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CONTROLLABILITY, REACHABILITY AND STRONG CONNECTIVITY OF DISCRETE-TIME SYSTEMS WITH POSITIVELY LIMITED CONTROLS

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Abstract

In this paper the author studies the controllability, reachability and strong connectivity of discrete-time linear systems with positively limited controls. Necessary and sufficient conditions are given to test these properties for a single-input system.

§1. Introduction

The problems of discrete-time systems have been studied by many researchers during the last twenty years. Kalman[1] introduced the concept of complete controllability and derived an elegant algebraic test for this property when controls are unconstrained. The positive controller problem studied in [2—4] for continuous-time systems has been resolved in [5] for the single-input discrete-time case. [6] gives a test for complete controllability of linear systems with control constraint.

In this paper we examine the controllability, reachability and strong connectivity of single-input discrete-time systems of the form

$$X_{k+1}=AX_k+bu_k, \quad k=0, 1, \cdots, \tag{1}$$

where $X_k\in R^n$, $u_k\in[0, l]$, $l>0$, A and B are real constant $n\times n$, $n\times m$ matrices, respectively. Later we consider $u_k\in[a, l]$, $l>a>0$.

§ 2. Perliminaries

For system (1) we can obtain the solution after k steps:

$$X_k=A^kX_0+\sum_{i=1}^{k}A^{k-i}bu_{i-1}. \tag{2}$$

Definition 1. *The system* (1) *is said to be completely controllable with* $u_k\in[0, l]$ *if any state in it may be carried to zero in finite steps by a suitable choice of the*

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input sequenece $\{u_i \in [0, l]\}$.

Definition 2. *The system* (1) *is said to be completely reachable with* $u_k \in [0, l]$ *if any state in it can be reached from zero in finite steps by a suitable choice of the input sequence* $\{u_i \in [0, l]\}$.

Definition 3. *The system* (6) *is said to be strongly connected with* $u_k \in [0, l]$ *if it is realizable to transfer any state into any other state in finite steps by a suitable choice of the input sequence* $\{u_i \in [0, l]\}$.

The following result as a lemma is from [6].

Lemma 1. *The system* (1) *is completely controllable with* $u_k \in [-l, l]$, $k=0, 1, \cdots$ *if and only if*

1) rank $[A^{n-1}b, \cdots Ab, b] = n$,

2) $|\lambda_i(A)| \leqslant 1$ *for all* $\lambda_i(A)$,

where $\lambda_i(A)$ *represents the i–th eigenvalue of* A.

Remark. The conclusions of Lemma 1 do not relate with the magnitude of $l>0$.

Other two lemmas are quoted from [5].

Lemma 2. *The system* (1) *is completely controllable with* $u_k \in [0, \infty)$ $k=0, 1, \cdots$ *if and only if*

1) rank $[A^{n-1}b \cdots, Ab, b] = n$,

2) *A has no real eigenvalues* $\lambda_i(A) \geqslant 0$.

Remark. The conclusions of Lemma 2 do not change when $u_k \in [a, \infty)$, $a>0$ $k=0, 1, \cdots$.

Lemma 3. *If A has no real eigenvalues* $\lambda_i(A) \geqslant 0$, *then for A there exists an annihilating polynomial with strictly positive coefficients.*

§ 3. Controllability

Now we give the main result in this section, it is essential to the later development.

Theorem 1. *The system* (1) *with* $u_k \in [0, l]$, $k=0, 1, \cdots$ *is completely controllable on* R^n *if and only if*

1) rank$[A^{n-1}b, \cdots Ab, b] = n$,

2) $|\lambda_i(A)| \leqslant 1$ *for all* $\lambda_i(A)$,

3) *A has no real eigenvalues* $\lambda_i(A) \geqslant 0$.

Proof The necessity of 1) follows from the result for unconstrained u. The necessary conditions 2) and 3) are based on Lemmas 1—2.

The proof of sufficiency proceeds in three parts.

Part 1: By Lemma 3, the condition 3) implies that there exists a polynomial with strictly positive coefficients, e. g.,

$$\omega(\lambda)=a_0\lambda^{k_1-1}+a_1\lambda^{k_1-2}+\cdots+a_{k_1-1},\ a_i>0,\ i=0,\ 1,\ \cdots,\ k_1-1,$$

such that

$$\omega(A)=a_0A^{k_1-1}+a_1A^{k_1-2}+\cdots+a_{k_1-1}I=0. \tag{4}$$

Let $D=\max\{a_0,a_1,\ \cdots a_{k_1-1}\}$, $d=\min\{a_0,\ a_1,\ \cdots,\ a_{k_1-1}\}$. We select two positive constants which satisfy the following conditions:

$$\begin{cases} E/\varepsilon\geqslant D/d, \\ \varepsilon+E=1, \end{cases} \tag{5}$$

and then determine a proportional constant η which satisfies

$$\varepsilon l\leqslant\eta a_i\leqslant El \quad \text{for all } i=0,\ 1,\ \cdots k_1-1. \tag{6}$$

The existence of η is fairly clear from (5).

Postmultiplying equation (4) by $b\eta$, we have

$$A^{k_1-1}b\eta a_0+A^{k_1-2}b\eta a_1+\cdots+Ab\eta a_{k_1-2}+b\eta a_{k_1-1}=0. \tag{7}$$

Part 2: By Lemma 1, the conditions 1) and 2) imply that the system (1) is completely controllable with $u_k\in[-\varepsilon l,\ \varepsilon l]$, $k=0,\ 1,\ \cdots$, i. e., for any given state $X\in R^n$ there exists a step number k_2 and an input sequence $\bar{u}_0\ \bar{u}_1\cdots\bar{u}_{k_2-1}$ such that it satisfies

$$A^{k_2}X+A^{k_2-1}b\bar{u}_0+A^{k_2-2}b\bar{u}_1+\cdots+Ab\bar{u}_{k_2-2}+b\bar{u}_{k_2-1}=0, \tag{8}$$

where

$$-\varepsilon l\leqslant\bar{u}_i\leqslant\varepsilon l,\ i=0,\ 1,\ \cdots,\ k_2-1. \tag{9}$$

We want $k_1=k_2$. If it is not, the following method can be used:

(i) $k_1>k_2$. Let the new sequence $\hat{u}_0\ \hat{u}_1\cdots\hat{u}_{k_1-1}$ be

$$\bar{u}_0\cdots\bar{u}_{k_2-1}\quad \underbrace{00\cdots0}_{(k_1-k_2)\text{ zero inputs}}.$$

Then we have

$$A^{k_1}X+A^{k_1-1}b\hat{u}_0+A^{k_1-2}b\hat{u}+\cdots+Ab\hat{u}_{k_1-2}+b\hat{u}_{k_1-1}=0, \tag{10}$$

and equation (8) gives place to equation (10).

(ii) $k_1<k_2$. Selecting a positive integer m which satisfies $mk_1+k_1>k_2$, we can obtain a new annihilating polynomial with strictly positive coefficients for A

$$\omega'(\lambda)=\lambda^{mk_1}\omega(\lambda)+\lambda^{(m-1)k_1}\omega(\lambda)+\cdots+\lambda^{k_1}\omega(\lambda)+\omega(\lambda).$$

It is fairly clear that

$$\omega'(A)=A^{mk_1}\omega(A)+A^{(m-1)k_1}\omega(A)+\cdots+A^{k_1}\omega(A)+\omega(A)=0. \tag{11}$$

Postmultiplying Equation (11) by $b\eta$, we have

$$A^{(m+1)k_1-1}b\eta a_0+A^{(m+1)k_1-2}b\eta a_1+\cdots+Ab\eta a_{k_1-2}+b\eta a_{k_1-1}=0. \tag{12}$$

It may be used instead of equation (7), and turns into above (i).

In brief, it is reasonable to assume $k_1=k_2=k$.

Part 3: Adding equation (7) to equation (8), and merging the coefficients of term A^ib into one, we obtain

$$A^kX+A^{k-1}bu_0+A^{k-2}bu_1+\cdots+bu_{k-1}=0, \tag{13}$$

where $u_i=\bar{u}_i+\eta a_i$, $i=0, 1, \cdots, k-1$.

From (6), (9) and (5) we can verify

$$0\leqslant u_i\leqslant l, \quad i=0, 1, \cdots, k-1.$$

By Definition 1, it is to say that the system (1) with $u_k\in[0, l]$, $k=0, 1, \cdots$ is completely controllable on R^n.

Remark. 1) The conclusions of Theorem 1 do not relate with the magnitude of $l>0$;

2) If we now restrict our input $u_k\in[a, l]$, $l>a>0$, the conclusions, in fact, have no alteration.

§ 4. Reachability

First, we introduce the concept of inverse system. It proves to be a happy one, for we see that our criteria yield the Duality Theorem.

Definition 4. *The system*

$$X_{k+1}=A^{-1}X_k-A^{-1}bu_k, \quad k=0, 1, \cdots \tag{14}$$

is called an inverse of the system (1) *when* $\det A\neq 0$.

Obviously, the systems (A, b) and $(A^{-1}, -A^{-1}b)$ are each other's inverse.

From equation (2), the controllable set and reachable set of (A, b) with $u_k\in[0, l]$, $k=0, 1, \cdots$ in k step times are given by

$$\mathscr{C}^k_{(A,b)}\triangleq\left\{X_0: X_0=-[A^{-1}b, A^{-2}b, \cdots A^{-k}b]\begin{bmatrix}u_0\\ \vdots\\ u_{k-1}\end{bmatrix}, \quad u_i\in[0, l], i=0, 1, \cdots, k-1\right\}$$

and

$$\mathscr{R}^k_{(A,b)}\triangleq\left\{X: X=[A^{k-1}b, \cdots, Ab, b]\begin{bmatrix}u_0\\ \vdots\\ u_{k-1}\end{bmatrix}, \quad u_i\in[0, l], i=0, 1, \cdots, k-1\right\},$$

respectively, when $\det A\neq 0$.

Before proceeding with the main result of this section, we state a lemma which is useful to the later theorem.

Lemma 4. 1) $\mathscr{C}^k_{(A,b)}=\mathscr{R}^k_{(A^{-1},-A^{-1}b)}$,

2) $\mathscr{R}^k_{(A,b)}=\mathscr{C}^k_{(A^{-1},-A^{-1}b)}$.

Proof Provided we note

$$-[A^{-1}b, A^{-2}b, \cdots, A^{-k}b]\begin{bmatrix}u_0\\ \vdots\\ u_{k-1}\end{bmatrix}$$

$$=[(A^{-1})^{k-1}(-A^{-1}b),\ \cdots,\ (A^{-1})^{-1}(-A^{-1}b),\ (-A^{-1}b)]\begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix},$$

$$[A^{k-1}b,\ \cdots,\ Ab,\ b]\begin{bmatrix} u_0 \\ \vdots \\ u_{k-1} \end{bmatrix}$$

$$=-[(A^{-1})^{-1}(-A^{-1}b),\ (A^{-1})^{-2}(-A^{-1}b),\ \cdots,\ (A^{-1})^{-k}(-A^{-1}b)]\begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix},$$

it is easy to see this lemma is true.

By Lemma 4, obviously, we have

Theorem 2. *The system* (1) *with* $u_k \in [0, l]$, $k=0, 1, \cdots$, *is completely reachable on* R^n *if and only if*

1) rank $[A^{n-1}b, \cdots Ab, b]=n$,

2) $|\lambda_i(A)| \geqslant 1$ *for all* $\lambda_i(A)$,

3) *A has no real eigenvalues* $\lambda_i(A) \geqslant 0$.

We omit the straightforward proof.

Remark. 1) The conclusions of Theorem 2 do not relate with the magnitude of $l>0$;

2) If we restrict our input $u_k \in [a, l]$, $l>a>0$, the conclusions, in fact, have no alteration.

§ 5. Strong Connectivity

From the following lemma we will derive our main result for this section–Theorem 3.

Lemma 5. *The system* (1) *with* $u_k \in [0, l]$, $k=0, 1, \cdots$ *is strongly connected on* R^n *if and only if it is both completely controllable and completely reachable with* $u_k \in [0, l]$.

We omit the straightforward proof.

Theorem 3. *The system* (1) *with* $u_k \in [0, l]$, $k=0, 1, \cdots$ *is strongly connected on* R^n *if and only if*

1) rank$[A^{n-1}b, \cdots, Ab, b]=n$,

2) $|\lambda_i(A)|=1$ *for all* $\lambda_i(A)$,

3) *A has no real eigenvalue* 1.

Remark. 1) The conclusions of Theorem 3 do not relate with the magnitude of $l>0$;

2) If we restrict our input $u_k \in [a, l]$, $l>a>0$, the conclusions, in fact, have

no alteration.

§ 6. Conclusion

The controllability, reachability and strong connectivity of linear discrete-time systems with control constraint are three different concepts. Besides the controllability matrix, they are connected with system pole location, respectively. The problem of positive controller with amplitude constraint has been completely resolved for the single-input discrete-time case. For multiinput systems the corresponding result may be conjectured, but the development is not straightforward.

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