, holomorphic function $f(z)\not\equiv 0$ in the domain $\{z=x+iy: |y|<\alpha\}$ such that $f(z)$ has a Dirichlet expansion $\sum_{n=0}^{+\infty} d_n e^{-u_n z}$ in the halfplane $x>x_f$ if and only if $\frac{\alpha}{\pi}\log r-\sum_{u_n<r}\frac{2}{u_n}$ has a finite upperbound on $[1, +\infty)$, where α is a positive constant, $x_f(<+\infty)$ is the abscissa of convergence of $\sum_{n=1}^{+\infty} d_n e^{-u_n z}$ and the infinite sequence $\{u_n\}$ satisfies $\lim_{n\to+\infty}(u_{n+1}-u_n)>0$. We also point out some necessary conditions and sufficient ones Such that a bounded holomorphic function in an angular(or half-band) domain is identically zero if an infinite sequence of its derivatives and itself vanish at some point of the domain. Here some results are generalizations of those in [4].

P. Malliavin[1] transforms many problems(for example: Moment problems, quasianalyticity problems, asymptotic expansion problems of analytic functions, etc.) into corresponding Watson's problems and establishes the well-known Malliavin's theorem(M-theorem) of uniqueness for functions meromorphic in th halfplane. Yu Jia-Youg[6,7] extends the M-theorem to the case of zeros and poles of higher orders. Fuchs[3] gives another proof of M-theorem. As consequences of the M-theorem, we have the following reshlts:

Theorem 1. *Suppose that* $0<u_1<u_2<u_3<\cdots$, $\lim_{n\to\infty}(u_{n+1}-u_n)>0$. *Then there exists a function* $f(z)\not\equiv 0$, *which is holomorphic and bounded in* $\{z=x+iy: |y|<\alpha\}$ $(\alpha>0)$ *and which has a Dirichlet expansion*

$$f(z)=\sum_{n=1}^{+\infty} d_n e^{-u_n z} \quad (x>x_f), \tag{1.1}$$

if and only if

$$\overline{\lim_{r\to\infty}}\left(\frac{2\alpha}{\pi}\log r-2\sum_{u_n<r}\frac{1}{u_n}\right)<+\infty, \tag{1.2}$$

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where $x_f + \infty$ is the abscissa of convergence.

Proof We first suppose that $\alpha=\frac{\pi}{2}$, $u_{n+1}-u_n \geqslant h>0$ $(n=1, 2, \cdots)$, $u_1 \geqslant h$, where h is a constant. If $f(z)\not\equiv 0$ is holomorphic and bounded in $\{z=x+iy: |y|<\alpha\}$ and satisfies (1.1), we can also suppose that $x_f<-\frac{1}{n}$ $\Big($otherwise, we consider the function $f\left(z+x_0+\frac{1}{h}\right)$, where $x_0>x_f$ is a constant$\Big)$, then there exists a constant $M>0$ such that

$$|f(z)|\leqslant M, \quad |y|<\frac{\pi}{2},$$

$$|d_n e^{-u_n(-1/h)}|\leqslant M, \quad n=1, 2, \cdots.$$

There exists, for any positive integer p, a positive integer m such that $u_{m-1}<p\leqslant u_m$. Hence, if $x>0$, we have (set $u_0=0$, $d_0=0$)

$$|f(z)-\sum_{u_n<p} d_n e^{-u_n z}|=\left|\sum_{n=m}^{+\infty} d_n e^{-u_n z}\right|\leqslant M\sum_{n=m}^{+\infty} e^{-1/n u_n} e^{-u_n x}$$

$$\leqslant M e^{-px}\sum_{n=m}^{+\infty} e^{-n}\leqslant M(1-e^{-1})^{-1}e^{-px};$$

if $x\leqslant 0$, $|y|<\frac{\pi}{2}$, we have

$$|f(z)-\sum_{u_n<p} d_n e^{-u_n z}|\leqslant M+M\sum_{u_n<p} e^{-u_n/h} e^{|x| u_n|}$$

$$\leqslant M\sum_{n=0}^{m-1} e^{-u_n/h} e^{|x|p}\leqslant M(1-e^{-1})^{-1}e^{-px}.$$

Therefore (set $C=M(1-e^{-1})^{-1}$)

$$|C^{-1}f(z)-\sum_{u_n<p} C^{-1}d_n e^{-u_n z}|\leqslant e^{-px}\left(|y|<\frac{\pi}{2},\ p=1, 2, \cdots\right).$$

Since $C^{-1}f(z)\not\equiv 0$, in virtue of a result of Malliavin[[1], p.213] Watson's problem $W(1, k(r))$ admits a non-zero solution (where $k(r)=\log r-2\sum_{u_n<r}\frac{1}{u_n}$, $(r>u_1)$; $k(r)=0$, $0<r\leqslant u_1)$, i. e., there exists a function $g(z)\not\equiv 0$, which is holomorphic in the half-plane $x>0$ and satisfies $|g(z)|\leqslant e^{-xk(r)}$ $(r=|z|, x>0)$. If $k(r)$ has no finite upper bound on $(0, +\infty)$, then there would be a sequence $\{R_n\}$ of positive real numbers such that

$$\lim_{n\to+\infty} R_n=+\infty, \quad \text{tlim}_{n\to+\infty} k(R_n)=+\infty.$$

But for any positive n, the function $g_n(z)=g(z)e^{zk(R_n)}$ is holomorphic in the holf-plane $x>0$ and satisfies

$$|g_n(R_n e^{i\theta})|\leqslant 1, \quad |\theta|<\frac{\pi}{2},$$

$$\lim_{x\to 0^+}|g_n(x+iy)|\leqslant 1, \quad |y|\leqslant R_n.$$

By the Maximum Modulus Principle

$$|g_n(z)|\leqslant 1, \quad |z|\leqslant R_n, \quad x>0$$

and therefore

$$|g_n(z)|\leqslant e^{-xk(R_n)}, \quad |z|\leqslant R_n, \quad x>0.$$

Taking $n\to+\infty$, we see that $g(z)\equiv 0$ in the half-plane $x>0$, which is in contradiction with the hypothesis that $g(z)\not\equiv 0$. Therefore (1.2) holds.

Conversely, if (1.2) holds, then there exists a positive constant C_1 such that $k(r)\leqslant C_1 (r>0)$. Set

$$U(r)=\begin{cases}2\sum\limits_{v_n<r}\frac{1}{u_n}, & r>u_1,\\ 0, & 0<r\leqslant u_1,\end{cases}$$

$$G(z)=\prod_{n=1}^{\infty}\left[\frac{1+\frac{z}{u_n}}{1+\frac{z}{u_n}}\right]e^{\frac{2z}{u_n}}.$$

Then we see from Theorem 4.5.III in [2] (p. 126) (and from Chap. 7.4 (4.3; 3') in [8] (p. 469)) that $G(z)$ is a meromorphic function in the z-plane and satisfies

$$|G(z)|\geqslant\exp(-C_2x+xU(r)),\ z\in W,$$

and that gamma function $\Gamma(z)$ satisfies

$$|\Gamma(z+1)|\leqslant\exp\left(C_3+\log|z+1|+x\log\left|z-\left|\frac{\pi}{2}\right|y\right|\right)^*,\ x>0,$$

where C_2 and C_3 are positive constant. Set $C=C_1+C_2+C_3$, then the function $g(z)=(1+z)^{-3}(G(z))^{-1}\Gamma(z+1)e^{-Cz}$ is meromorphic in the z-plane and satisfies

$$|g(z)|\leqslant(1+y^2)^{-1}e^{-x|y|/2},\ z\in W.$$

Set

$$f_n(z)=\int_{-\infty}^{+\infty}g(2^{-1}(u_n+u_{n+1})+it)e^{-itz}\,dt,\quad n=1,2,\cdots,$$

$$f_0(z)+=\int_{-\infty}^{+\infty}g(it)e^{-itz}\,dt.$$

Then $f(z)$ is holomorphic in $\left\{z=x+iy:|y|<\frac{\pi}{2}\right\}$ and satisfies

$$|f_n(z)|\leqslant\exp(-2^{-1}(u_n+u_{n+1})x),\ \ |y|<\frac{\pi}{2},\ n=1,2,\cdots, \tag{1.3}$$

$$|f_n(z)|\leqslant\pi,\ \ |y|<\frac{\pi}{2}. \tag{1.4}$$

For any $u'>u''>0$, we have

$$\lim_{|t|\to\infty}\int_{u''}^{u'}g(u+it)e^{-z(u+it)}\,du,\ |y|<\frac{\pi}{2}.$$

By the residue theorem we have

$$f_0(z)=\sum_{k=1}^{n}d_ke^{-u_nz}+f_n(z),\ |y|<\frac{\pi}{2},\ n=1,2,\cdots,$$

where d_k is the residue of $g(z)$ at u_k, $k=1,2,\cdots$. By (1.3), if $x>0$, $|y|<\frac{\pi}{2}$, we have

$$\lim_{n\to\infty}f_n(z)=0.$$

Hence

$$f_0(z)=\sum_{k=1}^{+\infty}d_ke^{-u_kz},\ x>0,\ |y|<\frac{\pi}{2}.$$

This implies that $\sum_{k=1}^{+\infty} d_k e^{-u_k z}$ is holomorphic in the half-plane $x>0$. We establish the theorem, if we put

$$f(z)=\begin{cases}\sum_{k=1}^{+\infty} d_k e^{-u_k z}, & x>0,\\ f_0(z), & x\leqslant 0,\ |y|<\frac{\pi}{2}.\end{cases}$$

For the case $\alpha\neq\frac{\pi}{2}$, let $u'_u=\frac{2\alpha}{\pi}u_n(n=1, 2,\cdots)$, $f_1(z)=f\left(\frac{2\alpha}{\pi}z\right)$. Since

$$\varlimsup_{r\to+\infty}\left(\log r-2\sum_{u'_n<r}\frac{1}{u'_n}\right)<+\infty$$

holds if and only if

$$\varlimsup_{r\to+\infty}\left(\frac{2\alpha}{\pi}\log r-2\sum_{u_n<r}\frac{1}{u_n}\right)<+\infty,$$

we see that the theorem also holds in the general case.

Let $\{u_n\}$ and $\{v_n\}$ be two complexmentary incresing sequences of positive integers ([2], p. 54), we see that $\sum_{u_n<r}\frac{1}{u_n}+\sum_{v_n<r}\frac{1}{v_n}=\sum_{n<r}\frac{1}{n}=\log r+O(1)$, $r\to\infty$. Therefore, as a consequence of Theorem 1, we have the following corollary:

Corollary 1. *There exists a function $f(z)\not\equiv 0$ such that $f(z)$ is holomorphic and boundea in $\{z=x+iy: |\arg z|<a\}\cup\{0\}$ $(0<a\leqslant\pi)$ and that $f(0)=f^{(v_n)}(0)=0$ if and only if*

$$\log_{r\to+\infty}\left(\sum_{v_n<r}\frac{1}{v_n}-\left(1-\frac{a}{\pi}\right)\log r\right)<+\infty.$$

Remark. Corollary 1 is a generlization of the result III in [4].

If we add a half-band to each side of the angular domain $\{z=x+iy: |\arg z|\leqslant a\}$ $(0\leqslant a\leqslant\pi)$, and write $\log_0 x=x$ and $\log_k x=\log(\log_{k-1}x)$, k being a positive integer and x being sufficiently large, we have the following theorem:

Theorem 2. *Suppose $0\leqslant a<\pi$. If there exist an integer $p\geqslant 2$ and a real number $b>1$ such that*

$$\varlimsup_{r\to+\infty}\left(2\sum_{v_n<r}\frac{1}{v_n}-2\left(1-\frac{a}{\pi}\right)\log r+\log_2 r+\cdots+b\log_p r\right)<+\infty, \tag{2.1}$$

then there exist a function $f(z)\not\equiv 0$ and a constant $a_f>0$ such that $f(z)$ is holomorphic and bounded in the domain $G_a=\bigcup_{z\in H_a}D(z, a_f)$, where $H_a=\{z=re^{i\theta}: |\theta|\leqslant a, r\geqslant 0\}$, $D(z, a_f)=\{z': |z'-z|<a_f\}$ and satisfies

$$f(0)=f^{(v_n)}(0)=0.$$

Conversely, if there exists an integer $p\geqslant 2$ such that

$$\varliminf_{r\to\infty}\left(2\sum_{v_n<r}\frac{1}{v_n}-2\left(1-\frac{a}{\pi}\right)\log r+\log_2 r+\cdots+\log_p r\right)>-\infty, \tag{2.2}$$

then there exists no such function $f(z)\not\equiv 0$ with the above properties.

Remark. **1)** The latter part of Theorem 2, if $a=0$, is the one of the result I in [4].

Remaak. 2) We see from Corollary 1 that if the condition (2.2) is replaced by

$$\overline{\lim_{r\to+\infty}}\left(2\sum_{v_n<r}\frac{1}{v_n}-2\left(1-\frac{a}{\pi}\right)\log r\right)=+\infty,$$

then Theorem 2 also holds.

Before proving Theorem 2, we first compare it with Corollary 1. Suppose that $\pi>a>>0$, $C>0$, $p\geqslant 2$ and $r_0>0$ ($\log_p r_0>0$) satisfy

$$2\left(1-\frac{a}{\pi}\right)\log r-\log_2 r-\cdots-\log_p r-C\leqslant 2\sum_{v_n<r}\frac{1}{v_n}\leqslant 2\left(1-\frac{a}{\pi}\right)\log r+C,\ r\geqslant r_0.$$

Corollary 1 implies that there exists a function $f(z)\not\equiv 0$ such that $f(z)$ is holomorphic and bounded in the domain $\{z=x<iy\colon |\arg z|\leqslant a\}\cup\{0\}$ and satisfies $f(0)=f^{(v_n)}(0)=0$. Theorem 2 implies that if $f(z)$ can be analytically continued to $G_a=\bigcup_{z\in H_a}D(z,\ a_f)$, where $H_a=\{z=re^{i\theta}\colon |\theta|\leqslant a,\ r\geqslant 0,\ D(z'\colon|z'-z|<a_f\},\ a_f>0$, then $f(z)$ is unbounded in G_a.

In order to prove Theorem 2, we first introduce a definition: Let G be a set of z-plane and $f(z)$ be defined in G. Then $f(z)$ is said to be differentiable on G, if, at each point $z_0\in G$,

$$\lim_{z\to z_0,z\in G}\frac{f(z)-f(z_0)}{z-z_0}$$

exists. This limit is denoted by $f'(z_0)$. As usual, the differentiability of higher order and infinite differentiability on G can be similarly defined. The following Lemma 1 is a generlization of Lemma in [4].

Lemma 1. *Let $\{M_n\}$, $n=0,\ 1,\ 2,\ \cdots$ be a sequence of positive real numbers. Suppose that for $a(0\leqslant a\leqslant\pi)$, $f(z)\not\equiv 0$ is infinitely differentiable on $G=\{z=re^{i\theta}\colon|\theta|\leqslant a,\ r\geqslant 0\}$ and satisfies*

$$\{f^{(n)}(z)|\leqslant M_n,\quad n=0,\ 1,\ 2,\ \cdots,\quad z\in G. \tag{3.1}$$

Then there exists a holomorphic function $g(z)\not\equiv 0$ in the half-plane $x>0$ such that

$$g(n)=(-1)^n f^{(n)}(0)^*,\ n=1,\ 2,\ \cdots, \tag{3.2}$$

$$|g(n+b+iy)|\leqslant(M_{n-1}+M_n+M_{n+1}+M_{n+2})C|2+iy|^3e^{(\pi/2-a)|y|},$$
$$0<b<1,\ n=1,\ 2,\ \cdots, \tag{3.3}$$

where C is a positive constant.

Proof We see from the proof in [1] (p. 210) and [4] that there exists a holomorphic function $(z)\not\equiv 0$ such that (3.2) holds and that

$$|g(n+b+iy)|=\left|\frac{1}{\Gamma(-b-iy)}\int_0^{+\infty}f^{(n)}(t)-f^{(n)}(0))t^{-i-b-iy}\,dt\right|$$
$$=\left|\frac{\exp(-a|y|)}{\Gamma(-b-iy)}\int_0^{+\infty}(\bar f_y^{(n)}(t)-\bar f_y^{(n)}(0))t^{-1-b-iy}\,dt\right|$$

holds for any positive integer n and b $(0<b<1)$, where

$$\tilde{f}_y(t)=\begin{cases} f(te^{-ia}), & t\geqslant 0,\ y>0, \\ f(te^{ia}), & t\geqslant 0,\ y\leqslant 0. \end{cases}$$

Integration by parts gives us

$$|g(n+b+iy)|=\frac{\exp(-a|y|)}{|\Gamma(2-b-y)|}\Big|(b+iy)(\tilde{f}_y^{(n)}(0)-\tilde{f}_y^{(n)}(0))-\tilde{f}_y^{(n+1)}(1)$$
$$+\int_0^1 f_y^{(n+2)}(t)t^{1-b-iy}\,dt+(b+iy-1)\Big[f_y^{(n-1)}(1)(-b-iy)-f_y^{(n)}(0)$$
$$+(b+iy)(b+1+iy)\int_1^{+\infty} f_y^{(n-1)}(t)t^{-2-b-iy}\,dt\Big]\Big|,$$
$$0<b<1,\ n=1,\ 2,\ \cdots.$$

We can see that (3.3) is also valid.

The following Lemma 1 can be considered as the converse of Lemma 1.

Lemma 2. *Let $\{M_n\}$ $(n=0, 1, 2,\cdots)$ be a sequence of positive numbers. Suppose that for a $(0\leqslant a\leqslant\pi)$ $g(z)$ is holomorphic in the half-plane $x>0$ and satisfies*

$$|g(n+b+iy)|\leqslant M_n\exp\left(\left(\frac{\pi}{2}-a\right)|y|\right),\quad 0<b<1,\ n=0,\ 1,\ 2,\ \cdots. \tag{4.1}$$

Then

$$f(z)=\int_{-\infty}^{+\infty}\left(\frac{1}{2}+it+1\right)^{-3}\left(\frac{1}{2}+it\right)g\left(\frac{1}{2}+it\right)\Gamma\left(\frac{1}{2}\ it\right)z^{\frac{1}{2}+it}\,dt$$

is infinitely differentiable in $G=\{z=re^{i\theta}: |\theta|\leqslant a, r\geqslant 0\}$ and there exists a positive constant B such that

$$|f(z)|\leqslant BM_0,\ |f^{(n)}(z)|\leqslant Be^n\max(M_n,\ M_{n-1}),\ z\in G,\ n=1,\ 2,\ \cdots, \tag{4.2}$$
$$f(0)=0,\ f^{(n)}(0)=(n+1)^{-3}2\pi i(-1)^{n+1}g(n)n,\ n=1,\ 2,\ \cdots. \tag{4.3}$$

Moreover, if $g(z)\not\equiv 0$, then $f(z)\not\equiv 0$.

Proof Since ([8], p. 469. (4.3.3'))

$$|\Gamma(z+1)^{-1}\leqslant|z+1|^{-x-\frac{1}{2}}\exp\left(\frac{\pi}{4}+x+\frac{1}{2}|y|\right),\quad x>0, \tag{4.4}$$

$$\left|\sin\pi\left(n+\frac{1}{2}+iy\right)\right|\geqslant\frac{1}{2}\exp(\pi|y|),\quad n=0,\ 1,\ 2,\ \cdots, \tag{4.5}$$

$$|\sin\pi(x+iy)|\geqslant\frac{1}{4}\exp(\pi|y|)\quad(|y|\geqslant 2), \tag{4.6}$$

it follows that there exists a positive constant B_1 such that the holomorphic function $h(z)=(z+1)^{-3}z\Gamma(-z)g(z)$ in the halfplane $x>0$ satisfies

$$\left|h\left(n+\frac{1}{2}+it\right)\frac{\partial^k}{\partial z^k}(z^{n+\frac{1}{2}+it})\leqslant B_1(1+y^2)^{-1}M_ne^n|z|^{n+\frac{1}{2}-k},\ x>0,\ k;\ n=0,\ 1,\ 2,\cdots\right).$$

This implies that

$$f_n(z)=\int_{-\infty}^{+\infty}h\left(n+\frac{1}{2}+it\right)z^{n+\frac{1}{2}+it}\,dt$$

is $(n+1)$ times differentable on $\{z:*|\arg z|\leqslant a,\ z\neq 0\}$ and n times differentiable on G and it satisfies

$$|f_n^{(k)}(z)| \leqslant B_1 e^n M_n |z|^{n+\frac{1}{2}-k}, 0 \leqslant k \leqslant n+1, |\arg z| \leqslant a, z \neq 0 \tag{4.7}$$

and that, for any v'', $v'(v''>v'>0)$,

$$\lim_{|t|\to\infty} \int_{v'}^{v''} h(v+it) z^{n+it} dv = 0, \quad z \in G.$$

It follows from the residue theorem and Cauchy's theorem that, for any v_1, v_0 $\left(0<v_1<v_0<\frac{1}{2}\right)$ and $n=1, 2, \cdots$,

$$\begin{aligned} f_{v_0}(z) &= \int_{-\infty}^{+\infty} h(v_0+it) z^{v_0+it} dt = \int_{-\infty}^{+\infty} h(v_1+it) z^{v_1+it} dt \\ &= \sum_{k=1}^{n} \frac{2\pi i(-1)^{k+1} g(k) k}{(k+1)k!} z^k + f_n(z), \; z \in G. \end{aligned} \tag{4.8}$$

We conclude from (4.3)—(4.6) that there exists a positive constant C_{21}, which is independ on v_1 and h such that

$$|f_{v_0}(z)| \leqslant C_2 M_0 |z|^{v_1}, \; z \in G.$$

Taking $v_1 \to 0$, we obtain

$$f_{v_0}(z)| \leqslant C_2 M_0, \; z \in G.$$

We see from (4.1), (4.7) and (4.8) that (4.2) and (4.3) are valid with $f(z) f_{v_0}(z)$.

The proof of Theorem 2 Suppose that (2.1) holds. Then there exist two constants $C>0$ and r_0 ($\log_p r_0 > 0$) such that

$$k^{\cdot}(r) = \sup_{1 \geqslant r' \geqslant r} \left(2 \sum_{v_n' < r'} \frac{1}{v_n} - \left(1 - a \frac{2}{\pi}\right) \log r'\right)$$

$$\leqslant C + \log r - \log_2 r + \cdots + \log_{p-1} r + b \log_p r), \; r \geqslant r_0$$

and ([2], p. 122)

$$M(t) = \sup_{n \geqslant 1} (nt - n \log n) \leqslant C e^t (t \geqslant 0).$$

Therefore, we have

$$\int_{r_0}^{+\infty} M(k^{\cdot}(r)) \frac{dr}{r^2} \leqslant C e^C \int_{r_0}^{+\infty} (r \log r \cdots \log_{p-2}{}^r (\log_{p-1} r)^b)^{-1} dr < +\infty.$$

It follows from Theorem 5.8 in [1], (p. 205) that there exists a holomorphic function $g(z)$ in the half-plane $x>0$ such that

$$g(v_n) = 0, |g(n+b+iy)| \leqslant n^n \exp\left(\left(\frac{\pi}{2} - a\right)|y|\right), \; 0<b<1, \; n=1, 2, \cdots,$$

$$|g(b+iy)| \leqslant \exp\left(\left(\frac{\pi}{2} - a\right)|y|\right), \; 0<b<1.$$

We see from Lemma 2 that there exists a constant B and an infinitely diefferentiable function $\tilde{f}(z) \not\equiv 0$ in $\{z: |\arg z| \leqslant a\}$ such that for $z: |\arg z| \leqslant a$,

$$|\tilde{f}(z)| \leqslant B, \tilde{f}(0) = \tilde{f}^{(v_n)}(0) = 0, \; |\tilde{f}^{(n)}(z)| \leqslant B n^n e^n, n=1, 2, \cdots.$$

We conclude that, at each point $z_0 \in \{z: |\arg z| \leqslant a\}$, $f_0(z) = \sum_{n=1}^{-\infty} \frac{\tilde{f}^{(n)}(z-z_0)}{n!} (z-z_0)^n$ is holomorphic in $\{z: |z-z_0| \leqslant (2e^2)^{-1}\}$ and is equal to $\tilde{f}(z)$ in

$$\{z: |z-z_0| \leqslant (2e^2)^{-1}\} \cap \{z: |\arg z\{ \leqslant a\}$$

and satisfies

$$|f_0(z)|\leqslant\sum_{0=n}^{+\infty}B(n!)^{-1}(\mathrm{en})^n(2e^2)^{-n}\leqslant 2B(|z-z_0|\leqslant(2e^2)^{-1}).$$

Therefore $\tilde{f}(z)$ can be analytically continued to a bounded, holomorphic function $f(z)$ in the domain $G_a=\bigcup_{z\in H_a}D(z,\ (2e^{-1})^{-1})$.

Conversely, if (2.2) holds and if there exists a function $f(z)\not\equiv 0$, which is holomorphic and bounded in the domain $G_a=\bigcup_{z\in H_a}D(z,\ a_f)$, where $H_a=\{z=re^{i\theta}:|\theta|\leqslant a,\ r\geqslant 0\}$, $a_f>0$, it follows, then, from (2.2), ([2], p. 122) and Cauchy's integral formula that there exist constants $M>0$, $B>0$, $B_1>0$, $C>0$ and $h>0(h<\min(1,a))$, $r_0(\log_p r_0>0)$ such that

$$|f^{(n)}(z)|\leqslant n!Mh^{-n}=M_n,\ M_{n+4}\leqslant B_1^n n^n(|\arg z|\leqslant a,\ n=1,\ 2,\ 3,\cdots),$$
$$M(t)=\sup_{n\geqslant 1}(nt-\log M_{n+4})\geqslant Be^t(t\geqslant 10),$$

$$k.(r)=\inf_{r'\geqslant r}\left(2\sum_{v'_n<r'}\frac{1}{v'_n}-\left(1-a\frac{2}{\pi}\right)\log r'\right)\geqslant\log r-\log_2 r-\cdots-\log_p r-C,\quad r\geqslant r_0,$$

where $a'_n=v_{n+2}-2(n=1,\ 2,\ \cdots)$. It follows from Lemma 1 that there existsa holomorphic function $g(z)\not\equiv 0$ in the half-plane $x>0$ such that there exists a positive constant C_1

$$g(v_n)=(-1)v_n f^{(v_n)}(0)=0,\ n=1,\ 2,\ \cdots,$$
$$|g(n+b+iy)|\leqslant C_1|2+iy|^3M_{n+2}\exp\left(\left(\frac{\pi}{2}-a\right)|y|\right),\ 0<b<1,\ n=0,\ 1,\ 2,\cdots.$$

We see that the holomorphic function $\tilde{g}(z)=C_1^{-1}(4+z)^{-3}g(z+2)$ in the half-plane $x>0$ satisfies

$$\tilde{g}(v'_n)=0\quad n=1,\ 2,\ \cdots,$$
$$|\tilde{g}(n+b+iy)|\leqslant M_{n+4}\exp\left(\left(\frac{\pi}{2}-a\right)|y|\right),\quad 0<b<1,\ n=0,\ 1,\ 2,\cdots.$$

But for any real number a, there exists $r_a>r_0$ such that

$$k\cdot(r)-a>10,\quad r>r_a,$$

$$\int_{r_a}^{+\infty}M(k.(r)-a)\frac{dr}{r^2}\geqslant B\int_{r_a}^{+\infty}\exp(k.(r)-a)\frac{dr}{r^2}$$
$$\geqslant B\exp(-a-C)\int_{r_a}^{+\infty}(r\log r\cdots\log_{p-1}r)^{-1}dr=+\infty.$$

We conclude from Theorem 5.8 in [1] (p. 205) that $\tilde{g}(z)\equiv 0$, i. e., $g(z)\equiv 0$, which is in contradiction with $g(z)\not\equiv 0$, and this proves Theorem 2.

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