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A CLASS OF UNIFORM AMARTS INDEXED BY DIYECTED SETS

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Abstract

The main purpose of this paper is to extend the concept of Bellow's uniform amarts and E-valued martingales indexed by directed sets, and to give a necessary and sufficient condition for the strong sto. convergence of the net $(x_\tau)_{\tau\in T^s}$, this condition is also a necessary and sufficient condition for the strong ess. convergence of the net $(x_t)_{t\in D}$ when the stochastic basis satisfies the Vitali cndition V.

In this paper we introducea class of uniform amarts in a σ-finite measure space indexed by directed sets as a generalization of Bellow's uniform amarts and E-valued martingales indexed by directed sets, and characterize the strong stochastic convergence of and strong essential convergence of an adapted process.

§ 1. Introduction

Let (Ω, F, μ) be a fixed σ-finite measure space and D a directed set filtering to the right. A family $(F_t)_{t\in D}$ of σ-algebras, contained in F satisfying $F_s \subset F_t (s \leqslant t)$, is called a stochastic basis. Let E be a Banach space with norm $|\cdot|$, and $(X_t)_{t\in D}$ be a family of strongly measurable E-valued r. v.'s adapted to $(F_t)_{t\in D}$. Throughout this paper, functions, sets, and r. v'. s are considered equal if they are equal almost surely. A function $\tau:\Omega\to D$ is called a countable (simple)stopping time with respect to $(F_t)_{t\in D}$ if $(\tau=t)\in F_t$ for all $t\in D$ and $R(\tau)\triangleq\{t\in D \mid \text{there exists } \omega\in\Omega:\tau(\omega)=t\}$ is a countable (finite) subset of D. The set of all countable (simple) stopping times will be denoted by $T^c(T^s)$. Under the natural order, T^c and T^s are directed sets filtering to the right. Let $\tau\in T^c$ and $(X_t)_{t\in D}$ be an adapted process. Define the r. v. $X_\tau=\sum_{t\in R(\tau)} I_{(\tau=t)}X_t$ and the σ-algebra $F_\tau=\{A\in F \mid \text{for any } t\in R(\tau),\ A(\tau=t)\in F_t\}$, and write $E^\tau(\cdot)$ for conditional expectation $E(\cdot\mid F_\tau)$. It is easy to see that X_τ is F_τ-measurable. We denote as usual by $L^1_E=L^1_E(\Omega, F, \mu)$ a space of all Bochner integrable E-valued r. v.'s. For $X\in L^1_E$, we write $\|X\|_1=\int_\Omega |X(\omega)|\,d\mu(\omega)=E|X|$.

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When $D=1, 2, \cdots, \triangleq N$, and μ is a probability measure, Bellow [1, 2] introduced the concept of uniform amarts to extend E-valued martingales and quasi-martingales. She called an adapted sequence $(X_n)_{n\in N}$ an uniform amart, if for each $\tau\in T^s$, $X_\tau\in L_E^1$ and

$$\operatorname*{Lim}_{\substack{\tau,\sigma\in T^s\\ \tau\leq\sigma}} \|X_\tau - E^\tau X_\sigma\|_1 = 0.$$

She proved that if $(X_n)_{n\in N}$ is an L_E^1-bounded uniform amart and E has the Radon-Nikodym property, then $(X_n)_{n\in N}$ converges strongly almost surely and $\sup_T \mathrm{E}\|X_\tau\|_1 <\infty$. For a general directed set D Millet and Sucheston[8] extened Chatterji's theorem[4] and showed that if E has the Radon-Nikodym property and if the stochastic basis $(F_t)_{t\in D}$ satisfies the Vitali condition V, then every L_E^1-bounded martingale $(X_t)_{t\in D}$ with respect to $(F_t)_{t\in D}$ converges essentially.

The main purpose of this paper is to extend the concept of uniform amarts and E-valued martingales indexed by directed sets, and to give a necessary and sufficient condition for the strong stochastic convergence of the net $(X_\tau)_{\tau\in T^s}$. By Theorem 12.3 of [8], this condition is also a necessary and sufficient condition for the strong essential convergence of the net $(X_t)_{t\in D}$ when the stochastic basis $(F_t)_{t\in D}$ satisfies the Vitali condition V.

§ 2. A Class of Uniform Amarts

In this section we introdnce a class of uniform amarts and give a characterization of the class of uniform amarts by the Riesz decomposition. Suppose $T\subset T^c$ and $\tau\in T^c$, we denote $T(\tau)=\{\sigma\in T\,|\,\tau\leq\sigma\}$, and write

$$\underline{\underline{T}}=\{T\,|\,T \text{ is a directed subset of } T^c \text{ and } T(t)\neq\emptyset \text{ for each } t\in D\}.$$

Definition 2.1[8]. *Let $T\in\underline{\underline{T}}$, we say that T has the localization property if for each finite family $(\tau_j)_{j\in J}\subset T$ and each finite partition of Ω, $(A_j)_{j\in J}$ with $A_j\in F_{\sigma_j}$ for $j\in J$, the stopping time $\tau=\tau_j(\omega)$ for $\omega\in A_j$, $j\in J$ belongs to T.*

Let

$$\underline{\underline{L}}=\{T\in\underline{\underline{T}}\,|\,\text{T has the localization property}\}.$$

Definition 2.2. *Let $T\in\underline{\underline{L}}$. $(X_t)_{t\in D}$ is a*

a) *T-uniform amart if $\|X_\sigma\|_1<\infty$ for each $\tau\in T$ and*

$$\operatorname*{Lim}_{\substack{\tau,\sigma\in T\\ \tau\leq\sigma}} \|X_\sigma - E^\tau X_\sigma\|_1 = 0;$$

b) *T-uniform potential if*

$$\operatorname*{Lim}_{T} \|X_\tau\|_1 = 0.$$

Since the conditional expectation contracts the norm $\|\cdot\|_1$, and L_E^1 is complete, we thus have

Lemma 2.3. *Let $T\in L$ and $(X_t)_{t\in D}$ be a T-uniform amart. Then for any $\tau\in T^c$ with $T(\tau)\neq\emptyset$, the net $(E^\tau X_\sigma)_{\sigma\in T}$ converges in L^1_E.*

Definition 2.4. *Let $T\in T$ and $(X_t)_{t\in D}$ be an E-valued martingale. $(X_t)_{t\in D}$ is called a T-regular martingale if for any τ, $\sigma\in T$ and $\tau\leqslant\sigma$, $\|X_\sigma\|_1<\infty$ and* $E^\tau X_\sigma=X_\tau$ a. s.

Definition 2.5. *A sub-σ-algebra B of F is called σ-finite if there exists $(\Omega_n)\subset B$ such that $\Omega_n\uparrow\Omega$ and $\mu(\Omega_n)<\infty$.*

In the case $T=T^s$ and $D=N$, the following Riesz decomposition theorem is due to Bellow[1].

Theorem 2.6. *For $T\in\underline{\underline{L}}$, $(X_t)_{t\in D}$ is a T-uniform amart if and only if $(X_t)_{t\in D}$ admits a unique decomposition, $X_t=Y_t+Z_t$, for $t\in D$, where $(Y_t)_{t\in D}$ is a $(T\cup D)$-regular martingale and (Z_t) is a T-uniform potential. Furthermore, for each $t\in D$ Y_t is the L^1_E limit of the net $(E^tX_\tau)_{\tau\in T}$. If there exists a $\tau^*\in T$, F_{τ^*} is σ-finite, then Y_t is also the strong essential limit of the net $(E^t X_\tau)_{\tau\cap T}$.*

Proof The sufficiency is obvious. Now we prove the necessity. For each $t\in D$ and $\sigma\in T$, denote the L^1_E limits of the net $(E^t X_\tau)_{\tau\in T}$ and the net $(E^\sigma X_\tau)_{\tau\in T}$ respectively by Y_t and $Y(\sigma)$. It is easy to show that $y(\sigma)=\sum_{t\in R(\sigma)} Y_tI_{(\sigma=t)}=Y_\sigma$ and $(Y_t)_{t\in D}$ is a $(T\cup D)$-regular martingale. Let $Z_t=X_t-Y_t$, we shall show that $(Z_t)_{t\in D}$ is a T-uniform potential. In fact, for any $\varepsilon>0$ there exits $\tau_0\in T$ such that $\sup_{\substack{\tau,\sigma\in T(t_0)\\ \tau\leqslant\sigma}}\|X_\tau-E^\tau X_\sigma\|_1<\varepsilon$. For any $\sigma\in T(\tau_0)$, choose $\sigma'\in T(\sigma)$, $\|Y_\sigma-E^\sigma X_{\sigma'}\|_1<\varepsilon$, then

$$\|Z_\sigma\|_1=\|X_\sigma-Y_\sigma\|_1\leqslant\|X_\sigma-E^\sigma X_{\sigma'}\|_1+\|E^\sigma X_{\sigma'}-Y_\sigma\|_1<2\varepsilon,$$

which implies $\lim_T\|Z_\sigma\|_1=0$, i. e., $(Z_t)_{t\in D}$ is a T-uniform potential. If $(X_t)_{t\in D}$ has another decomposition: $X_t=Y^1_t+Z^1_t$, where $(Y^1_t)_{t\in D}$ is a $(T\cup D)$-regular martingale and $(Z^1_t)_{t\in D}$ a T-uniform potential, then

$$\lim_T\|Y_\tau-Y^1_\tau\|_1=\lim_T\|Z_\tau-Z^1_\tau\|_1\leqslant\lim_T\|Z_\tau\|_1+\lim_T\|Z^1_\tau\|_1=0,$$

and for any τ, $\sigma\in(T\cup D)$, $\tau\leqslant\sigma$,

$$\|Y_\tau-Y^1_\tau\|_1=\|E^\tau(Y_\sigma-Y^1_\sigma)\|_1\leqslant\|Y_\sigma-Y^1_\sigma\|_1,$$

hence for each $t\in D$

$$\|Y_t-Y^1_t\|_1\leqslant\lim_T\|Y_\tau-Y^1_\tau\|_1=0.$$

It then follows that $Y_t=Y^1_t$ a. s.

It remains to show that for each $t\in D$, Y_t is the strong essential limit of the net $(E^tX_\tau)_{\tau\in T}$ if there exists a $\tau^*\in T$ and F_{τ^*} is σ-finite. We shall prove it by contradiction. Suppose it were not true, then there would exist $t_0\in D$, $\varepsilon>0$, and $A\in F_{\tau^*}$ $\mu(A)<\infty$ such that

$$\mu(\text{ess}\lim\sup_T A_\tau)=a>0,$$

where $A_\tau=A(|E^{t_0}X_\tau-Y_{t_0}|>\varepsilon_0)$. For each $\tau\in T(t_0)\cap T(\tau^*)$, there exists $(\tau_n)\subset T(\tau)$ $\text{ess}\sup\limits_{T(\tau)} A_\sigma=\bigcup\limits_n A_{\tau_n}$. Choose $k\in N$ $\mu\left[\bigcup\limits_1^k A_{\tau_n}\right]>a/2$. Let $\sigma=\tau_1$ on A_{τ_1}, $\sigma=\tau_n$ on $A_{\tau_n}\backslash\left(\bigcup\limits_1^{n-1} A_{\tau_i}\right)$ for $1<n<k$, and $\sigma=\tau_k$ on $\Omega\backslash\left(\bigcup\limits_1^{k-1} A_{\tau_n}\right)$. Then $\sigma\in T(\tau)$, and

$$\mu(A_\sigma)\geqslant\mu\left(\bigcup_1^k A_{\tau_n}\right)>a/2.$$

Yet Y_{t_0} is the L_E^1 limit of the net$(E^{t_0}X_\tau)_{\tau\in T}$, and it is a contradiction.

Definition 2.7. *Suppose $T\in L$, we say that $(X_t)_{t\in D}$ satisfies condition $A(T)$ $(B(T))$ if* $\lim\inf\limits_T\|X_\tau\|_1<\infty$ $(\lim\sup\limits_T\|X_\tau\|_1<\infty)$.

The following theorem improves Theorem 1 in [1].

Theorem 2.8. *Let $T\in L$ and $(X_t)_{t\in D}$ be a T-uniform amart. Then*

a) *Condition $A(T)$ is equivalent to condition $B(T)$;*

b) *Under condition $A(T)$, $(|X_t|)_{t\in D}$ is a real-vallued T-amart, i. e.,* $\lim\limits_T\|X_\tau\|_1$ *exists in R^1.*

Proof It is sufficient to show b). Suppose that $\lim\inf\limits_T\|X_\tau\|_1=a<\infty$. By Theorem 2.6, $X_t=Y_t+Z_t$, where$(Y_t)_{t\in D}$ is $(T\cup D)$-regular martingale and $(Z_t)_{t\in D}$ is a T-uniform potential. For any τ, $\sigma\in T$ and $\tau\leqslant\sigma$, $\|Y_\sigma\|_1\geqslant\|E^\tau Y_\sigma\|_1=\|Y_\tau\|_1$, hence

$$\lim_T\|Y_\tau\|_1=\lim_T\|X_\tau-Z_\tau\|_1\leqslant\lim\inf_T\|X_\tau\|_1+\lim_T\|Z_\tau\|_1=a<\infty,$$

$$\lim_T\|Y_\tau\|_1=\lim_T\|X_\tau-Z_\tau\|_1\geqslant\lim\sup_T\|X_\tau\|_1-\lim_T\|Z_\tau\|_1=\lim\sup_T\|X_\tau\|_1,$$

whence $\lim\sup\limits_T\|X_\tau\|_1=\lim\inf\limits_T\|X_\tau\|_1=a<\infty$, $(|X_t|)_{t\in D}$ is a real-valued T-amart.

Using the proof of Theorm 2.2 in [8], we can obtain the following cofinal optional sampling property for T-uniform amarts, where $T=T^s$ or T^c.

Theorem 2.9. *The class of $T^s(T^c)$-uniform amarts has cofinal optional sampling property, i. e., for any $T^s(T^c)$-uniform amart $(X_t)_{t\in D}$, and for any cofinal subset T of T^s (T^c) (i. e., $T\subset T^s(T^c)$ and for any $\tau\in T^s(T^c)$ $T(\tau)\neq\emptyset$), the process $(X_\tau, F_\tau)_{\tau\in T}$ is a $\tilde{T}^s$ $(\tilde{T}^c)$-uniform amart, where $\tilde{T}^s$ $(\tilde{T}^c)$ is the set of all simple (countable) stopping times with respect to $(F_\tau)_{\tau\in T}$.*

§ 3. Convergence in the absence of the Aitali condition V.

In the following sections we assume that the σ-algebra$F_{-\infty}=\wedge_t F_t$ is σ-finite.

Definition 3.1. *Let $(U_t)_{t\in D}$ be a family of E-valued r. v.' s. We say that $(U_t)_{t\in D}$ converges strongly stochastically if there exists an E-valued r. v. X such that for any $\varepsilon>0$ and $A\in F$, $\mu(A)<\infty$,*

$$\lim_D\mu[A(|U_t-X|>\varepsilon)]=0.$$

Theorem 3.2. *Suppose that $T\in L$, $(X_t)_{t\in D}$ is a T-uniform amart satisfying condition $A(T)$, and E has the Radon-Nikodym property. Then $(X_\tau)_{\tau\in T}$ converges strongly stochastically.*

Proof By Theorem 2.6, we can write $X_t=Y_t+Z_t$, where $(Y_t)_{t\in D}$ is a $(T\cup D)$-regular martingale and $(Z_t)_{t\in D}$ a T-uniform potential. It is easy to see that $(Z_\tau)_{\tau\in T}$ converges strongly stochastically to zero. We need to show the convergence of the net $(Y_\tau)_{\tau\in T}$. Using the argument of Theorem 2.8, $\sup_T\|Y_\tau\|_1=\lim_T\|Y_\tau\|_1=\lim\inf_T\|X_\tau\|_1<\infty$, thus, for every increasing sequence $(\tau_n)\subset T$, $(Y_{\tau_n}, F_{\tau_n})_{n\geq 1}$ is an L^1_E-bounded martingale which converges according to Chatterji's theorem (see, e. g, [9] p. 112) almost surely in the norm topology, hence strongly stochastically. Since the strong stochastic convergence is defined by a complete metric, this implies that $(Y_\tau)_{\tau\in T}$ converges strongly stochastically to Y.

Defination 3.3.[7] *Let $(U_t)_{t\in D}$ be a family of E-valued r. v.'s. We say that $(U_t)_{t\in D}$ is terminally uniformly integrable if given any $\varepsilon>0$ there exist an $S\in D$, a positive number C and an element H in F of finite measare such that*

$$\sup_{D(S)}\left[\int_{(|U_t|>C)}|U_t|\,d\mu+\int_{(\Omega/H)}|U_t|\,d\mu\right]<\varepsilon.$$

Theorem 3.4. *Suppose $T\in L$ and $(X_\tau)_{\tau\in T}$ is terminally uniformly integrable. Among the following assertions:*

(a) *the net $(X_\tau)_{\tau\in T}$ converges in L^1_E;*

(b) *the net $(X_\tau)_{\tau\in T}$ converges strongly stochastically;*

(c) *$(X_t)_{t\in D}$ is a T-uniform amart,*

we have (a)$\Leftrightarrow$(b)$\Rightarrow$(c). *If, in addition, E has the Radon-Nikodym property, then the assertions* (a), (b), *and* (c) *are equivalent.*

Proof It is clear that, under the hypothesis of the theorem (a)$\Leftrightarrow$(b). (b)$\Rightarrow$(c): Suppose that $(X_\tau)_{\tau\in T}$ converges strongly stochastically to X. Since $(X_\tau)_{\tau\in T}$ is terminally uniformly integrable, $\lim\sup_T\|X_\tau\|_1<\infty$. Thus, by Fatou's lemma, $\|X\|_1<\infty$. It is easy to see that $(X_\tau-E^\tau X)_{\tau\in T}$ converges strongly stochastically to zero. Since

$$|X_\tau-E^\tau X|\leqslant|X_\tau|+|E^\tau X|\leqslant|X_\tau|+E^\tau|X|,$$

$(X_\tau-E^\tau X)_{\tau\in T}$ is terminally uniformly integrable, hence

$$\lim_T\|X_\tau-E^\tau X\|_1=0.$$

For any $\varepsilon>0$, there exists a $\tau_0\in T$ such that

$$\sup_{T(\tau_0)}\|X_\tau-E^\tau X\|_1<\varepsilon.$$

If $\tau,\sigma\in T(\tau_0)$ and $\tau\leqslant\sigma$, then

$$\|X_\tau-E^\tau X_\sigma\|_1\leqslant\|X_\tau-E^\tau X\|_1+\|E^\tau X-E^\tau X_\sigma\|_1$$
$$\leqslant\varepsilon+\|E^\tau(E^\sigma X-X_\sigma)\|_1\leqslant\varepsilon+\|X_\sigma-E^\sigma X\|_1<2\varepsilon,$$

whence $(X_t)_{t\in D}$ is a T-uniform amart.

If E has the Radon-Nikodym property, (c)⇒(b) by Theorem 3.2.

§ 4. A Characterization of Strong Stochastic Covergence of $(X_\tau)_{\tau\in T^s}$

Definition 4.1.[3] *Let $T\in \underline{\underline{L}}$, we say that T has the density if for any $\varepsilon>0$ and any A in F of finite measure, there exists a $t\in D$ such that for any $\tau\in T^c(t)$ there is $\tau'\in T$ with $\mu[A(\tau\neq\tau')]<\varepsilon$.*

Lemma 4.2. *The following assertions are equivalent:*

(a) *$(X_\tau)_{\tau\in T^c}$ converges strongly stochastically to X;*

(b) *$(X_\tau)_{\tau\in T^s}$ converges strongly stochastically to X;*

(c) *There exists a $T\in L$ such that T has the density and $(X_\tau)_{\tau\in T}$ converges strongly stochastically to X.*

Proof (a)⇒(b). For any $\varepsilon>0$ and A in F of finite measre, there exists a $\sigma\in T^c$ such that

$$\sup_{T^c(\sigma)} \mu[A(|X_\tau-X|>\varepsilon)]<\varepsilon.$$

Suppose $R(\sigma)=(t_n)$. Choose k, t and σ' satisfying $k\in N$, $t\in \bigcap_1^k D(t_i)$, and $\sigma'\in T^c(\sigma)\cap T^c(t)$ respectively so that

$$\mu\left\{A\left[\bigcup_{k+1}^{\infty}(\sigma=t_n)\right]\right\}<\varepsilon.$$

For any $\tau \cap T^s(t)$, let $\tau'=\tau$ on $\left[\bigcup_1^k(\sigma=t_n)\right]$, and $\tau'=\sigma'$ on $\left[\bigcup_{k+1}^{\infty}(\sigma=t_n)\right]$, then $\tau'\in T^c(\sigma)$, and

$$\mu[A([X_\tau-X|>\varepsilon)]\leqslant\mu[A(\tau\neq\tau')]+\mu[A(|X_{\tau'}-X|>\varepsilon)]<\varepsilon+\varepsilon=2\varepsilon,$$

which implies that $(X_\tau)_{\tau\in T^s}$ converges strongly stochastically to X.

Clearly, T^s has the density, (b)⇒(c).

(c)⇒(a). For given $\varepsilon>0$ and A in F of finite measure there exists a $\sigma\in T$ such that

$$\sup_{T(\sigma)} \mu[A(|X_\tau-X|>\varepsilon)]<\varepsilon.$$

Since T has the density, there is a $t\in D$ as given in Definition 4.1. Then, for any $\tau\in T^c(\sigma)\cap T^c(t)$, there is a $\tau'\in T$, $\mu[A(\tau\neq\tau')]<\varepsilon$.

Take $\tau''\in T(\sigma)\cap T(\tau')$, and let $\sigma'=\tau'$ on $(\tau'\geqslant\sigma)$, and $\sigma'=\tau''$ on $(\tau'\not\geqslant\sigma)$, then $\sigma'\in T(\sigma)$ and $\mu[A(\tau\neq\sigma')]\leqslant\mu[A(\tau\neq\tau')]<\varepsilon$. Hence

$$\mu[A(|X_\tau-X|\geqslant\varepsilon)]\leqslant u[A(|X_{\sigma'}-X|>\varepsilon)]+\mu[A(\tau\neq\sigma')]<2\varepsilon.$$

It follows that $(X_\tau)_{\tau\in T^c}$ converges strongly stochastically to X.

Theorem 4.3. *Suppose that $(X_t)_{t\in D}$ is an E-valued adapted process, and E has the Radon-Nikodym property. Then the net $(X_\tau)_{\tau\in T^s}$ $(X_\tau)_{\tau\in T^c})$ converges strongly* *stochastically to a Bochner integrable r. v. if and only if there exists a* $T\in\underline{\underline{L}}$ *such that* T *has the density and* $(X_t)_{t\in D}$ *is a* T*-uniform amart, satisfying* $\sup_T\|X_\tau\|_1<\infty$.

Proof By Theorem 3.2 and Lemma 4.2 the sufficiency is obvious. As to the necessity, suppose that $(X_\tau)_{\tau\in T^c}$ converges strongly stochastically to $X\in L^1_E$. Take (H_n) of disjoint sets in $F_{-\infty}$ such that for each $n\geqslant 1$, $\mu(H_n)<\infty$ and $\Omega_n=\bigcup_1^n H_k\uparrow\Omega$. For any $\varepsilon>0$ let

$$U^{(\varepsilon)}=\sum_n\frac{\varepsilon}{2^n(\mu(H_n)+1)}\cdot I_{H_n},$$

then $U^{(\varepsilon)}$ is an $F_{-\infty}$-measurable positive r. v. Let

$$T=\{\tau\in T^c\,|\,|X_\tau-E^\tau X|\leqslant U^{(1)}\},$$

then $\sup_T\|X_\tau\|_1\leqslant EU^{(1)}+\|E^\tau X\|_1\leqslant EU^{(1)}+\|X\|_1<\infty$. For any $0<\varepsilon\leqslant 1$ let $A^\varepsilon_\tau=(|X_\tau-E^\tau X|>U^{(\varepsilon)})$, $B^\varepsilon_\tau=(|X_\tau-E^\tau X|\leqslant U^{(\varepsilon)})$. Since $(X_\tau-E^\tau X)_{\tau\in T^c}$ converges strongly stochastically to zero, $(I_{A^\varepsilon_\tau})_{\tau\in T^c}$ converges stochastically to zero. For any $\sigma\in T^c$, we can choose $\tau_1\in T^c(\sigma)$, $\mu[\Omega_1A^\varepsilon_{\tau_1}]<1/2$. Having chosen $\tau_1,\cdots,\tau_n$, take $\tau_{n+1}\in T^c(\tau_n)$, $\mu[\Omega_{n+1}A^\varepsilon_{\tau_{n+1}}]<1/2^{n+1}$, then

$$\Omega=\bigcup_1^\infty B^\varepsilon_{\tau_n}.$$

Let $\tau=\tau_1$ on $B^\varepsilon_{\tau_1}$, and $\tau=\tau_n$ on $\left[B^\varepsilon_{\tau_n}/\bigcup_1^{n-1}B^\varepsilon_{\tau_i}\right]$ for $n>1$. Then $\tau\in T^c(\sigma)$, and $|X_\tau-E^\tau X|\leqslant U^{(\varepsilon)}$, particularly, $\tau\in T(\sigma)$, hence $T\in\underline{\underline{T}}$. It is clear that T has the localization property, thus $T\in\underline{\underline{L}}$. Now we prove that T has the density. For any $\varepsilon>0$ and $A\in F$, $\mu(A)<\infty$, take $\sigma\in T$ such that

$$\sup_{T^c(\sigma)}\mu(AA^1_\tau)<\frac{\varepsilon}{2}.$$

Suppose $R(\sigma)=(t_n)$, choose $k\in N$,

$$\mu\left\{A\left[\bigcup_{k+1}^\infty(\sigma=t_n)\right]\right\}<\frac{\varepsilon}{2}.$$

Take $t\in\bigcap_1 D(t_n)$, then for any $\tau\in T^c(t)$, take $\sigma'\in T(\tau)\cap T(\sigma)$, and let

$$\tau'=\begin{cases}\tau, \text{ on } B^1_\tau\left[\bigcup_1^k(\sigma=t_n)\right]\\ \sigma', \text{ on } \Omega\backslash B^1_\tau\left[\bigcup_1^k(\sigma=t_n)\right]\end{cases}$$

$$\tau''=\begin{cases}\tau, \text{ on } \bigcup_1^k(\sigma=t_n),\\ \sigma', \text{ on } \Omega\backslash\bigcup_1^k(\sigma=t_n),\end{cases}$$

then $\tau'\in T(\sigma)$, $\tau''\in T^c(\sigma)$, and

$$\mu[A(\tau'\neq\tau)]\leqslant\mu\left\{A\left[\bigcup_{k+1}^\infty(\sigma=t_n)\right]\right\}+\mu(AA^1_{\tau''})<\varepsilon.$$

Therefore T has the density. It remains to show that $(X_t)_{t\in D}$ is a T-uniform amart. For any $\varepsilon>0$, choose $k\in N$ and $\tau_0\in T$ such that

$$\sum_{k+1}^{\infty}\frac{1}{2^n}<\varepsilon, \sup_{T^c}(\tau_0)\mu(\Omega_k A_\tau^\varepsilon)<\varepsilon.$$

Then if $\tau, \sigma\in T(\tau_0)$ and $\tau\leqslant\sigma$,

$$\|X_\tau-E^\tau X\|_1\leqslant E(U^{(1)}I_{\Omega/\Omega_k})+E(U^{(\varepsilon)}I_{\Omega_k}B_\tau^\varepsilon)+\mu(\Omega_k A_\tau^\varepsilon)$$
$$<\sum_{k+1}^{\infty}\frac{1}{2^n}+\varepsilon\cdot\sum_{1}^{k}\frac{1}{2^n}+\varepsilon<3\varepsilon.$$

Similarly

$$\|X_\sigma-E^\sigma X\|_1<3\varepsilon.$$

Hence

$$\|X_\tau-E^\tau X_\sigma\|_1\leqslant\|X_\tau-E^\tau X\|_1+\|E^\tau X-E^\tau X_\sigma\|_1$$
$$<3\varepsilon+\|E^\tau(E^\sigma(X-X_\sigma))\|_1\leqslant3\varepsilon+\|X_\sigma-E^\sigma X\|_1<6\varepsilon,$$

which implies that $(X_t)_{t\in D}$ is a T-uniform amart.

§5. Convergence under the Vitali Condition V.

Krickeberg[7] introduced the Vitali condition V on a stochastic basis to assure essential convergence of $L^1_{R^1}$-bounded martingales. A stochastic basis $(F_t)_{t\in D}$ is said to satisfy the Vitali condition V if the following holds[7,9]

For every A in F of finite measure, for every family of $A_t\in F_t(t\in D)$ such that $A\subset\text{ess}\lim\sup_D A_t$, and for any $\varepsilon>0$, there exist finitely many indices $t_1, t_2, \cdots t_n\in D$ and sets $B_i\in F_{t_i}$ $(i=1, 2, \cdots, n)$ such that

$$B_i\subset A_{t_i} \text{ for } i=1, 2, \cdots, n,$$
$$B_i\cap B_j=\emptyset \text{ for } i\neq j,$$

and

$$\mu\Big(A\backslash\bigcup_1^n B_i\Big)\leqslant\varepsilon.$$

We remark that if D is totally ordered, then the Vitali condition V holds[7,9]. Millet and Sucheston[8] showed that $(F_t)_{t\in D}$ satisfies the Vitali condition V if and only if for every Banach space E and for every E-valued adapted process $(X_t)_{t\in D}$, the strong stochastic convergence of $(X_\tau)_{\tau\in T^s}$ implies the strong essential convergence of $(X_t)_{t\in D}$. Using this result and Lemma 4.2, we have

Theorem 5.1. *For a stochastic basis $(F_t)_{t\in D}$ the following* assertions are equivalent:

(a) *$(F_t)_{t\in D}$ satisfies the Vitali condition V;*

(b) *For every Banach space E and every E-valued adapted process $(X_t)_{t\in D}$, the strong stochastic convergence of $(X_\tau)_{\tau\in T^c}$ to a limit X implies the strong essential* *convergence of* $(X_t)_{t\in D}$ *to* X;

(c) *Same as* (b) *except that* T^c *is being replaced by* T, *where* $T\in \underline{\underline{L}}$ *and* T *has the density.*

By Theorems 5.1, 3.2, 3.4, and 4.3, we obtain

Theorem 5.2. *Uuder the assumptions of Theorem* 3.2, *if* $(F_t)_{t\in D}$ *satisfies the Vitali condition* V, *and if* T *has the density, then* $(X_t)_{t\in D}$ *converges strongly essentially.*

Therem 5.3. *Suppose that* $T\in \underline{\underline{L}}$, T *has the density, and* $(X_\tau)_{\tau\in T}$ *is terminally uniformly integrable. Then the strong essential convergence of* $(X_t)_{t\in D}$ *implies that* $(X_t)_t)_{t\in D}$ *is a T-uniform amart. If, in addition,* E *has the Radon-Nikodym property and* $(F_t)_{t\in D}$ *satisfies the Vitali condition* V, *then the converse is also true.*

Theorem 5.4. *Suppose that*$(X_t)_{t\in D}$ *is an E-valued adapted process,* E *has the Radon-Nikodym property, and* $(F_t)_{t\in D}$ *satisfies the Vitali condition* V. *Then* $(X_t)_{t\in D}$ *converges strongly essentially to a Bochner integrable r. v. if and only if there exists a* $T\in \underline{\underline{L}}$ *such that* T *has the density and* $(X_t)_{t\in D}$ *is a T-uniform amart satisfying* $\sup_T \|X_\tau\|_1<\infty$.

Remark. In the case that $T=T^s$, $D=N$, and that μ is aprobability measure, Theorem 5.2 is due to [1], Theorem 5.4 is due to [3], and Theorem 5.3 is an improvement of a result in [1]. Theorem 5.2 is also a generalization of Theorem 12.4 in [8].

§ 6. The Real-Valued Case

In this section we assume $E=R^1$.

Definition 6.1. *Let* $T\in \underline{\underline{L}}$, *and* $(X_t)_{t\in D}$ *be a real-valued adapted process. We say that* $(X_t)_{t\in D}$ *is a T-amart, if* $\lim_T EX_\tau$ *exists in* R^1.

Lemma 6.2. *Let* $T\in \underline{\underline{L}}$. $(X_t)_{t\in D}$ *is a T-amart if and only if* $(X_t)_{t\in D}$ *is a T-uniform amart.*

Proof For any $\tau,\ \sigma\in T,\ \tau\leqslant\sigma,\ |EX_\tau-EX_\sigma|=|E(X_\tau-E^\tau X_\sigma)|\leqslant E|X_\tau-E^\tau X_\sigma|=\|X_\tau-E^\tau X_\sigma\|_1$, the sufficiency is obvious. Now we prove the necessity. For any $\varepsilon>0$, choose $\tau_1\in T$,

$$\sup_{T(\tau_0)}|E(X_\tau-X_\sigma)|<\varepsilon.$$

For any $\tau,\ \sigma\in T(\tau_0),\ \tau\leqslant\sigma$, let $\tau'=\tau$ on $(X_\tau-E^\tau X_\sigma\geqslant 0)$, and $\tau'=\sigma$ on $(X_\tau-E^\tau X_\sigma<0)$. Then $E(X_\tau-E^\tau X_\sigma)^+=EX_{\tau'}-EX_\sigma<\varepsilon$. Similary, $E(X_\tau-E^\tau X_\sigma)^-<\varepsilon$. Hence $\|X_\tau-E^\tau X_\sigma\|_1<2\varepsilon$, and$(X_t)_{t\in D}$ is a T-uniform amart.

Thus, the results in section 2, 3, and 5 are also true with T-uniform amart replaced by T-amart, and R^1 in lieu of E.

In the following we give another characterization of convergence of real-valued adapted process.

We say that a real-valued process $(U_t)_{t\in D}$ stochastically converges to *a* r. v. X which take values in $\bar{R}^1$, if for any $\varepsilon>0$, $M>0$, and A in F of finite measure

$$\lim_D\{\mu[A(|U_t-X|>\varepsilon)(|X|<\infty)+\mu[A(U_t<M)(X=+\infty)]$$
$$+\mu[A(U_t>-M)(X=-\infty)]\}=0.$$

It is easy to see that $(U_t)_{t\in D}$ stochastically converges to *a* r. v. X if and only if for any finite positive r. v. λ, $((-\lambda)\vee X_t\wedge\lambda)_{t\in D}$ stochastically converges to $(-\lambda)\vee X\wedge\lambda$.

Definition 6.2.[5] *Let $(X_t)_{t\in D}$ be a real-valued adapted process. $T\in\underline{\underline{L}}$. We say that $(X_t)_{t\in D}$ is a TS-martingale, if for any $F_{-\infty}$-measurable positive* r. v. λ in $L^1_{R^1}$, *$((-\lambda)\vee X_t\wedge\lambda)$ is a T-amart.*

From Theorems 2.4 and 5.3, we obtain

Theorem 6.3. *Let $T\in\underline{\underline{L}}$ and $(X_t)_{t\in D}$ be a real-valued adapted process. Then $(X_\tau)_{\tau\in T}$ stochastically converges if and only if $(X_t)_{t\in D}$ is a TS-martingale.*

Theorem 6.4. *Suppose that $(F_t)_{t\in D}$ satisfies Vitali condition V, and $(X_t)_{t\in D}$ be a real-valued adapted process. Then $(X_t)_{t\in D}$ essentially converges if and only if there exists a $T\in\underline{\underline{L}}$ such that T has the density and $(X_t)_{t\in D}$ is a TS-martingale.*

§ 7. A Class of Ordered Amarts

A τ in T^s is called an ordered stopping time if $R(\tau)$ is a totally ordered subset of D. All ordered stopping times are denoted by T'. Given σ, τ in T', we writed $\sigma\leqslant 1\leqslant\tau$ if $\tau=\sigma$ or, if there exists an $S\in D$ such that $\sigma\leqslant S\leqslant\tau$. For the partial order$\leqslant 1$, T' is a directed set filtering to the right. We say that $(F_t)_{t\in D}$ satisfies the Vitali condition V' if for every A in F of finite measurs, for every family of $A_t\in F_t(t\in D)$ such that $A\subset \operatorname{ess}\limsup_D A_t$, and for every $\varepsilon>0$, there exists *a* $\tau\in T'$ such that $\mu(A\backslash A_\tau)<\varepsilon$, where $A_\tau=\bigcup_t(\tau=t)A_t$. This has been shown by Krickeberg to be sufficient for essential convergence of $L^1_{R^1}$-bounded submartingales[6]. Millet and Sucheston[8] showed that the Vitali condition V' is necessary and sufficient for essential convergence of $L^1_{R'}$-bounded ordered amarts (i. e., $\lim_{T'} EX_\tau$ exists in R^1), and it is also equivalent that for every Banach space E and for every E-valued adapted process $(X_t)_{t\in D}$, the strong stochastic convergence of $(X_\tau)_{\tau\in T'}$ implies the strong essential convergence of $(X_t)_{t\in D}$.

Let

$$\underline{\underline{T}}'=\{T\,|\,T \text{ is a directed subset of } T' \text{ and } T'(t)\neq\emptyset \text{ for any } t\in D\}.$$

Definition 7.0. *Let $T\in\underline{T}'$, we say that T has the monotone localization property, if for each finite family $\tau_1\leqslant 1\leqslant\tau_2\leqslant 1\leqslant\cdots\leqslant 1\leqslant\tau_n\subset T'$ and finite partition of Ω, (A_j) $1\leqslant j\leqslant n$ with $A_j\in F_{\tau_j}$ $(1\leqslant j\leqslant n)$, the stopping time $\tau=\tau_j(\omega)$ for $\omega\in A\tau_j(1\leqslant j\leqslant n)$, belongs to T.*

Let

$$L'=\{T\in T' \mid \underline{T} \text{ has the monotone localization property}\}.$$

Definition 7.1. *Let $T\in\underline{L}'(X_t)_{t\in D}$ is a*

(a) *T-uniform ordered amart if $\|X_\tau\|_1<\infty$ for each $\tau\in T$ and*

$$\lim_{\substack{\tau,\sigma\in T\\ \tau\leqslant 1\leqslant\sigma}} \|X_\tau-E^\tau X_\sigma\|_1=0;$$

(b) *T-uniform orded potential if $\lim_T\|X_\tau\|_1=0$.*

For T-uniform ordered amarts we have

Theorem 7.2. *Let $T\in\underline{L}'$. $(X_t)_{t\in D}$ is a T-uniform ordered amart if and only if $(X_t)_{t\in D}$ admits a unique decomposition, $X_t=Y_t+Z_t$, for $t\in D$, where $(Y_t)_{t\in D}$ is a martingale and $(Z_t)_{t\in D}$ is a T-uniform ordered potential. Forthermore, for each $t\in D$, Y_t is the L^1_E limit of $(E^tX_\tau)_{\tau\in T}$.*

Theorem 7.3. *Identicat to Theorems 2.8, 2.9, 3.2, 3.4, 5.2, 5.3, 6.3, and 6.4, with T' replacing T^s and T^c, L' replacing L, V' replacing V, and T-unifrom ordered amart replacing T-uniform amart.*

Proofs of Theorems 7.2.and 7.3 are analogous to that of Theorems 2.6, 2.8, 2.9, 3.2, 3.4, 5.2, 5.3, 6.3, and 6.4.

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