Chin. Ann. of Math.
7B (3) 1986

A THEOREM OF LIOUVILLE'S TYPE ON HARMONIC MAPS WITH FINITE OR SLOWLY DIVERGENT ENERGY

HU HESHENG (胡和生)* PAN YANGLIAN (潘养廉)*

Abstract

Some theorems of Liouville's type on harmonic maps from Euclidean space of conformal flat space with finite or slowly divergent energy have been obtained by the first-named author and H. C. J. Sealey, respectively. In this paper, a more general theorem is proved, which includes their results as special cases. The technique is to use a conservation law for harmonic maps.

§ 1. Introduction

In [1], the first-named author proved the theorem: Let φ: $R^n \to M^m$ be a harmonic map of $n(n\neq 2)$-dimensional Euclidean space R^n into an m-dimensional Riemannian manifold M^m. Suppose that the energy $e(\varphi)$ of φ is finite or slowly divergent. Then φ is a constant map. Here "slowly divergent energy" means that $\int_{R^n} e(\varphi)d^n(x) = \infty$ and $\int_{R^n} \frac{e(\varphi)}{\psi(r)} d^n x<\infty$, where $\psi(r)$ is a positive, continuous function of r satisfying

$$\int_a^\infty \frac{dr}{r\psi(r)}=\infty \quad \text{(for a certain constant } a>0). \tag{1}$$

On the other hand, H. C. J. Sealey in [2] proved the theorem: Let $M^n(n\geqslant 3)$ be a conformal flat space with metric form $ds^2=f^2(x)\ (dx'^2+ \cdots +dx^{n^2})$. If $L(f) \equiv \sum_i x^i \frac{\partial \log f}{\partial x^i} \geqslant -1$, then any harmonic map with finite energy from M^n into any Riemannian manifold must be a constant map.

Sealey has pointed out that the condition $L(f)\geqslant -1$ has a geometric significance. In fact, if S_r denotes the level surface $\{x\in R^n \mid \sum(x^i)^2=r^2\}$, then $L(f)\geqslant -1$ holds if and only if the mean curvature normal of S_r with respect to $ds=f^2(x)\sum_i (dx^i)^2$ is never pointing away from zero.

In this paper, using a similar technique as in [1], we will prove the following

Manuscript received April 12, 1984.
* Institute of Mathematics, Fudan University, Shanghai, China.

more general theorem which includes both the above theorems as special cases.

Main theorem. *Let $M^n (n\geqslant 3)$ be a Riemannian manifold with metric form $ds^2=f_1^2(x)(dx^1)^2+f_2^2(x)(dx^2)^2+\cdots+f_n^2(x)(dx^n)^2$ satisfying the following conditions:*

(A) $L(f_i)=\sum_j x^j \dfrac{\partial \log f_i}{\partial x^j}\geqslant -1, \quad i=1, \cdots, n.$

(B) *There exists a positive constant K such that*

$$\max_{1\leqslant i,j\leqslant n}\frac{f_i}{f_j}\leqslant K.$$

(C) *For any index $1\leqslant i\leqslant n$, and index $j_1\neq j_2\neq\cdots\neq j_{n-2}$,*

$$\sum_{k=1}^{n-2}(1+L(f_{j_k}))\geqslant 1+L(f_i).$$

Then, any harmonic map φ with finite or slowly divergent energy from M^n into any Riemannian manifold must be a constant map, where "slowly divergent energy" means that $\int_{M^n} e(\varphi)dV=\infty$ and $\int_{M^n}\dfrac{e(\varphi)}{\psi(r)}dV<\infty$, where $\psi(r)$ is a positive, continuous function of r satisfing

$$\int_a^\infty \frac{dr}{r\psi(r)}=\infty \quad (\textit{for a certain constant } a>0).$$

Remark 1. In the case of $f_1=\cdots=f_n=f$, the conditions (B) and (C) are trivial and this theorem is a generalization of Sealey's result as well as Hu's previous result.

Remark 2. We point out that the condition (A) also has the geometric significance as that in Sealey's case.

Remark 3. Theorem 1 includes essentially the case where M^n is a direct product manifold of p conformal flat manifolds $M_1\times\cdots\times M_p$.

§ 2. Preliminary

Let M^n be as above and S_r the level surface $\{x\in M^n | \sum_i (x^i)^2=r^2\}$.

Since there exists at least an $x^i\neq 0$ on S_r, say $x^n\neq 0$, we denote the induced metric of S_r from M^n by $g'_{ab}dx^a dx^b$, where $a, b, c, \cdots=1, \cdots, n-1$, and the volume element of S_r by dh. A straightforward computation shows

$$g'_{ab}=f_a^2\delta_{ab}+\left(\frac{f_n}{x^n}\right)^2 x^a x^b. \tag{2}$$

Thus, it is easy to show that

$$\det(g'_{ab})=\left(\prod_{i=1}^n f_i\right)\cdot\frac{\Phi}{(x^n)^2}=\det(g_{ij})\frac{\Phi}{(x^n)^2}, \tag{3}$$

where $\Phi=\sum_{i=1}^n\left(\dfrac{x^i}{f_i}\right)^2$.

Since $dx^n = d\sqrt{r^2 - \sum\limits_a (x^a)^2} = \dfrac{r\,dr - \sum\limits_a x^a dx^a}{x^n}$, and the volume element dV of M^n is $\sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dx^n$, we have from (3) the following Lemma 1.

Lemma 1. *On S_r, it holds that $dV = \dfrac{r}{\sqrt{\Phi}}\, dh \wedge dr$.*

Now suppose that φ is a harmonic map from M^n into any Riemannian manifold $(N^m, \tilde{g})$. The stress-energy tensor S of φ is a (1, 1)-type tensor with components $S^j_i = e(\varphi)\delta^j_i - \sum\limits_{l,\alpha,\beta} g^{jl}\tilde{g}_{\alpha\beta}\varphi^\alpha_{,i}\varphi^\beta_{,l}$, where $e(\varphi)$ is the energy density of φ and $\alpha, \beta, \gamma = 1, \cdots, m$ (cf. [3]).

It is well known that the divergence of S vanishes, i. e.,

$$\sum_j S^j_{i,j} = 0. \tag{4}$$

Here the comma stands for the covariant derivative.

Lemma 2. *It holds that $\sum\limits_{i,j,k} \left\{ {i \atop jk} \right\} x^k S^j_i = \sum\limits_{i,k} x^k \dfrac{\partial \log f_i}{\partial x^k} S^i_i$, where $\left\{ {i \atop jk} \right\}$ is the second Christoffel symbol of M^n.*

Proof Since $g_{ij} = f_i^2 \delta_{ij}$, from computation we have

$$\left\{ {i \atop jk} \right\} = \frac{1}{2}\left(\delta^i_j \frac{\partial \log f_i f_j}{\partial x^k} + \delta^i_k \frac{\partial \log f_i f_k}{\partial x^j} - \delta_{jk} \frac{1}{f_i^2} \frac{\partial f_j f_k}{\partial x^i} \right). \tag{5}$$

Thus

$$\sum_{i,j,k} \left\{ {i \atop jk} \right\} x^k S^j_i = \sum_{i,k} x^k \frac{\partial \log f_i}{\partial x^k} S^i_i + \sum_{i,j} \frac{x^i}{f_i} \frac{\partial f_i}{\partial x^j} S^j_i - \sum_{i,j} \frac{f_j}{f_i^2} \frac{\partial f_j}{\partial x^i} x^j S^j_i. \tag{6}$$

But, on the other hand, we have

$$\sum_{i,j} \frac{x^i}{f_i} \frac{\partial f_i}{\partial x^j} S^j_i = \frac{1}{2} e(\varphi) \sum_k \frac{x^k}{f_k} \frac{\partial f_k}{\partial x^k} - \sum_{i,j} \frac{x^i}{f_i f_j^2} \frac{\partial f_i}{\partial x^j} \tilde{g}_{\alpha\beta}\varphi^\alpha_{,i}\varphi^\beta_{,j}, \tag{7}$$

$$-\sum_{i,j} \frac{f_j}{f_i^2} \frac{\partial f_j}{\partial x^i} x^j S^j_i = -\frac{1}{2} e(\varphi) \sum_k \frac{x^k}{f_k} \frac{\partial f_k}{\partial x^k} + \sum_{i,j} \frac{x^j}{f_j f_i^2} \frac{\partial f_j}{\partial x^i} \tilde{g}_{\alpha\beta}\varphi^\alpha_{,j}\varphi^\beta_{,i}. \tag{8}$$

From (6), (7) and (8), the lemma is obvious.

Lemma 3. *If condition* (A) *and* (C) *are satisfied, then*

$$\sum_{i=1}^n (1 + L(f_i)) S^i_i \geq 0. \tag{9}$$

Proof For simplicity, we denote $a_i = (1 + L(f_i))$. Thus, $a_i \geq 0$. Let p be a point in M^n. If, at the point p, $S^i_i \geq 0$ for any index i, then (9) holds obviously. Otherwise, since $S^i_i = \dfrac{1}{2}\left(\sum\limits_{j \neq i} \dfrac{1}{f_j^2} \tilde{g}_{\alpha\beta}\varphi^\alpha_{,j}\varphi^\beta_{,j} - \dfrac{1}{f_i^2} \tilde{g}_{\alpha\beta}\varphi^\alpha_{,i}\varphi^\beta_{,i} \right)$, it is clear that $S^i_i < 0$ holds only for one index i, say, $S^n_n < 0$, and in this case we have

$$\frac{1}{f_n^2} \tilde{g}_{\alpha\beta}\varphi^\alpha_{,n}\varphi^\beta_{,n} > \sum_{i=1}^{n-1} \frac{1}{f_i^2} \tilde{g}_{\alpha\beta}\varphi^\alpha_{,i}\varphi^\beta_{,i}. \tag{10}$$

Without loss of generality, we can assume $0 \leq a_1 \leq \cdots \leq a_{n-1}$ at point p. Now, we have

$$\sum_{i=1}^{n}(1+L(f_i))S_i^i=\frac{1}{2}\left\{\sum_{i=1}^{n-1}\left(\sum_{j\neq i}\frac{a_i}{f_j^2}\tilde{g}_{\alpha\beta}\varphi_{,j}^{\alpha}\varphi_{,j}^{\beta}-\frac{a_i}{f_i^2}\tilde{g}_{\alpha\beta}\varphi_{,i}^{\alpha}\varphi_{,i}^{\beta}\right)\right.$$
$$\left.+\sum_{j=1}^{n-1}\frac{a_n}{f_j^2}\tilde{g}_{\alpha\beta}\varphi_{,j}^{\alpha}\varphi_{,j}^{\beta}-\frac{a_n}{f_n^2}\tilde{g}_{\alpha\beta}\varphi_{,n}^{\alpha}\varphi_{,n}^{\beta}\right\}. \tag{11}$$

In the case $a_n<a_{n-1}$, we have

$$\text{RHS of (11)}=\frac{1}{2}\left\{\sum_{i=1}^{n-2}\sum_{j\neq i}\frac{a_i}{f_j^2}\tilde{g}_{\alpha\beta}\varphi_{,j}^{\alpha}\varphi_{,j}^{\beta}+\sum_{j=1}^{n-2}\frac{a_{n-1}}{f_j^2}\tilde{g}_{\alpha\beta}\varphi_{,j}^{\alpha}\varphi_{,j}^{\beta}-\underset{\substack{\vee\\0}}{\sum_{i=1}^{n-2}\frac{a_i}{f_i^2}\tilde{g}_{\alpha\beta}\varphi_{,i}^{\alpha}\varphi_{,i}^{\beta}}\right.$$
$$\left.-\frac{a_{n-1}}{f_{n-1}^2}\tilde{g}_{\alpha\beta}\varphi_{,n-1}^{\alpha}\varphi_{,n-1}^{\beta}+\underset{\substack{\vee\\0}}{\sum_{j=1}^{n-1}\frac{a_n}{f_j^2}\tilde{g}_{\alpha\beta}\varphi_{,j}^{\alpha}\varphi_{,j}^{\beta}}+\underset{\substack{\vee\\0}}{\frac{a_{n-1}}{f_n^2}\tilde{g}_{\alpha\beta}\varphi_{,n}^{\alpha}\varphi_{,n}^{\beta}-\frac{a_n}{f_n^2}\tilde{g}_{\alpha\beta}\varphi_{,n}^{\alpha}\varphi_{,n}^{\beta}}\right\}$$
$$\geqslant\frac{1}{2}\left(\sum_{i=1}^{n-2}a_i-a_{n-1}\right)\frac{1}{f_{n-1}^2}\tilde{g}_{\alpha\beta}\varphi_{,n-1}^{\alpha}\varphi_{,n-1}^{\beta}\geqslant 0. \tag{12}$$

When $a_n\geqslant a_{n-1}$, we have

$$\text{RHS of (11)}=\frac{1}{2}\left\{\underset{\substack{\vee\\0}}{\sum_{i=1}^{n-1}\sum_{j\neq i,n}\frac{a_i}{f_j^2}\tilde{g}_{\alpha\beta}\varphi_{,j}^{\alpha}\varphi_{,j}^{\beta}-\sum_{i=1}^{n-1}\frac{a_i}{f_i^2}\tilde{g}_{\alpha\beta}\varphi_{,i}^{\alpha}\varphi_{,i}^{\beta}}+\underset{\substack{\vee\\0}}{\sum_{j=1}^{n-1}\frac{a_n}{f_j^2}\tilde{g}_{\alpha\beta}\varphi_{,j}^{\alpha}\varphi_{,j}^{\beta}}\right.$$
$$\left.+\sum_{i=1}^{n-1}\frac{a_i}{f_n^2}\tilde{g}_{\alpha\beta}\varphi_{,n}^{\alpha}\varphi_{,n}^{\beta}-\frac{a_n}{f_n^2}\tilde{g}_{\alpha\beta}\varphi_{,n}^{\alpha}\varphi_{,n}^{\beta}\right\}$$
$$\geqslant\frac{1}{2}\left(\sum_{i=1}^{n-1}a_i-a_n\right)\frac{1}{f_n^2}\tilde{g}_{\alpha\beta}\varphi_{,n}^{\alpha}\varphi_{,n}^{\beta}\geqslant 0. \tag{13}$$

Thus the lemma is proved.

From the above proof, it is not difficult to prove the following lemma.

Lemma 4. *If conditions* (A) *and* (C) *are satisfied, then* $\sum_i(1+L(f_i)S_i^i\equiv 0$ *holds if and only if* φ *is a constant map.*

§ 3. Proof of Main Theorem

In this section, we use Einstein summation convention. Let $B_r=\{x\in M^n\mid\sum_i(x^i)^2\leqslant r^2\}$. From (4), we have

$$0=\int_{B_r}x^iS_{i,j}^j\,dV=\int_{B_r}(x^iS_i^j)_{,j}\,dV-\int_{B_r}\left(\frac{\partial x^i}{\partial x^j}+\left\{\begin{matrix}j\\jk\end{matrix}\right\}x^k\right)S_i^j\,dV. \tag{14}$$

Noting that the unit outward normal vector of S_r is

$$W=\frac{1}{r}\sum_{i=1}^{n}\frac{x^i}{f_i}\frac{\partial}{\partial x^i},$$

and using the integral formula

$$\int_{B_r}\operatorname{div}X\,dV=\int_{S_r}\langle X,W\rangle\,dh$$

and Lemma 2, (14) is reduced to

$$0=\frac{1}{r}\int_{S_r}\sum_{i,j} f_j S_i^j x^i x^j\,dh-\int_{B_r}\sum_i(1+L(f_i)S_i^i\,dV. \tag{15}$$

By using schwartz inequality, we have

$$\sum_{i,j} f_j S_i^j x^i x^j=e(\varphi)\sum_j f_j(x^j)^2-\sum_{j,k}\frac{1}{f_j}\,\tilde{g}_{\alpha\beta}(\varphi^\alpha_{,j}x^j)(\varphi^\beta_{,k}x^k)$$
$$\leqslant e(\varphi)\sum_j f_j(x^j)^2\leqslant e(\varphi)r^2\sqrt{\sum_i f_i^2}. \tag{16}$$

From (15) and (16), we obtain

$$r\int_{S_r} e(\varphi)\sqrt{\sum_i f_i^2}\,dh\geqslant\int_{B_r}\sum_i(1+L(f_i))S_i^i\,dV. \tag{17}$$

If φ is not a constant, from Lemma 3 and Lemma 4, we claim that there exist two positive numbers R_0 and ε such that, for $r\geqslant R_0$,

$$\int_{B_r}\sum_i(1+L(f_i))S_i^i\,dV>\varepsilon. \tag{18}$$

Let $\psi(r)$ be a positive continuous function of r satisfying

$$\int_a^\infty\frac{dr}{r\psi(r)}=\infty \quad \text{(for a certain constant } a>0). \tag{19}$$

In consequence of Lemma 1 and (18), integrating (17), we have

$$\int_{R_0}^R\frac{\varepsilon}{r\psi(r)}\,dr\leqslant\int_{R_0}^R\int_{S_r}\frac{e(\varphi)}{\psi(r)}\sqrt{\sum_i f_i^2}\,dh\,dr\leqslant\int_0^R\int_{S_r}\frac{e(\varphi)}{\psi(r)}\sqrt{\sum_i f_i^2}\,dh\,dr$$
$$=\int_0^R\int_{S_r}\frac{e(\varphi)}{\psi(r)}\frac{\sqrt{\sum_i f_i^2}\sqrt{\Phi}}{r}\cdot\frac{r}{\sqrt{\Phi}}\,dh\,dr$$
$$=\int_{B_R}\frac{e(\varphi)}{\psi(r)}\frac{\sqrt{\sum_i f_i^2}\sqrt{\Phi}}{r}\,dV,\ \text{for } R>R_0. \tag{20}$$

Furthermore, from condition (B), we have

$$\frac{\sqrt{\sum_i f_i^2}\sqrt{\Phi}}{r}=\sqrt{\left(\sum_i\left(\frac{f_i}{r}\right)^2\left(\sum_j\left(\frac{x^j}{f_j}\right)^2\right)\right)}\leqslant\sqrt{\left(\sum_i f_i^2\right)\left(\sum_j\frac{1}{f_j^2}\right)}\leqslant\sqrt{n^2K^2}\leqslant nK.$$

Thus, (20) reduces to

$$\int_{R_0}^R\frac{\varepsilon}{r\psi(r)}\,dr\leqslant nK\int_{B_R}\frac{e(\varphi)}{\psi(r)}\,dV,\ \text{for } R>R_0. \tag{21}$$

Letting $R\to\infty$ in (21), the left hand side of (21) approaches infinite, but the right hand side of (21) is finite, since φ is of finite energy or with slowly divergent energy. This contradiction proves our thoerem.

References

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