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ON ADMISSIBILITY OF VARIANCE COMPONENTS ESTIMATES

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Abstract

Suppose that there is a variance components model

$$\begin{cases} E \underset{n\times 1}{Y} = \underset{n\times p}{X} \underset{p\times 1}{\beta}, \\ DY = \sigma_2^2 V_1 + \sigma_2^2 V_2, \end{cases}$$

where β, σ_1^2 and σ_2^2 are all unknown, X, $V>0$ and $V_2>0$ are all known, $r(X)<n$. The author estimates simultaneously (σ_1^2, σ_2^2). Estimators are restricted to the class $\mathscr{D}=\{d(A_1A_2)=(Y'A_1Y, Y'A_2Y), A_1\geqslant 0, A_2\geqslant 0\}$. Suppose that the loss function is $L(d(A_1, A_2), (\sigma_1^2, \sigma_2^2))=\frac{1}{\sigma_1^4}(Y'A_1Y-\sigma_1^2)+\frac{1}{\sigma_2^4}(Y'A_2Y-\sigma_2^2)^2$. This paper gives a necessary and sufficient condition for $d(A_1, A_2)$ to be an equivariant $\mathscr{D}$-admissible estimator under the restriction $V_1=V_2$, and a sufficient condition and a necessary condition for $d(A_1, A_2)$ to be equivariant $\mathscr{D}$-admissible without the restriction.

§ 1. Introduction

Suppose tnat the distribution of random variable X has density $p_\theta(x)$ with respect to a σ finite measure μ, where $\theta\in\Theta$ is an unknown parameter, X and θ may be multidimensional. Let $h(\theta)$ be the function of parameter θ to be estimated with its estimate $d(X)$, a function of observation X. The expression $L(h(\theta), d(X))$ denotes the loss function, whose expected value $R(d, \theta)=E[L(h(\theta), d(X))|\theta]$ is called the risk function of the estimator $d(X)$ of $h(\theta)$. An estimator $d_0(X)$ of $h(\theta)$ is said to be better than $d_1(X)$, if

$$R(d_0, \theta)\leqslant R(d_1, \theta)$$

for all $\theta\in\Theta$, and for at least one point, say θ_0, of Θ

$$R(d_0, \theta_0)<R(d_1, \theta_0).$$

If there is no estimator better than $d_0(X)$, then $d_0(X)$ is said to be an admissible estimater. Suppose that we confine our estimators to a certain class $\mathscr{D}$. If $d_0(X)\in\mathscr{D}$ and $d_0(X)$ is admissible within $\mathscr{D}$, $d_0(X)$ is called $\mathscr{D}$-admissible. In recent years, admissibility of point estimator has received much attention.

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For linear models, the admissible problem of estimable linear function of regression coefficients in a class of linear estimators was solved by Cohen[1] and Rao[2]. For some specific linear models, Wu Qiguang, Cheng Ping and Li Guoying gave a necessary and sufficient condition for the admissible estimator of error variance in a class of estimators of nonegative definite quadratic form[3]. In this paper, we deal with the admissibility of variance components estimators in a variance components modle.

Let

$$\underset{n\times 1}{Y}=\underset{n\times p}{X}\ \underset{p\times 1}{\beta}+\underset{n\times p_1}{X_1}\ \underset{p_1\times 1}{\varepsilon_1}+\underset{n\times p_2}{X_2}\ \underset{p_2\times 1}{\varepsilon_2},$$

where X, X_1 and X_2 are known, $n\geqslant p$, $\varepsilon_1'=(\varepsilon_{11}, \varepsilon_{12}, \cdots, \varepsilon_{1p_1})$, $\varepsilon_2'=(\varepsilon_{21}, \varepsilon_{22}, \cdots, \varepsilon_{2p_2})$, $\varepsilon_{11}, \varepsilon_{12}, \cdots, \varepsilon_{1p_1}, \varepsilon_{21}, \varepsilon_{22}, \cdots, \varepsilon_{2p_2}$ are independent of each other, and

$$E(\varepsilon_{1i})=0,\ E(\varepsilon_{1i}^2)=\sigma_1^2,\ E(\varepsilon_{1i}^3)=0,\ E(\varepsilon_{1i}^4)=3\sigma_1^4,\ i=1,2,\cdots,p_1,$$

$$E(\varepsilon_{2j})=0,\ E(\varepsilon_{2j}^2)=\sigma_2^2,\ E(\varepsilon_{2j}^3)=0,\ E(\varepsilon_{2j}^4)=3\sigma_2^4,\ j=1,2,\cdots,p_2,$$

$\beta\in R^p$, $0<\sigma_1^2, \sigma_2^2<\infty$, are all unknown. Set $X_1X_1'=V_1>0$, $X_2X_2'=V_2>0$ (We use the convention $M>0$ and $M\geqslant 0$ to denote that the square matrix M is positive definite and nonnegative definite respectively). Above hypothesis is denoted by H. Under the hypothesis we obtain a model

$$\begin{cases} EY=X\beta, \\ DY=\sigma_1^2V_1+\sigma_2^2V_2\triangleq V. \end{cases}$$

For the model, the estimators of linear combinations of σ_1^2 and σ_2^2 which have various best properties have been discussed[4,5,6]. Here we need to estimate simultaneously $(\sigma_1^2; \sigma_2^2)$. The estimators are restricted to the class $\mathscr{D}=\{d(A_1, A_2)=(Y'A_1Y,\ Y'A_2Y),\ A_1\geqslant 0,\ A_2\geqslant 0\}$. Suppose that the loss function $L(d(A_1, A_2), (\sigma_1^2, \sigma_2^2))=\frac{1}{\sigma_1^4}(Y'A_1Y-\sigma_1^2)^2+\frac{1}{\sigma_2^4}(Y'A_2Y-\sigma_2^2)^2$. We obtain the risk function

$$\begin{aligned}
&R(d(A_1, A_2), \beta, \sigma_1^2, \sigma_2^2)\\
&=\xi_1'X'A_1X\xi_1)^2+4\xi_1'X'A_1\frac{V}{\sigma_1^2}A_1X\xi_1+2\xi_1'X'A_1X\xi_1\operatorname{tr}\left(A_1\frac{V}{\sigma_1^2}\right)\\
&\quad+\left[\operatorname{tr}\left(A_1\frac{V}{\sigma_1^2}\right)-1\right]^2+2\operatorname{tr}\left(A_1\frac{V}{\sigma_1^2}\right)^2-2\xi_1'X'A_1X\xi_1+(\xi_2'X'A_2X\xi_2)^2\\
&\quad+4\xi_2'X'A_2\frac{V}{\sigma_2^2}A_2X\xi_2+2\xi_2'XA_2X\xi_2\operatorname{tr}\left(A_2\frac{V}{\sigma_2^2}\right)+\left[\operatorname{tr}\left(A_2\frac{V}{\sigma_2^2}\right)-1\right]^2\\
&\quad+2\operatorname{tr}\left(A_2\frac{V}{\sigma_2^2}\right)^2-2\xi_2'X'A_2X\xi_2,
\end{aligned}\tag{1.1}$$

where $\xi_1=\beta/\sigma_1$, $\xi_2=\beta/\sigma_2$, and $\operatorname{tr}(M)$ denotes the trace of the square matrix M.

In § 2 of this paper we suppose that the rank of matrix X, $r(X)<n$. Under hypothesis H we consider the equivariant $\mathscr{D}$-admissibility of estimator for (σ_1^2, σ_2^2). We give here a necessary and sufficient condition for $d(A_1, A_2)$ to be an equivariant $\mathscr{D}$-admissible estimator under restriction $V_1=V_2$, and a sufficient condition and a

necessary condition for $d(A_1, A_2)$ to be equivariant $\mathscr{D}$-admissible without the restriction.

§ 2. Equivariant $\mathscr{D}$-Admissible Estimator Under Condition $r(X)<n$ in Variance Component Model

First, we prove a few lemmas.

Lemma 2.1. *If M is a nonnegative definite matrix which is not zero-matrix, then*

$$\frac{1}{r(M)}[\operatorname{tr}(M)]^2 \leqslant \operatorname{tr}(M^2) \tag{2.1}$$

and a ntcessary and sufficient condition under which the equality sign holds true is that the non-zero-eigenvalues of M are all equal.

Proof Because M is not zero-matrix, $r(M)>0$ and formula (2.1) has exact meaning. Suppose that the non-zero-eigenvalues of M are $\lambda_1, \lambda_2, \cdots, \lambda_l$. It follows from Schwarz inequality that

$$\frac{1}{r(M)}[\operatorname{tr}(M)]^2=\frac{1}{l}\Big(\sum_{i=1}^{l}\lambda_i\Big)^2 \leqslant \sum_{i=1}^{l}\lambda_i^2=\operatorname{tr}(M^2)$$

and a necessary and sufficient condition under which the equality sign holds is that every eigenvalue is equal to the same constant.

Lemma 2.2. *Suppose that A and B are two square matrices of equal orders. If $A\geqslant B\geqslant 0$, then $\operatorname{tr}(A)\geqslant\operatorname{tr}(B)$ and $\operatorname{tr}(A^2)\geqslant\operatorname{tr}(B^2)$ (Here we use the convention $A\geqslant B$ to denote $A-B\geqslant 0$).*

Proof Because of $A-B\geqslant 0$, $\operatorname{tr}(A)-\operatorname{tr}(B)=\operatorname{tr}(A-B)\geqslant 0$, consequently $\operatorname{tr}(A)\geqslant\operatorname{tr}(B)$. From $A\geqslant B$ we obtain

$$A^2\geqslant A^{\frac{1}{2}}BA^{\frac{1}{2}} \quad \text{and} \quad B^{\frac{1}{2}}AB^{\frac{1}{2}}\geqslant B.$$

Therefore

$$\operatorname{tr}(A^2)\geqslant\operatorname{tr}(A^{\frac{1}{2}}BA^{\frac{1}{2}})=\operatorname{tr}(B^{\frac{1}{2}}AB^{\frac{1}{2}})\geqslant\operatorname{tr}(B^2).$$

Lemma 2.3. *Under hypothesis H, if $r(X)<n$ and $V_1=V_2=I_n$, the equivariant $\mathscr{D}$-admissible estimator of (σ_1^2, σ_2^2) has the following form*

$$(Y'[a_1(I-XX^+)]Y,\ Y'[a_2(I-XX^+)]Y),$$

where a_1 and a_2 are nonnegative real numbers (Sign X^+ denotes the generalized inverse matrix of X in the sense of Moore).

Proof Suppose that there is a transformation in the sample space

$$Y\to Y+X\alpha,$$

where α is any vector in R^p. It leads to a corresponding transformation in the parameter space

$$\beta\to\beta+\alpha.$$

and

$$(\sigma_1^2,\ \sigma_2^2)\to(\sigma_1^2,\ \sigma_2^2).$$

In the decision space of estimators for $(\sigma_1^2,\ \sigma_2^2)$ there are

$$(Y+X\alpha)'A_1(Y+X\alpha)=Y'A_1Y$$

and

$$(Y+X\alpha)'A_2(Y+X\alpha)=Y'A_2Y. \tag{2 2}$$

That is, $2Y'A_1X\alpha+\alpha'X'A_1X\alpha=0$ and $2Y'A_2X\alpha+\alpha'X'A_2X\alpha=0$ for all Y and α. Because of this formula, it is clear that a necessary and sufficient condition under which formula (2.2) holds is $A_1X=0$ and $A_2X=0$. Namely, the equivariant estimator $d(A_1,\ A_2)$ for $(\sigma_1^2,\ \sigma_2^2)$ satisfies $A_1Y=0$ and $A_2X=0$.

Thus the risk function is equal to (see (1.1))

$$\begin{aligned}&R(d(A_1,\ A_2),\ \beta,\ \sigma_1^2,\ \sigma_2^2)\\&=\left[\left(1+\frac{\sigma_2^2}{\sigma_1^2}\right)\operatorname{tr}A_1-1\right]^2+2\operatorname{tr}\left[A_1\left(1+\frac{\sigma_2^2}{\sigma_1^2}\right)\right]^2+\left[\left(1+\frac{\sigma_1^2}{\sigma_2^2}\right)\operatorname{tr}A_2-1\right]^2\\&\quad+2\operatorname{tr}\left[A_2\left(1+\frac{\sigma_1^2}{\sigma_2^2}\right)\right]^2.\end{aligned}$$

For any $d(A_1,\ A_2)\in\mathscr{D}$ which satisfies $A_1X=0$ and $A_2X=0$, where A_1 is a non-zero-matrix, we write $A_1^*=a_1(I-XX^+)$, where $a_1=\frac{1}{n-r(X)}\operatorname{tr}[A_1^{\frac{1}{2}}(I-XX^+)A_1^{\frac{1}{2}}]$. Obviously, $A_1^*X=0$. It follows from the fact that a trace of idempotent matrix is equal to its rank and $(XX^+)^2=XX^+$, that

$$a_1[n-r(X)]=\operatorname{tr}[A_1^{\frac{1}{2}}(I-XX^+)A_1^{\frac{1}{2}}]=\operatorname{tr}[A_1(I-XX^+)]=\operatorname{tr}(A_1),$$

and

$$\begin{aligned}\operatorname{tr}(A_1^*)&=\operatorname{tr}[a_1(I-XX^+)]=a_1[n-\operatorname{tr}(XX^+)]=a_1[n-r(XX^+)]\\&=a_1[n-r(X)]=\operatorname{tr}(A_1).\end{aligned}$$

Obviously, $\operatorname{tr}[(A_1^*)^2]=a_1^2[n-r(X)]$. Therefore

$$\begin{aligned}&R(d(A_1,\ A_2),\ \beta,\ \sigma_1^2,\ \sigma_2^2)-R(d(A_1^*,\ A_2),\ \beta,\ \sigma_1^2,\ \sigma_2^2)\\&=2\left(1+\frac{\sigma_2^2}{\sigma_1^2}\right)^2\operatorname{tr}(A_1^2)-2\left(1+\frac{\sigma_2^2}{\sigma_1^2}\right)^2\frac{1}{n-r(X)}\{\operatorname{tr}[A_1^{\frac{1}{2}}(I-XX^+)A_1^{\frac{1}{2}}]\}^2.\end{aligned}$$

Let $r[A_1^{\frac{1}{2}}(I-XX^+)A_1^{\frac{1}{2}}]=q$. In view of $q\leqslant n-r(X)$ and Lemma 2.1,

$$\begin{aligned}&R(d(A_1,\ A_2),\ \beta,\ \sigma_1^2,\ \sigma_2^2)-R(d(A_1^*,\ A_2),\ \beta,\ \sigma_1^2,\ \sigma_2^2)\\&\geqslant2\left(1+\frac{\sigma_2^2}{\sigma_1^2}\right)^2\operatorname{tr}[A_1^{\frac{1}{2}}(I-XX^+)A_1^{\frac{1}{2}}]^2-2\left(1+\frac{\sigma_2^2}{\sigma_1^2}\right)^2\\&\quad\cdot\frac{1}{q}\{\operatorname{tr}[A_1^{\frac{1}{2}}(I-XX^+)A_1^{\frac{1}{2}}]\}^2\\&\geqslant0\end{aligned}\tag{2.3}$$

for all β, σ_1^2 and σ_2^2. A necessary and sufficient condition under which the equality sign in formula (2.3) holds is that $q=n-r(X)$ and $A_1^{\frac{1}{2}}(I-XX^+)A_1^{\frac{1}{2}}$ has equal non-zero-eigenvalues

$$\lambda_1=\lambda_2=\cdots=\lambda_q. \tag{2.4}$$

Therefore, if $d(A_1, A_2)$ is an equivariant $\mathscr{D}$-admissible estimator for (σ_1^2, σ_2^2), then the equality sign in (2.3) holds and consequently fomula (2.4) is also ture. Set $\lambda_1=\lambda_2=\cdots=\lambda_q=a_1>0$. Because $(I-XX^+)A_1(I-XX^+)$ and $A_1^{\frac{1}{2}}(I-XX^+)A_1^{\frac{1}{2}}$ have the same eigenvalues, the eigenvalues of $\frac{1}{a_1}(I-XX^+)A_1(I-XX^+)$ are equal to 1 (altogether $q=n-r(X)$ in number) and 0, and it is a symmetric idempotent matrix. It shows that $\frac{1}{a_1}(I-XX^+)A_1(I-XX^+)$ is an orthogonal projection operator along $\mu(XX^+)$ to $\mu(I-XX^+)$ (Symbol $\mu(M)$ denotes a linear space spanned by the column vectors of matrix M). It is well-known that $I-XX^+$ is an orthogonal projection operator along $\mu(XX^+)$ to $\mu(I-XX^+)$. On account of the uniqueness of orthogonal projection operator, we obtain

$$\frac{1}{a_1}(I-XX^+)A_1(I-XX^+)=I-XX^+.$$

Therefore

$$\begin{aligned}A_1&=(I-XX^++XX^+)A_1(I-XX^++XX^+)\\&=(I-XX^+)A_1(I-XX^+)+XX^+A_1XX^+\\&\quad+(I-XX^+)A_1XX^++XX^+A_1(I-XX^+)\\&=(I-XX^+)A_1(I-XX^+)=a_1(I-XX^+).\end{aligned}$$

When A_1 is a zero-matrix, the conclusion is clear.

Similarly, it can be shown that A_2 has the form of $a_2(I-XX^+)$, where $a_2\geq 0$.

Theorem 2.1. *Under hypothesis H, if $r(X)<n$, $V_1=V_2=I_n$, then a necessary and sufficient condition for $d(A_1, A_2)$ to be an equivariant $\mathscr{D}$-admissible estimator for (σ_1^2, σ_2^2) is that $d(A_1, A_2)$ has the form of $(Y'[a_1(I-XX^+)]Y, Y'[a_2(I-XX^+)]Y)$, where $a_1\geq 0$, $a_2\geq 0$ and $a_1+a_2\leq\frac{1}{n-r(X)+2}$.*

Proof (1) Necessity:

According to Lemma 2.3, it is known that if $d(A_1, A_2)$ is an equivariant $\mathscr{D}$-admissible estimator for (σ_1^2, σ_2^2), then $A_i=a_i(I-XX^+)$ with $a_i\geq 0$, $i=1, 2$. For the sake of simplicity we set $d(a_1, a_2)=(Y'[a_1(I-XX^+)]Y, Y'[a_2(I-XX^+)]Y)$ and denote its risk function by $R(d(a_1, a_2))$. Through calculation we obtain

$$\begin{aligned}R(d(a_1, a_2))&=[(1+k)a_1(n-r(X))-1]^2+2(1+k)^2a_1^2(n-r(X))\\&\quad+\left[\left(1+\frac{1}{k}\right)a_2(n-r(X))-1\right]^2+2\left(1+\frac{1}{k}\right)^2a_2^2(n-r(X)),\end{aligned}$$

where $k=\sigma_2^2/\sigma_1^2$, $0<k<+\infty$.

If there is an other estimator $d(b_1, b_2)$, then

$$\begin{aligned}&R(d(a_1, a_2))-R(d(b_1, b_2))\\&=(1+k)(a_1-b_1)(n-r(X))[(1+k)(a_1+b_1)(n-r(X)+2)-2]\\&\quad+\left(1+\frac{1}{k}\right)(a_2-b_2)(n-r(X))\left[\left(1+\frac{1}{k}\right)(a_2+b_2)(n-r(X)+2)-2\right].\end{aligned}\tag{2.5}$$

First, we prove that if $d(a_1, a_2)$ is an equivariant $\mathscr{D}$-admissible estimator, then it is true that

$$0\leqslant a_1\leqslant\frac{1}{n-r(X)+2} \quad\text{and}\quad 0\leqslant a_2\leqslant\frac{1}{n-r(X)+2}.$$

Were this conclusion not right, we should have (1) $a_1>\frac{1}{n-r(X)+2}$ and/or (2) $a_2>\frac{1}{n-r(X)+2}$. We shall only prove the case $a_1>\frac{1}{n-r(X)+2}$. Choose b_1 and b_2 such that $a_1>b_1>\frac{1}{n-r(X)+2}$, $b_2=a_2$. By virtue of (2.5) we obtain

$$\begin{aligned}&R(d(a_1, a_2))-R(d(b_1, b_2))\\&\quad=(1+k)(a_1-b_1)(n-r(X))[(1+k)(a_1+b_1)(n-r(X)+2)-2]>0\end{aligned}$$

for all $k>0$, that is, $d(b_1, b_2)$ is better than $d(a_1, a_2)$. This conclusion contradicts the fact that $d(a_1, a_2)$ is an equivariant $\mathscr{D}$-admissible estimator.

Secondly we prove that when $0<a_1\leqslant\frac{1}{n-r(X)+2}$, $0<a_2\leqslant\frac{1}{n-r(X)+2}$ but $a_1+a_2>\frac{1}{n-r(X)+2}$, $d(a_1, a_2)$ is not an equivariant $\mathscr{D}$-admissible estimator.

Write $n-r(X)+2=C$. Make $d(b_1, b_2)$, where $b_1=a_1-\delta$, $0<\delta<a_1$, $b_2=a_2-\delta x$, $0<x\delta<a_2$. (2.6)

Then (see (2.5))

$$\begin{aligned}&R(d(a_1, a_2))-R(d(b_1, b_2))\\&\quad=(1+k)\delta(n-r(X))[(1+k)(2a_1-\delta)c-2]\\&\qquad+\frac{1}{k^2}(1+k)x\delta(n-r(X))[(1+k)(2a_2-x\delta)c-2k]\end{aligned}$$

or

$$\begin{aligned}&\frac{k^2}{(1+k)\delta(n-r(X))}[R(d(a_1, a_2))-R(d(b_1, b_2))]\\&\quad=k^2[(1+k)(2a_1-\delta)c-2]+x[(1+k)(2a_2-x\delta)c-2k]\\&\quad=k^3(2a_1-\delta)c+[(2a_1-\delta)c-2]k^2+[(2a_2-x\delta)c-2]xk+x(2a_2-x\delta)c.\end{aligned} \tag{2.7}$$

In order to show that there exist δ and x, which satisfy (2.6) and make (2.7) positive for all $k>0$, we consider the following cubic equation

$$k^3+\frac{[(2a_1-\delta)c-2]}{(2a_1-\delta)c}k^2+\frac{[(2a_2-x\delta)c-2]x}{(2a_1-\delta)c}k+\frac{x(2a_2-x\delta)}{2a_1-\delta}=0. \tag{2.8}$$

Write

$$a(x, \delta)=\frac{[(2a_1-\delta)c-2]}{(2a_1-\delta)c},\quad b(x, \delta)=\frac{[(2a_2-x\delta)c-2]x}{(2a_1-\delta)c},$$

$$e(x, \delta)=\frac{(2a_2-x\delta)x}{2a_1-\delta}.$$

Make transformation $k=l-\frac{1}{3}a(x, \delta)$. Then (2.8) becomes

$$l^3+p(x, \delta)l+q(x, \delta)=0, \tag{2.9}$$

where $p(x, \delta)=-\frac{1}{3}a^2(x, \delta)+b(x, \delta)$, $q(x, \delta)=\frac{2}{27}a^3(x, \delta)-\frac{1}{3}a(x, \delta)b(x, \delta)$

$+e(x, \delta)$. By calculation we obtain

$$\frac{q^2(x,0)}{4}+\frac{p^3(x,0)}{27}$$
$$=\frac{x}{108(a_1c)^4}\{4a_1c(a_2c-1)^3x^2+[-(a_1c-1)^2(a_2c-1)^2+27a_1^2a_2^2c^4$$
$$-18a_1a_2(a_1c-1)(a_2c-1)c^2]x+4(a_1c-1)^3a_2c\}$$
$$\triangleq\frac{x}{108(a_1c)^4}(Ax^2+Bx+D). \tag{2.10}$$

On the basis of $0<a_1c<1$, $0<a_2c<1$ and $(a_1+a_2)c>1$, we obtain $A<0$, $\Delta=B^2-4AD>0$ and $B>\sqrt{\Delta}$. Therefore $Ax^2+Bx+D=0$ has two unequal positive roots $\frac{-B\pm\sqrt{\Delta}}{2A}$. For any $x^*\in\left(\frac{-B+\sqrt{\Delta}}{2A}, \frac{-B-\sqrt{\Delta}}{2A}\right)$, there is always

$$\frac{q^2(x^*,0)}{4}+\frac{p^3(x^*,0)}{27}>0. \tag{2.11}$$

It is seen easily that when $a_1c=1$ or $a_2c=1$, also there exists $x^*>0$ which makes formula (2.11) true. Because $\frac{q^2(x^*,\delta)}{4}+\frac{p^3(x^*,\delta)}{27}$ is a continuous function of δ with $0\leqslant\delta<a_1$ and $0\leqslant x^*\delta<a_2$, there exists $\delta^*>0$ (with $\delta^*<a_1$ and $\delta^*x^*<a_2$) which satisfies

$$\frac{q^2(x^*,\delta^*)}{4}+\frac{x^3(x^*,\delta^*)}{27}>0.$$

According to the property of cubic equation, if $p^*=p(x^*, \delta^*)$ and $q^*=q(x^*, \delta^*)$, then equation $l^3+p^*l+q^*=0$ has only one real root. That is, cubic equation

$$k^3+a^*k^2+b^*k+e=0 \tag{2.12}$$

also has only one real root, where $a^*=a(x^*, \delta^*)$, $b^*=b(x^*, \delta^*)$, $e^*=e(x^*, \delta^*)$. But on account of $e^*=\frac{(2a_2-x^*\delta^*)x^*}{2a_1-\delta^*}>0$, equation (2.12) must have a negative root, that is $k^3+a^*k^2+b^*k+e^*>0$ for all $k>0$. Write $b_1^*=a_1-\delta^*$, $b_2^*=a_2-\delta^*x^*$. Then

$$R(d(a_1, a_2))-R(d(b_1^*, b_2^*))$$
$$=\frac{(1+k)\delta^*(n-r(X))(2a_1-\delta^*)c}{k^2}(k^3+a^*k^2+b^*k+e^*)>0,$$

for all $k>0$, that is, $d(b_1^*, b_2^*)$ is better than $d(a_1, a_2)$. Consequently, when $0<a_1\leqslant\frac{1}{c}$, $0<a_2\leqslant\frac{1}{c}$ and $a_1+a_2>\frac{1}{c}$, $d(a_1, a_2)$ is not an equivariant $\mathscr{D}$-admissible estimator.

(2) Sufficiency (by the Bayes method):

Since we are discussing the problem of admissibility, the same conclusion will be reached, whether under risk $R(d(a_1, a_2))$ or under risk $k^2\cdot R(d(a_1, a_2))$. Write $\tilde{R}(d(a_1, a_2))=k^2R(d(a_1, a_2))$. Then

$$\tilde{R}(d(a_1, a_2))=k^2[(1+k)a_1(n-r(X))-1]^2+2k^2(1+k)^2a_1^2(n-r(X))$$
$$+k^2\left[\left(1+\frac{1}{k}\right)a_2(n-r(X))-1\right]^2+2k^2\left(1+\frac{1}{k}\right)^2a_2^2(n-r(X))$$

$$=(k^2+3k^3+k^4)(n-r(X))(n-r(X)+2)a_1^2-2(k^2+k^3)(n-r(X))a_1$$
$$+2k^2+(1+2k+n^2)(n-r(X))(n-r(X)+2)a_2^2$$
$$-2(k+k^2)(n-r(X))a_2.$$

Suppose that $\eta(k)$ is a certain prior distribution of k with $M_i=\int_0^\infty k^i d\eta(k)$ $(i=1, 2, 3, 4)$. Then the Bayes risk of $d(a_1, a_2)$ with respect to $\eta(k)$ is

$$\widetilde{R}_\eta(d(a_1, a_2))=\int_0^\infty \widetilde{R}(d(a_1, a_2))d\eta(k)$$
$$=(M_2+2M_3+M_4)(n-r(X))ca_1^2-2(M_2+M_3)(n-r(X))a_1+2M_2$$
$$+(1+2M_1+M_2)(n-r(X))ca_2^2-2(M_1+M_2)(n-r(X))a_2. \tag{2.13}$$

Formula (2.13) has the absolute minimum point at

$$\begin{cases} a_1=\dfrac{M_2+M_3}{(M_2+2M_3+M_4)c} \\ a_2=\dfrac{M_1+M_2}{(1+2M_1+M_2)c}. \end{cases}$$

Now we shall show that for a_1, a_2 under the condition $a_1>0$, $a_2>0$ and $a_1+a_2\leq\frac{1}{c}$, $d(a_1, a_2)$ is the only Bayes estimator corresponding to a certain prior distribution $\eta(k)$ of k.

Case I: $a_1>0$, $a_2>0$ and $a_1+a_2=\frac{1}{c}$.

Set $M_1=\frac{a_2c}{1-a_2c}$. Clearly, $M_1>0$. Suppose that $\eta(k)$ possesses a one-point distribution which satisfies $P\{k=M_1\}=1$. Then $M_1=E(k)$, $M_2=E(k^2)=M_1^2$, $M_3=E(k^3)=M_1^3$ and $M_4=E(k^4)=M_1^4$. It is easy to show directly that $d(a_1, a_2)$ is the only Bayes estimator for (σ_1^2, σ_2^2) with respect to $\eta(k)$.

Case II: $a_1>0$, $a_2>0$ and $a_1+a_2<\frac{1}{c}$.

We give proof for the case $1-2a_1c>0$.

From $a_2=\frac{M_1+M_2}{(1+2M_1+M_2)c}$ we obtain

$$M_2=\frac{a_2c+(2a_2c-1)M_1}{1-a_2c}. \tag{2.14}$$

In order to find a range of M_1, which satisfies $M_2>M_1^2$, we consider equation

$$(1-a_2c)M_1^2-(2a_2c-1)M_1-a_2c=0.$$

Its two roots are -1 and $\frac{a_2c}{1-a_2c}(>0)$. Therefore when $0<M_1<\frac{a_2c}{1-a_2c}$, it i true that

$$(1-a_2c)M_1^2-(2a_2c-1)M_1-a_2c<0,$$

namely

$$M_2=\frac{a_2c+(2a_2c-1)M_1}{1-a_2c}>M_1^2. \tag{2.15}$$

After M_1 and M_2 are found, make

$$M_3=lM_2^{3/2}. \tag{2.16}$$

($l>1$, is to be determined). From $a_1=\dfrac{M_2+M_3}{(M_2+2M_3+M_4)c}$, we obtain

$$M_4=\frac{(1-a_1c)M_2+(1-2a_1c)M_3}{a_1c}=\frac{(1-a_1c)M_2+(1-2a_1c)lM_2^{3/2}}{a_1c}. \tag{2.17}$$

We shall select suitable l and M_1 such that M_1, M_2, M_3 and M_4 not only satisfy (2.14), (2.15), (2.16) and (2.17), but also satisfy

$$M_4(M_2-M_1^2)+M_2(M_1M_3-M_2^2)-M_3(M_3-M_1M_2)>0$$

and

$$lM_1-M_2^{\frac{1}{2}}>0. \tag{2.18}$$

Write $\overline{M}_4=\dfrac{(1-a_1c)M_2+(1-2a_1c)M_2^{3/2}}{a_1c}$. From $1-2a_1c>0$ we know $\overline{M}_4<M_4$. Substitute (2.16) into the first formula of (2.18), we get

$$\begin{aligned}&M_4(M_2-M_1^2)+M_2(M_1M_3-M_2^2)-M_3(M_3-M_1M_2)\\&=M_4(M_2-M_1^2)+2M_1M_2\cdot lM_2^{3/2}-M_2^3-l^2M_2^3\\&=-l^2M_2^3+2M_1M_2^{5/2}\cdot l+M_4(M_2-M_1^2)-M_2^3\\&>-l^2M_2^3+2M_1M_2^{5/2}l+\overline{M}_4(M_2-M_1^2)-M_2^3.\end{aligned} \tag{2.19}$$

The two zeros of expression (2.19) taken as a function of l are

$$l_1=\frac{M_1M_2-\sqrt{(M_2-M_1^2)(\overline{M}_4-M_2^2)}}{M_2^{3/2}}$$

and

$$l_2=\frac{M_1M_2+\sqrt{(M_2-M_1^2)(\overline{M}_4-M_2^2)}}{M_2^{3/2}}.$$

If $(M_2-M_1^2)(\overline{M}_4-M_2^2)>0$, then expression (2.19)$>0$, when $l_1<l<l_2$. Write

$$\widetilde{M}_1=\frac{a_2c}{1-a_2c},\quad \widetilde{M}_2=\frac{a_2c+(2a_2c-1)\widetilde{M}_1}{1-a_2c}=\left(\frac{a_2c}{1-a_2c}\right)^2=\widetilde{M}_1^2,\quad \widetilde{M}_3=\widetilde{M}_2^{3/2},$$

and

$$\widetilde{M}_4=\frac{\widetilde{M}_2(1-a_1c)+\widetilde{M}_3(1-2a_1c)}{a_1c}=\frac{(a_2c)^2(1-a_1c)(1-a_2c)+(a_2c)^3(1-2a_1c)}{a_1c(1-a_2c)^3}.$$

It is seen easily that $\widetilde{M}_4>(\widetilde{M}_1)^4$.

The value of $M_1^2\overline{M}_4-M_2^3$ as a function of M_1 at the point $\widetilde{M}_1=\dfrac{a_2c}{1-a_2c}$ is equal to

$$\widetilde{M}_1^2\widetilde{M}_4-\widetilde{M}_1^6=\widetilde{M}_1^2(\widetilde{M}_4-\widetilde{M}_1^4)>0.$$

Therefore there exists $M_{10}\in(0,\widetilde{M}_1)$ which satisfies $M_{10}^2\overline{M}_{40}-M_{20}^3>0$, where $\overline{M}_{40}$ and M_{20} are values of $\overline{M}_4$ and M_2 respectively at the point M_{10}. Thus we obtain

$$M_{20}^2(M_{20}-M_{10}^2)^2<M_{10}^2(\overline{M}_{40}-M_{20}^2)(M_{20}-M_{10}^2).$$

Consequently $\dfrac{M_{20}^{1/2}}{M_{10}}<\dfrac{M_{10}M_{20}+\sqrt{(\overline{M}_{40}-M_{20}^2)(M_{20}-M_{10}^2)}}{M_{20}^{3/2}}$. Take l_0 such that it satisfies

$$1<\operatorname{Max}\left\{\frac{M_{20}^{1/2}}{M_{10}},\ \frac{M_{10}M_{20}-\sqrt{(M_{40}-M_{20}^2)(M_{20}-M_{10}^2)}}{M_{20}^{3/2}}\right\}<l_0$$

$$<\frac{M_{10}M_{20}+\sqrt{(M_{40}-M_{20}^2)(M_{20}-M_{10}^2)}}{M_{20}^{3/2}}.$$

Such an l_0 validates not only (2.19)>0 but also the second formula of (2.18), hence both formulae of (2.18) hold.

Thus we obtain M_{10}, $M_{20}=\frac{a_2c+(2a_2c-1)M_{10}}{1-a_2c}$, $M_{30}=l_0M_{20}^{3/2}$ and

$$M_{40}=\frac{(1-a_1c)M_{20}+(1-2a_1c)l_0M_{20}^{3/2}}{a_1c}.$$

They possess the following property

$$0<M_{10}<\frac{a_2c}{1-a_2c},\quad M_{20}>M_{10}^2,$$

$$M_{10}M_{30}-M_{20}^2=M_{10}\cdot l_0\cdot M_{20}^{3/2}-M_{20}^2=M_{20}^{3/2}(l_0M_{10}-M_{20}^{1/2})>0$$

and

$$M_{40}(M_{20}-M_{10}^2)+2M_{10}M_{20}M_{30}-M_{20}^3+M_{30}^2$$
$$>M_{40}(M_{20}-M_{10}^2)+2M_{10}M_{20}M_{30}-M_{20}^3-M_{30}^2>0.$$

Consequently, all the five principal minor determinants of

$$\begin{pmatrix} 1 & 0 & M_{10} & 0 & M_{20} \\ 0 & M_{10} & 0 & M_{20} & 0 \\ M_{10} & 0 & M_{20} & 0 & M_{30} \\ 0 & M_{20} & 0 & M_{30} & 0 \\ M_{20} & 0 & M_{30} & 0 & M_{40} \end{pmatrix}$$

are positive. Hence there exists a random variable Z which satisfies $E(Z^{2s-1})=0$, $E(Z^{2s})=M_{s0}$, $s=1, 2, 3, 4$ (see Theorem 2.1.1 of [7] on page 43). By setting $k=Z^2$, the distribution of k will be denoted by $\eta(k)$. Then $E(k^s)=M_{s0}$, $s=1, 2, 3, 4$. Clearly, $k>0$. Thus for given a_1 and a_2, $d(a_1, a_2)$ is the only Bayes estimator corresponding to the prior distribution $\eta(k)$ of k.

When $1-2a_1c\leqslant 0$, because of $0<a_1c+a_2c<1$, it is necessary that $1-2a_2c>0$. From the symmetry of a_1 and a_2, $d(a_1, a_2)$ will surely be the only Bayes estimator for a certain prior distribution of $\frac{1}{k}$.

Combining the two cases, we see that when $a_1>0$, $a_2>0$ and $a_1+a_2\leqslant\frac{1}{c}$, $d(a_1, a_2)$ surely is the only Bayes estimator for a certain prior distribution $\left(\text{of } k \text{ or } \frac{1}{k}\right)$, and it is the equivariant $\mathscr{D}$-admissible estimator for (σ_1^2, σ_2^2).

Now we show that when $0\leqslant a_1\leqslant\frac{1}{c}$, $0\leqslant a_2\leqslant\frac{1}{c}$ and at least one of a_1 and a_2 is zero, $d(a_1, a_2)$ also is an equivariant $\mathscr{D}$-admissible estimator.

Without loss of generality we shall show only the case of $a_1=0$.

$$R(d(0, a_2))-R(d(b_1, b_2))$$
$$=(1+k)(n-r(X))(-b_1)[(1+k)b_1c-2]$$
$$+\frac{(1+k)}{k}(a_2-b_2)(n-r(X))\left[\left(1+\frac{1}{k}\right)(a_2+b_2)c-2\right].$$

If $b_1\neq 0$. then the coefficient $-b_1^2(n-r(X))c$ of term k^2 is negative. When k becomes large enough

$$R(d(0, a_2))-R(d(b_1, b_2))<0.$$

If $b_1=0$, then

$$R(d(0, a_2))-R(d(0, b_2))$$
$$=\left(1+\frac{1}{k}\right)(a_2-b_2)(n-r(X))\left[\left(1+\frac{1}{k}\right)(a_2+b_2)c-2\right].$$

It is seen easily that in the two cases $b_2>a_2$ and $b_2<a_2$,

$$R(d(0, a_2))-R(d(0, b_2))<0$$

holds, provided k becomes small and large enough respectively.

The above argument shows that when $a_1\geqslant 0$, $a_2\geqslant 0$ and $a_1+a_2\leqslant\frac{1}{c}$, $d(a_1, a_2)$ is an equivariant $\mathscr{D}$-admissible estimator.

Corollary 2.1. *Under hypothesis H, if $r(X)<n$ and $V_1=V_2\triangleq V_0>0$, then a necessary and sufficient condition for $d(A_1, A_2)$ to be equivariant $\mathscr{D}$-admissible estimator for (σ_1^2, σ_2^2) is that*

$$A_i=a_iV_0^{-\frac{1}{2}}[I-(V_0^{-\frac{1}{2}}X)(V_0^{-\frac{1}{2}}X)^+]V_0^{-\frac{1}{2}},\quad i=1, 2,$$

where $a_1\geqslant 0$, $a_2\geqslant 0$ and $a_1+a_2\leqslant\dfrac{1}{n-r(X)+2}$.

Theorem 2.2. *Under hypothesis H (V_1 may be unequal to V_2) and $r(X)<n$, if*

$$a_1=\frac{C_1M_2+M_3\mathrm{tr}(D_1)}{M_2C_1^2+2M_3C_1\mathrm{tr}(D_1)+M_4[\mathrm{tr}(D_1)]^2+2M_2C_1+4M_3\mathrm{tr}(D_1)+2M_4\mathrm{tr}(D_2^1)},$$
$$a_2=\frac{M_1\mathrm{tr}(D_2)+C_1M_2}{[\mathrm{tr}(D_2)]^2+M_2C_1^2+2M_1C_1\mathrm{tr}(D_2)+2\mathrm{tr}(D_2^2)+4M_1\mathrm{tr}(D_2)+2M_2C_1},$$

where

$$C_1=n-r(X),\ D_1=V_2^{\frac{1}{2}}V_1^{-\frac{1}{2}}[I-(V_1^{-\frac{1}{2}}X)(V_1^{-\frac{1}{2}}X)^+]V_1^{-\frac{1}{2}}V_2^{\frac{1}{2}},$$

$D_2=V_1^{\frac{1}{2}}V_2^{-\frac{1}{2}}[I-(V_2^{-\frac{1}{2}}X)(V_2^{-\frac{1}{2}}X)^+]V_2^{-\frac{1}{2}}V_1^{\frac{1}{2}}$ and M_1, M_2, M_3, M_4 are any set of real numbers, which satisfies

$$M_1>0,\ M_2-M_1^2>0,\ M_1M_3-M_2^2>0,\ M_4(M_2-M_1^2)+2M_1M_2M_3-M_3^2-M_2^3>0$$

or

$$M_1>0,\ M_2=M_1^2,\ M_3=M_1^3,\ M_4=M_1^4,$$

then $(Y'\{a_1V_1^{-\frac{1}{2}}[I-(V_1^{-\frac{1}{2}}X)(V_1^{-\frac{1}{2}}X)^+]V_1^{-\frac{1}{2}}\}Y,\ Y'\{a_2V_2^{-\frac{1}{2}}[I-(V_2^{-\frac{1}{2}}X)(V_2^{-\frac{1}{2}}X)^+]V_2^{-\frac{1}{2}}\}Y)$ is an equivariant $\mathscr{D}$-admissible estimator for (σ_1^2, σ_2^2).

By a method similar to that used in the proof of the sufficiency **of** Theorem 2.1, the above result is obtained easily.

Theorem 2.3. *Under hypothesis H (V_1 may be unequal to V_2), if $r(X)<n$, then a necessary condition for $d(A_1, A_2)$ to be an equivariant $\mathscr{D}$-admissible estimator for (σ_1^2, σ_2^2) is that*

1) $A_1X=0$, $A_2X=0$,

2) $2\lambda_1(A_1V_1)+\text{tr}(A_1V_1)\leqslant 1$, $2\lambda_1(A_2V_2)+\text{tr}(A_2V_2)\leqslant 1$

(Expression $\lambda_1(M)$ denotes the maximum eigenvalue of square matrix M).

Proof

1) The proof of $A_1X=0$ and $A_2X=0$ is similar to that of Lemma 2.3.

2) Because of the symmetry of A_1 and A_2, it suffices to show that $2\lambda_1(A_1V_1)+\text{tr}(A_1V_1)\leqslant 1$.

Let $2\lambda_1(A_1V_1)+\text{tr}(A_1V_1)>1$. Select an orthogonal matrix P such that $V_1^{\frac{1}{2}}A_1V_1^{\frac{1}{2}}=P\Lambda P'$, where $\Lambda=\text{diag}(\lambda_{11}, \cdots, \lambda_{1t}, 0, \cdots, 0)$, $\lambda_{11}=\lambda_1(A_1V_1)$ and $\lambda_{11}\geqslant\lambda_{12}\geqslant\cdots\geqslant\lambda_{1t}>0$ are non-zero-eigenvalues of $V_1^{\frac{1}{2}}A_1V_1^{\frac{1}{2}}$ (or A_1V_1) (Symbol diag $(C_1, C_2, \cdots, C_n)$ denotes a square matrix of order n, which possesses elements $C_1, C_2, \cdots, C_n$ along the principal diagonal and zero elsewhere).

Select α so that it satisfies $0<\alpha<\lambda_1(A_1V_1)$ and $3\alpha+\sum_{j=2}^{t}\lambda_{1j}>1$. By setting $D=\text{diag}(\alpha_1, \lambda_{12}, \cdots, \lambda_{1t}, 0, \cdots, 0)$, $B_1=V_1^{-\frac{1}{2}}PDP'V_1^{-\frac{1}{2}}$ and $B_2=A_2$, obviously, $\Lambda\geqslant D$. Now we show that $d(B_1, B_2)$ is better than $d(A_1, A_2)$.

$$\begin{aligned}
&R(d(A_1, A_2), \beta, \sigma_1^2, \sigma_2^2)-R(d(B_1, B_2), \beta, \sigma_1^2, \sigma_2^2)\\
&=2\text{tr}[A_1(V_1+kV_2)]^2-2\text{tr}[B_1(V_1+kV_2)]^2+\{\text{tr}[A_1(V_1+kV_2)]-1\}^2\\
&\quad-\{\text{tr}[B_1(V_1+kV_2)]-1\}^2\\
&=2k^2\cdot[\text{tr}(A_1V_2)^2-\text{tr}(B_1V_2)^2]+4k[\text{tr}(A_1V_2A_1V_1)-\text{tr}(B_1V_2B_1V_1)]\\
&\quad+2[\text{tr}(A_1V_1)^2-\text{tr}(B_1V_1)^2]\\
&\quad+\{\text{tr}(A_1V_1)+\text{tr}(B_1V_1)+k[\text{tr}(A_1V_2)+\text{tr}(B_1V_2)]-2\}\\
&\quad\cdot\{\text{tr}(A_1V_1)-\text{tr}(B_1V_1)+k[\text{tr}(A_1V_2)-\text{tr}(B_1V_2)]\}. \qquad (2.21)
\end{aligned}$$

From $\Lambda\geqslant D$, it follows that $A_1\geqslant B_1$ and $V_2^{\frac{1}{2}}A_1V_2^{\frac{1}{2}}\geqslant V_2^{\frac{1}{2}}B_1V_2^{\frac{1}{2}}$. According to Lemma 2.2 we obtain

$$\text{tr}(A_1V_2)=\text{tr}(V_2^{\frac{1}{2}}A_1V_2^{\frac{1}{2}})\geqslant\text{tr}(V_2^{\frac{1}{2}}B_1V_2^{\frac{1}{2}})=\text{tr}(B_1V_2) \qquad (2.22)$$

and

$$\text{tr}(A_1V_2)^2\geqslant\text{tr}(B_1V_2)^2. \qquad (2.23)$$

Also from

$$\begin{aligned}
\text{tr}(A_1V_2)-\text{tr}(B_1V_2)&=\text{tr}(V_1^{-\frac{1}{2}}P\Lambda P'V_1^{-\frac{1}{2}}V_2)-\text{tr}(V_1^{-\frac{1}{2}}PDP'V_1^{-\frac{1}{2}}V_2)\\
&=\text{tr}[V_1^{-\frac{1}{2}}P(\Lambda-D)P'V_1^{-\frac{1}{2}}V_2]\\
&=(\lambda_{11}-\alpha)\,\text{tr}[V_1^{-\frac{1}{2}}Pe\cdot e'P'V_1^{-\frac{1}{2}}V_2]
\end{aligned}$$

(where $e'=(1, 0, \cdots, 0)$), we have

$$\begin{aligned}
&\operatorname{tr}(A_1V_2A_1V_1)-\operatorname{tr}(B_1V_2B_1V_1)\\
&=\operatorname{tr}(V_1^{-\frac{1}{2}}P\Lambda P'V_1^{-\frac{1}{2}}V_2V_1^{-\frac{1}{2}}P\Lambda P'V_1^{-\frac{1}{2}}V_1)-\operatorname{tr}(V_1^{-\frac{1}{2}}PDP'V_1^{-\frac{1}{2}}V_2V_1^{-\frac{1}{2}}PDP'V_1^{-\frac{1}{2}}V_1)\\
&=\operatorname{tr}(\Lambda P'V_1^{-\frac{1}{2}}V_2V_1^{-\frac{1}{2}}P\Lambda)-\operatorname{tr}(DP'V_1^{-\frac{1}{2}}V_2V_1^{-\frac{1}{2}}PD)\\
&=\operatorname{tr}(V_1^{-\frac{1}{2}}P\Lambda^2P'V_1^{-\frac{1}{2}}V_2)-\operatorname{tr}(V_1^{-\frac{1}{2}}PD^2P'V_1^{-\frac{1}{2}}V_2)\\
&=(\lambda_{11}^2-\alpha^2)\operatorname{tr}(V_1^{-\frac{1}{2}}Pee'P'V_1^{-\frac{1}{2}}V_2)\\
&=(\lambda_{11}+\alpha)[\operatorname{tr}(A_1V_2)-\operatorname{tr}(B_1V_2)]. \qquad (2.24)
\end{aligned}$$

We note that $\operatorname{tr}(A_1V_1)=\sum_{j=1}^{t}\lambda_{1j}$, $\operatorname{tr}[(A_1V_1)^2]=\operatorname{tr}(\Lambda^2)=\sum_{j=1}^{t}\lambda_{1j}^2$, $\operatorname{tr}(B_1V_1)=\alpha+\sum_{j=2}^{t}\lambda_{1j}$, $\operatorname{tr}(B_1V_1)^2=\alpha^2+\sum_{j=2}^{t}\lambda_{1j}^2$. According to (2.21), (2.22), (2.23) and (2.24) we obtain

$$\begin{aligned}
&R(d(A_1, A_2), \beta, \sigma_1^2, \sigma_2^2)-R(d(B_1, B_2), \beta, \sigma_1^2, \sigma_2^2)\\
&\quad=2k^2[\operatorname{tr}(A_1V_2)^2-\operatorname{tr}(B_1V_2)^2]+4k(\lambda_{11}+\alpha)[\operatorname{tr}(A_1V_2)-\operatorname{tr}(B_1V_2)]\\
&\qquad+2(\lambda_{11}^2-\alpha^2)+\{\operatorname{tr}(A_1V_1)+\operatorname{tr}(B_1V_1)+k[\operatorname{tr}(A_1V_2)+\operatorname{tr}(B_1V_2)]-2\}\\
&\qquad\cdot\{\lambda_{11}-\alpha+k[\operatorname{tr}(A_1V_2)-\operatorname{tr}(B_1V_2)]\}\\
&\quad\geqslant 4k(\lambda_{11}+\alpha)[\operatorname{tr}(A_1V_2)-\operatorname{tr}(B_1V_2)]+2(\lambda_{11}-\alpha)(\lambda_{11}+\alpha)\\
&\qquad+\left\{\sum_{j=1}^{t}\lambda_{1j}+\sum_{j=2}^{t}\lambda_{1j}+\alpha+k[\operatorname{tr}(A_1V_2)+\operatorname{tr}(B_1V_2)]-2\right\}\\
&\qquad\cdot\{\lambda_{11}-\alpha+k[\operatorname{tr}(A_1V_2)-\operatorname{tr}(B_1V_2)]\}\\
&\quad\geqslant\left\{\sum_{j=1}^{t}\lambda_{1j}+\sum_{j=2}^{t}\lambda_{1j}+\alpha+k[\operatorname{tr}(A_1V_2)+\operatorname{tr}(B_1V_2)]-2+2(\lambda_{11}+\alpha)\right\}\\
&\qquad\cdot\{\lambda_{11}-\alpha+k[\operatorname{tr}(A_1V_2)-\operatorname{tr}(B_1V_2)]\}>0
\end{aligned}$$

for all $k>0$, that is, $d(B_1, B_2)$ is better than $d(A_1, A_2)$. This contradicts the fact that $d(A_1, A_2)$ is an equivariant $\mathscr{D}$-admissible estimator

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References

[1] Cohen, A., All admissible linear estimator of the mean vector, *Ann. Math. Statist.*, **37** (1966), 458—463.

[2] Rao, C.R., Estimation of parameters in a linear model, *Ann. Statist.*, **4**(1976), 1023—1037.

[3] Wu Qiguang, Cheng Ping and Li Guoying, Admissibility of quadratic estimates of error variance in linear models, *Scientia Sinica*, **7**(1981), 815—825.

[4] Rao, C. R., Estimation of variance and covariance components-MINRUE theory, *J. of Multivariate Analysis*, 1(1971), 257—275.

[5] Olsen, A., Seely, J. and Birkes, D., Invariant quadratic unbiased estimation for two variance components, *Ann. Statist.*, **4**(1976), 878—890.

[6] LaMotte, L.R., Quadratic estimation of variance components, *Biometrics*, **29**(1973), 311—330.

[7] Ахиезер, Н. И., Классическая проьлема маментов, москва, (1961).

[8] Rao, C.R., Linear statistical inference and its applications, John Wiley and Sons, New York (1973).