Chin. Ann. of Math.
7B (3) 1986

SEGAL ALGEBRA $A_{1,p}(G)$ AND ITS MULTIPLIERS

OUYANG GUANGZHONG (欧阳光中)*

Abstract

Let G be a locally compact abelian group and $A_p(G)$ the p-Fourier algebra of Herz. This pepar studies the space $A_{1,p}(G)=L_1(G)\cap A_p(G)$ with convolution product. It is proved that $A_{1,p}(G)$ is a character Segal algebra. Moreover, for the multipliers of $A_{1,p}(G)$ the author proves that $M(A_{1,p}(G), L_1(G))=M(G)$ and $M(A_{1,p}(G), A_{1,p}(G))=M(G)$ provided G is noncompact. If G is discrete, then $M(A_{1,p}(G), L_1(G))=A_{1,p}(G)$ and $M(A_{1,p}(G), A_{1,p}(G))=A_{1,p}(G)$.

Let G be a locally compact abelian group and $\hat{G}$ the character group of G. We denote by $A_p(G)$ a space consisting of all the following functions:

$$f=\sum_{i=1}^{\infty} u_i*\check{v}_i,\ u_i\in L_p(G),\ v_i\in L_q(G),$$

where $1<p,\ q<\infty,\ \dfrac{1}{p}+\dfrac{1}{q}=1,\ \check{v}_i(x)=v_i(x^{-1}),\ \forall x\in G$ and

$$\sum_{i=1}^{\infty}\|u_i\|_p\|v_i\|_q<\infty.$$

For each $f\in A_p(G)$, define the norm of f by

$$\|f\|_{A_p}=\inf\left\{\sum_{i=1}^{\infty}\|u_i\|_p\|v_i\|_q\right\},$$

where the infimum is taken over all possible representations for f.

In this paper, we introduce the algebra $A_{1,p}(G)$, the subspace of $A_p(G)$:

$$A_{1,p}(G)=L_1(G)\cap A_p(G).$$

The norm $\|\ \ \|_{1,p}$ of $A_{1,p}(G)$ is defined as follows:

$$\|f\|_{1,p}=\|f\|_1+\|f\|_{A_p},\quad \forall f\in A_{1,p}(G).$$

Obviously, $A_{1,p}(G)$ is nonempty since $C_c(G)*C_c(G)\subset A_{1,p}(G)$, where $C_c(G)$ is the space of all continuous functions with compact support in G. Herz[1] proved that $A_p(G)$ is a Banach algebra under pointwise multiplication, but is not an algebra under convolution. In a later paper[2] Herz also proved that $A_p(G)$ is a regular Tauberian algebra of the functions on G. Lai and Chen[3] proved that $A_{1,p}(G)$ is a commutative Banach algebra under pointwise multiplication, and moreover, the following statements are equivalent: (i) G is compact. (ii) $A_{1,p}(G)\subset L_r(G)$ for each

Manuscript received April 4, 1984. Revised October 3, 1984.

* Department of Mathematics, Fudan University, Shanghai, China.

$0<r\leqslant\infty$. (iii) $A_{1,p}(G)=A_p(G)$. (iv) $A_{1,p}(G)$ has a bounded approximate identity. (v) $A_{1,p}(G)$ has the factorization property. (vi) $A_{1,p}(G)$ has an identity, etc.

In this paper we will consider the space $A_{1,p}(G)$ under convolution product and study the multipliers of $A_{1,p}(G)$.

Theorem 1. *$A_{1,p}(G)$ is a character Segal algebra with convolution product.*

Note. A homogeneous Banach algebra $S(G)$ is called a character Segal algebra if it is dense in $L_1(G)$, and for each $\gamma\in\hat{G}$, $\gamma f\in S(G)$, $\|\gamma f\|_s=\|f\|_s$, where $\gamma f(x)=(x,\gamma)f(x)$.

Proof It suffices to prove the following facts since $A_{1,p}(G)$ is a commutative Banach algebra under multiplication and $L_1(G)$ possesses the homogeneous structures. These facts are

(a) $\forall f,g\in A_{1,p}(G)$, $f*g\in A_{1,p}(G)$,

$$\|f*g\|_{1,p}\leqslant\|f\|_{1,p}\|g\|_{1,p}.$$

(b) $A_{1,p}(G)$ is dense in $L_1(G)$.

(c) For each $f\in A_{1,p}(G)$ and each $\gamma\in\hat{G}$, $\gamma f\in A_{1,p}(G)$. Moreover

$$\|\gamma f\|_{1,p}=\|f\|_{1,p}.$$

We prove these facts now.

(a) For each f, $g\in A_{1,p}(G)$, evidently, $f*g\in L_1(G)$. Suppose

$$f=\sum_{i=1}^{\infty}u_i*\check{v}_i,\quad g=\sum_{i=1}^{\infty}s_i*\check{t}_i, \tag{1}$$

$$u_i,\ s_i\in L_p(G),\ v_i,\ t_i\in L_q(G),\ \frac{1}{p}+\frac{1}{q}=1,$$

$$\sum_{i=1}^{\infty}\|u_i\|_p\|v_i\|_q<\infty,\quad \sum_{i=1}^{\infty}\|s_i\|_p\|t_i\|_q<\infty. \tag{2}$$

Then

$$f*g=\sum_{i=1}^{\infty}(f*s_i)*\check{t}_i.$$

Since $f\in L_1(G)$ and $s_i\in L_p(G)$, $f*s_i\in L_p(G)$ and $\|f*s_i\|_p\leqslant\|f\|_1\|s_i\|_p$. But $t_i\in L_q(G)$. Therefore $f*g\in A_{1,p}(G)$.

Furthermore

$$\begin{aligned}\|f*g\|_{1,p}&=\|f*g\|_1+\|f*g\|_{A_p}\\&\leqslant\|f\|_1\|g\|_1+\sum_{i=1}^{\infty}\|f*s_i\|_p\|t_i\|_q\leqslant\|f\|_1\|g\|_1+\|f\|_1\sum_{i=1}^{\infty}\|s_i\|_p\|t_i\|_q\\&\leqslant\Big(\|f\|_1+\sum_{i=1}^{\infty}\|u_i\|_p\|v_i\|_q\Big)\Big(\|g\|_1+\sum_{i=1}^{\infty}\|s_i\|_p\|t_i\|_q\Big).\end{aligned} \tag{3}$$

For every $\varepsilon>0$, there exists u_i', v_i', s_i', t_i' $(i=1, 2, \cdots)$ such that (1), (2) and (3) hold together with

$$\sum_{i=1}^{\infty}\|u_i'\|_p\|v_i'\|_q\leqslant\|f\|_{A_p}+\varepsilon,$$

$$\sum_{i=1}^{\infty}\|s_i'\|_p\|t_i'\|_q\leqslant\|g\|_{A_p}+\varepsilon.$$

Since ε is arbitrary, we have

$$\|f*g\|_{1,p}\leqslant\|f\|_{1,p}\|g\|_{1,p}.$$

(b) Recall that $L_1(G)$ has the factorization property: $L_1(G)=L_1(G)*L_1(G)$. Hence for each $f\in L_1(G)$, there exists g, $h\in L_1(G)$ such that $f=g*h$. Furthermore since $C_c(G)$ is dense in $L(G)$, for every $\varepsilon>0$, there exist $\tilde{g}$, $\tilde{h}\in C_c(G)$, such that

$$\|g-\tilde{g}\|_1<\varepsilon,\quad \|h-\tilde{h}\|_1<\varepsilon.$$

Therefore

$$\|f-\tilde{g}*\tilde{h}\|_1=\|g*h-\tilde{g}*\tilde{h}\|_1<\varepsilon,$$

where $\tilde{g}*\tilde{h}\in C_c(G)*C_c(G)$. It follows that $C_c(G)*C_c(G)$ is dense in $L_1(G)$. By

$$C_c(G)*C_c(G)\subset A_{1,p}(G)\subset L_1(G),$$

(b) is proved.

(c) Let $\gamma\in\hat{G}$. Then γ is a continuous character on G, and $\forall x\in G$, $|(x,\ \gamma)|=1$. Suppose that

$$f=\sum_{i=1}^{\infty}u_i*\check{v}_i,\ u_i\in L_p(G),\ v_i\in L_q(G),\ \frac{1}{p}+\frac{1}{q}=1,$$

$$\sum_{i=1}^{\infty}\|u_i\|_p\|v_i\|_q\leqslant\|f\|_{A_p}+\varepsilon,$$

where $\varepsilon>0$ is arbitrary. Thus

$$\gamma f=\sum_{i=1}^{\infty}\gamma u_i*\check{v}_i.$$

It is easy to see that $\gamma u_i\in L_p(G)$ and $\|\gamma u_i\|_p=\|u_i\|_p$. Hence $\gamma f\in A_{1,p}(G)$. Moreover, by the fact $\|\gamma f\|_1=\|f\|_1$, it follows that

$$\|\gamma f\|_{1,p}\leqslant\|f\|_{1,p}+\varepsilon,$$

that is

$$\|\gamma f\|_{1,p}\leqslant\|f\|_{1,p}.$$

If $\|\gamma f\|_{1,p}<\|f\|_{1,p}$, then there exist $u_i'\in L_p(G)$, $v_i'\in L_q(G)$, $i=1, 2, \cdots$, $\frac{1}{p}+\frac{1}{q}=1$ such that

$$\gamma f=\sum_{i=1}^{\infty}u_i'*\check{v}_i',$$

$$\|\gamma f\|_{1,p}\leqslant\|\gamma f\|_1+\sum_{i=1}^{\infty}\|u_i'\|_p\|v_i'\|_q<\|f\|_{1,p}.$$

Repeating the above process and noting the fact $\gamma^{-1}\gamma=1$, we obtain

$$\|f\|_{1,p}=\|\gamma^{-1}(\gamma f)\|_{1,p}\leqslant\|\gamma f\|_{1,p}<\|f\|_{1,p}.$$

But this is a contradiction. Hence $\|\gamma f\|_{1,p}=\|f\|_{1,p}$. The proof is complete.

The following theorem will show that $A_{1,p}(G)$ is a proper subspace of $L_1(G)$ when the group G is nondiscrete.

Theorem 2. *The following are equivalent:*

(i) *G is discrete (or $\hat{G}$ is compact).*

(ii) $A_{1,p}(G)=L_1(G)$.

Proof (i)⇒(ii). It is known that if G is discrete, then any Segal algebra is the whole of $L_1(G)$[4, p.60]. Since $A_{1,p}(G)$ is a Segal algebra, we get $A_{1,p}=L_1(G)$.

(ii)⇒(i). It suffices to prove that if G is nondiscrete, there exists an element $u\in L_1(G)$ but $u\notin A_{1,p}(G)$.

Take a function

$$u(x)=\begin{cases}1, & x=e,\\ 0, & x\neq e,\end{cases}$$

where e is an identity of group G. It is obvious that $u\in L_1(G)$. On the other hand, if u is a continuous function, then the single point set $\{e\}$ is an open set of G. Hence G is discrete. It shows that u is discontionuous because G is nondiscrete. It is easy to see that $A_{1,p}(G)\subset C_0(G)$ since $u_i*\check{v}_i\in C_0(G)$ and $\|u_i*\check{v}_i\|_\infty\leqslant\|u_i\|_p\|v_i\|_q$, where $C_0(G)$ denotes the Banach space of all continuous functions on G vanishing at infinity under the norm $\|\ \|_\infty$. So $u\notin A_{1,p}(G)$. That is $A_{1,p}(G)\subsetneqq L_1(G)$ when G is nondiscrete.

We now study the multipliers of Segal algebra $A_{1,p}(G)$. Let G be a locally compact abelian group, $S_1(G)$ and $S_2(G)$ are two Segal algebras on G. Let T be a linear operator from $S_1(G)$ into $S_1(G)$. If T commutes with every translation operator τ_x, that is $T\tau_x=\tau_xT$, then T is called a multiplier of $S_1(G)$ into $S_2(G)$. We denote the collection of all the multipliers by $M(S_1(G), S_2(G))$, which is a Banach algebra.

The following theorem gives the relations between $M(A_{1,p}(G), L_1(G))$, $M(A_{1,p}(G), A_{1,p}(G))$ and $M(G)$, where $M(G)$ denotes the Banach algebra of bounded regular complex valued Borel measures on G.

Theorem 3. *Let G be a locally compact but noncompact abelian group. Then*

$$M(A_{1,p}(G), L_1(G))=M(G),$$
$$M(A_{1,p}(G), A_{1,p}(G))=M(G),$$
$$M(A_{1,p}(G), L_1(G))=M(A_{1,p}(G), A_{1,p}(G)),$$

where "=" means: isometric algebra isomorphism.

We will turn the proof into following two lemmas.

Lemma A. *Let G be a locally compact but noncompact abelian group. If*

$$T:A_{1,p}(G)\to L_1(G)$$

is a continuous linear operator, then the following are equivalent:

(i) $T\in M(A_{1,p}(G), L_1(G))$.

(ii) *There exists a unique measure $\mu\in M(G)$ such that*

$$Tf=\mu*f,\quad \forall f\in A_{1,p}(G),$$

where $M(G)$ denotes the Banach algebra of bounded regular complex valued Borel measures on G.

Moreover, the correspondence between T *and* μ *defines an isometric linear isomorphism from* $M(A_{1,p}(G), L_1(G))$ *onto* $M(G)$.

Proof If $\mu\in M(G)$ and $Tf=\mu*f$ for each $f\in A_{1,p}(G)$, then, clearly

$$\|Tf\|_1=\|\mu*f\|_1\leqslant\|\mu\|\,\|f\|_1\leqslant\|\mu\|\,\|f\|_{1,p}.$$

It is evident that $T\in M(A_{1,p}(G), L_1(G))$, $\|T\|\leqslant\|\mu\|$.

Conversely, suppose that $T\in M(A_{1,p}(G), L_1(G))$. For each $f\in A_{1,p}(G)$, since $Tf\in L_1(G)$ and G is a locally compact but noncompact group, for any $\varepsilon>0$, there exist $s_1, s_2, \cdots, s_{k-1}\in G$ such that

$$k\|Tf\|_1\leqslant\|Tf+\tau_{s_1}Tf+\cdots+\tau_{s_{k-1}}Tf\|_1+k\varepsilon.$$

T is a multiplier, that is, a continuous linear operator such that $\tau_{s_i}Tf=T\tau_{s_i}f$, $i=1, 2, \cdots, k-1$, so that

$$\begin{aligned}k\|Tf\|_1&\leqslant\|T(f+\tau_{s_1}f+\cdots+\tau_{s_{k-1}}f)\|_1+k\varepsilon\\&\leqslant\|T\|(\|f+\tau_{s_1}f+\cdots+\tau_{s_{k-1}}f\|_1+\|f+\tau_{s_1}f+\cdots+\tau_{s_{k-1}}f\|_{A_p})+k\varepsilon.\end{aligned}$$

By the homogeneity of $L_1(G)$,

$$\|f+\tau_{s_1}f+\cdots+\tau_{s_{k-1}}f\|_1\leqslant k\|f\|_1.$$

Hence

$$k\|Tf\|_1\leqslant\|T\|(k\|f\|_1+\|f+\tau_{s_1}f+\cdots+\tau_{s_{k-1}}f\|_{A_p})+k\varepsilon. \tag{4}$$

Let

$$f=\sum_{i=1}^{\infty}u_i*\check{v}_i,\quad u_i\in L_p(G),\quad v_i\in L_q(G),\quad \frac{1}{p}+\frac{1}{q}=1,$$

$$\sum_{i=1}^{\infty}\|u_i\|_p\|v_i\|_q<\infty.$$

For the above $\varepsilon>0$, there exists an integer N such that

$$\sum_{i=N+1}^{\infty}\|u_i\|_p\|v_i\|_q<\varepsilon.$$

Next let

$$g_N=\sum_{i=1}^{N}u_i*\check{v}_i.$$

We have

$$\|f-g_N\|_{A_p}=\Big\|\sum_{i=N+1}^{\infty}u_i*\check{v}_i\Big\|_{A_p}\leqslant\sum_{i=N+1}^{\infty}\|u_i\|_p\|v_i\|_q<\varepsilon. \tag{5}$$

Since $C_c(G)$ is dense in $L_p(G)$, there exist α_i, $\beta_i\in C_c(G)$, $i=1, 2, \cdots, N$, such that

$$\|u_i-\alpha_i\|_p<\frac{\varepsilon}{M},\quad \|v_i-\beta_i\|_q<\frac{\varepsilon}{M},$$

$$M=\max\Big(\sum_{i=1}^{N}\|u_i\|_p,\ \sum_{i=1}^{N}\|v_i\|_q,\ N\Big).$$

Let

$$\varphi=\sum_{i=1}^{N}\alpha_i*\check{\beta}_i,\quad \varphi\in C_c(G),$$

$$g_N-\varphi=\sum_{i=1}^{N}(u_i*\check{v}_i-\alpha_i*\check{\beta}_i)$$

$$=\sum_{i=1}^{N}u_i*(\check{v}_i-\check{\beta}_i)-\sum_{i=1}^{N}(u_i-\alpha_i)*(\check{v}_i-\check{\beta}_i)+\sum_{i=1}^{N}(u_i-\alpha_i)*\check{v}_i,$$

$$\|g_N-\varphi\|_{A_p}\leqslant\sum_{i=1}^{N}\|u_i\|_p\|v_i-\beta_i\|_q+\sum_{i=1}^{N}\|u_i-\alpha_i\|_p\|v_i-\beta_i\|_q+\sum_{i=1}^{N}\|u_i-\alpha_i\|_p\|v_i\|_q<3\varepsilon. \quad (6)$$

From (5) and (6)

$$\|f-\varphi\|_{A_p}\leqslant\|f-g_N\|_{A_p}+\|g_N-\varphi\|_{A_p}<4\varepsilon,$$

and it is easy to see that for any τ_s,

$$\|\tau_s f-\tau_s\varphi\|_{A_p}<4\varepsilon.$$

Hence

$$\|f+\tau_{s_1}f+\cdots+\tau_{s_{k-1}}f\|_{A_p}$$

$$\leqslant\|(f+\tau_{s_1}f+\cdots+\tau_{s_{k-1}}f)-(\varphi+\tau_{s_1}\varphi+\cdots+\tau_{s_{k-1}}\varphi)\|_{A_p}+\|\varphi+\tau_{s_1}\varphi+\cdots+\tau_{s_{k-1}}\varphi\|_{A_p}$$

$$\leqslant 4k\varepsilon+\|\varphi+\tau_{s_1}\varphi+\cdots+\tau_{s_{k-1}}\varphi\|_{A_p}. \quad (7)$$

Substituting (7) into (4), we get

$$k\|Tf\|_1\leqslant\|T\|(k\|f\|_1+4k\varepsilon+\|\varphi+\tau_{s_1}\varphi+\cdots+\tau_{s_{k-1}}\varphi\|_{A_p})+k\varepsilon, \quad (8)$$

where

$$\varphi=\sum_{i=1}^{N}\alpha_i*\check{\beta}_i,\quad \alpha_i,\ \beta_i\in C_c(G),$$

so that

$$\varphi+\tau_{s_1}\varphi+\cdots+\tau_{s_{k-1}}\varphi=\sum_{i=1}^{N}(\alpha_i+\tau_{s_1}\alpha_i+\cdots+\tau_{s_{k-1}}\alpha_i)*\check{\beta}_i,$$

$$\|\varphi+\tau_{s_1}\varphi+\cdots+\tau_{s_{k-1}}\varphi\|_{A_p}\leqslant\sum_{i=1}^{N}\|\alpha_i+\tau_{s_1}\alpha_i+\cdots+\tau_{s_{k-1}}\alpha_i\|_p\|\beta_i\|_q.$$

We can also choose $s_i(i=1, 2, \cdots, k-1)$ such that (8) holds and

$$\operatorname{supp}\tau_{s_e}\alpha_i\cap\operatorname{supp}\tau_{s_f}\alpha_i=\emptyset,$$

$$e\neq f,\quad e, f=0, 1, 2, \cdots, k-1.$$

Then

$$\|\alpha_i+\tau_{s_1}\alpha_i+\cdots+\tau_{s_{k-1}}\alpha_i\|_{A_p}\leqslant k^{\frac{1}{p}}\|\alpha_i\|_p,$$

$$\|\varphi+\tau_{s_1}\varphi+\cdots+\tau_{s_{k-1}}\varphi\|_{A_p}\leqslant k^{\frac{1}{p}}\sum_{i=1}^{N}\|\alpha_i\|_p\|\beta_i\|_q.$$

Substituting the last inequality into (8), we have

$$\|Tf\|_1\leqslant\|T\|\Big(\|f\|_1+4\varepsilon+k^{\frac{1}{p}-1}\sum_{i=1}^{N}\|\alpha_i\|_p\|\beta_i\|_q\Big)+\varepsilon.$$

Since ε and k are arbitrary and $p>1$, we obtain

$$\|Tf\|_1\leqslant\|T\|\,\|f\|_1.$$

This shows that T defines a multiplier of $A_{1,p}(G)$ considered as a subspace of $L_1(G)$ into $L_1(G)$. Since $A_{1,p}(G)$ is dense in $L_1(G)$, T determines a unique continuous linear operator T (we also use the notation T) from $L_1(G)$ into $L_1(G)$, whose norm remains the same. We now prove that T is a multiplier from $L_1(G)$ into $L_1(G)$.

For each $f\in L_1(G)$ and any $\varepsilon>0$, there exists $g\in A_{1,p}(G)$ such that $\|f-g\|_1<\varepsilon$. For each translation τ_s,

$$\|T\tau_s f-\tau_s Tf\|_1\leqslant\|T\tau_s f-T\tau_s g\|_1+\|\tau_s Tg-\tau_s Tf\|_1\leqslant 2\|T\|\|f-g\|_1<2\|T\|\varepsilon,$$

so that $T\tau_s f=\tau_s Tf$. Consequently T is a multiplier from $L_1(G)$ into $L_1(G)$.

By a well-known result on multipliers of $L_1(G)$, there exists a unique maesure $\mu\in M(G)$ such that

$$Tf=\mu*f,\quad \forall f\in L(G)$$

and $\|T\|=\|\mu\|$. This completes the proof.

From Lemma A, it is convenient to investigate the multiplieps from $A_{1,p}(G)$ into $A_{1,p}(G)$. In general, let $S(G)$ be a Segal algebra on G. Unni[5] proved that if T is a multiplier from $S(G)$ into $S(G)$, then there exists a unique pseudomeasure σ such that

$$Tf=\sigma*f \text{ for each } f\in S(G),$$

yet, in general the correspondence only appears to be a relation between $M(S(G), S(G))$ and some subset of all pseudomeasures. For $A_{1,p}(G)$ we have, however, the following lemma.

Lemma B. *Let G be a locally compact but noncompact abelian group. If*

$$T: A_{1,p}(G)\to A_{1,p}(G)$$

is a continuous linear operator, then the following are equivalent:

(i) $T\in M(A_{1,p}(G), A_{1,p}(G))$.

(ii) *There exists a unique measure $\mu\in M(G)$ such that*

$$Tf=\mu*f,\quad \forall f\in A_{1,p}(G).$$

Moreover, the correspondence between T and μ difines an isometric algebra isomorphism from $M(A_{1,p}(G), A_{1,p}(G))$ onto $M(G)$.

Proof Every $T\in M(A_{1,p}(G), A_{1,p}(G))$ will be identified with an element of $M(A_{1,p}(G), L_1(G))$ since

$$\|f\|_1\leqslant\|f\|_{1,p},\quad \forall f\in A_{1,p}(G).$$

By Lemma A, there exists a unique measure $\mu\in M(G)$ such that

$$Tf=\mu*f,\quad \forall f\in A_{1,p}(G),\quad \|T\|=\|\mu\|.$$

Conversely, for each $\mu\in M(G)$, $\mu*f\in A_{1,p}(G)$ since $A_{1,p}(G)$ is a Segal algebra. Therefore $A_{1,p}(G)$ is an ideal in the measure algebra $M(G)$. Moreover

$$\|\mu*f\|_{1,p}\leqslant\|\mu\|\|f\|_{1,p},\quad \forall f\in A_{1,p}(G).$$

Let

$$T: A_{1,p}(G)\to A_{1,p}(G),\quad Tf=\mu*f.$$

It is easy to see that $T\in M(A_{1,p}(G), A_{1,p}(G))$. Lemma B is proved.

From Lemma A and Lemma B we get Theorem 3 immediately.

In the case when G is a discrete group, Theorem 3 is simpler.

Theorem 4. *Let G be a discrete group. Then the following are equivalent:*

(i) $T\in M(A_{1,p}(G), L_1(G))$.

(ii) *There exists a unique function* $g_T\in A_{1,p}(G)$ *such that*

$$Tf=g_T*f,\quad \forall f\in A_{1,p}(G).$$

Moreover, this correspondence between T and g_T difines an algebra isomorphism from $M(A_{1,p}(G), L_1(G))$ onto $A_{1,p}(G)$, and the norms are equivalent.

Theorem 5. *Let G be a discrete group. Then the following are equivalent.*

(i) $T\in M(A_{1,p}(G), A_{1,p}(G))$.

(ii) *There exists a unique function* $g_T\in A_{1,p}(G)$ *such that*

$$Tf=g_T*f,\quad \forall f\in A_{1,p}(G).$$

Moreover, this correspondence between T and g_T difines an algebra isomorphism from $M(A_{1,p}(G), A_{1,p}(G))$ onto $A_{1,p}(G)$, and the two norms are equivalent.

The proofs of Theorem 4 and Theorem 5 are obvious. If G is a discrete group, then $A_{1,p}(G)=L_1(G)=M(G)$ and $g_T=T\delta$, where δ is an identity of $A_{1,p}(G)$. It remains to prove the norms are equivalent because the norm of $A_{1,p}(G)$ is not the same one of $L_1(G)=M(G)$. By

$$\|Tf\|=\|T\delta*f\|\leqslant\|T\delta\|_{1,p}\|f\|_{1,p}$$

we have

$$\|T\|\leqslant\|T\delta\|_{1,p}, \tag{9}$$

where $T\delta=g_T$.

On the other hand, since δ is an identity of $A_{1,p}(G)$,

$$\delta(x)=\begin{cases}1, & x=e,\\ 0, & x\neq e,\quad x\in G,\end{cases}$$

where e is an identity of group G. Hence

$$\|\delta\|_{1,p}=\|\delta\|_1+\|\delta*\delta\|_{A_p}\leqslant\|\delta\|_1+\|\delta\|_p\|\delta\|_q\leqslant 2.$$

Together with $\|T\|\leqslant\|T\delta\|_{1,p}$ we have

$$\|T\delta\|_{1,p}\leqslant\|T\|\,\|\delta\|_{1,p}\leqslant 2\|T\|. \tag{10}$$

Substituting (10) into (9), we obtian

$$\|T\|\leqslant\|T\delta\|_{1,p}=\|g_T\|_{1,p}\leqslant 2\|T\|.$$

The proof is complete.

References

[1] Herz, C., Theory of p-space with an application to convolution operators, *Trans. Amer. Soc.*, **154** (1971), 69—82.

[2] Herz, C., Harmonic synthesis for subgroups, *Ann. Inst.*, Fourier (Grenoble) **23** (1973), 91—123.

[3] Lai, H. & Chen, I., Harmonic analysis on the Fourier algebra $A_{1,p}(G)$, *J. Austral Math. Soc.*, (Series A) **30** (1981), 438—452.

[4] Wang, M. C., Homogeneous Banach Algebras, Marcel Dekker Inc., (1977).

[5] Unni, K. R., A note on multipliers of a Segal algebra, *Studia Math.*, **49** (1974), 125—127.