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THE GENERAL INITIAL-BOUNDARY VALUE PROBLEMS FOR LINEAR HYPERBOLIC-PARABOLIC COUPLED SYSTEM

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Abstract

In this paper, the author considers the general initial boundary value problem of hyperbolic-parabolic coupled system and gives the normal form of boundary conditions, sufficient and necessary for the BVP and the conjugate BVP to be well-posed. Furthermore, the author proves the existence of differentiable solution in Sobolev spaces of high order.

§ 1. Introduction

Let Ω be a domain in R^n, with boundary $\partial\Omega$ sufficiently smooth. In the region $\Omega\times R^1_+$, we consider the following hyperbolic-parabolic coupled system:

$$\begin{cases}\partial_t u=Pu+Av+F_1,\\ \partial_t v=Bu+Qv+F_2,\end{cases}\tag{1.1}$$

where u, F_1 are p-dimensional vectors, v, F_2 are q-dimensional vectors, P, Q, A, B are matrices whose elements are differential operators. The elements of A, B, Q are 1st order operators in R^n, and the elements of P are 2nd order operators in R^n, satisfying:

$$\begin{cases}1,\ \partial_t u=Pu+F_1 \text{ is a second order Petrovsky parabolic system;}\\ 2,\ \partial_t v=Qv+F_2 \text{ is a 1st order Kress' hyperbolic system.}\end{cases}$$

Here, a hyperbolic system is called Kreiss' hyperbolic if there exists the corresponding Kreiss' symmetrilizer. Specifically, the strictly hyperbolic systems and many symmetric hyperbolic systems encountered in physical applications are Kreiss' hyperbolic (cf. [6]).

In view of the various physical phenomena described by this kind of systems, e. g., the hydrodynamics of viscous compressible fluid and the radiative hydrodynamics, more and more attention is paid recently to its research. For the case $n=1$, there are already many discussion on the general boundary value

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problems of (1.1), and quite a few good results are available. For example, the summary article [5] has given a systematic and detailed exposition in this case. But for the case $n>1$, the existing literature is rather limited, because of the failure, in general, of the method of integration along the characteristics. Most of the articles now available are concentrated on the discussion of the hydrodynamics equation for viscous compressible fluids. For this concrete system of equations, in [12], Tani proved the existence and uniqueness of local solution for the Dirichlet boundary value problem, while in [8], under certain restrictions, Matsumura and Nishida have even proved the existence and uniqueness of global smooth solution. But their discussions are confined to the Dirichlet problem, and their methods, as pointed out in [14], depend on the fact that in the special system they considered, there is exactly one hyperbolic equation. For the more general system (1.1), where the hyperbolic system is assumed to be symmetric, in the frame of Friedrichs' admissibility for symmetric positive system, Zheng Songmu[14] has proved the existence and uniqueness of a local solution for the initial-boundary value problem with $u=0$ on the boundary.

In [11], Strikwerda has for the first time considered the general initial-boundary value problems for the system (1.1), i. e., the boundary conditions for the parabolic and hyperbolic variables are all of Lopatinsky's type. But in his paper, he only proved that L^2 energy estimate for the linear problem. We want in this paper to prove the existence of H^s solution and the corresponding energy estimate for the linear Strikwerda stable boundary value problem, in order to pave the way for the discussion of quasilinear problem in [7]. It is noteworthy to point out that, the general boundary value problem here includes (for the parabolic variables) the Dirichlet condition, Neumann condition and the conditions which are the mixture of the above two conditions. In dealing with such problems, in general, one cannot treat hyperbolic and parabolic system respectively and is obliged to consider them as a whole. And this kind of mixed problem is exactly what one meets in physics when considering the isothermal shock waves, which will be discussed in another paper.

In this paper, section 2 is devoted to the analysis of the result of Strikwerda in [11] and the typical form of the stable boundary conditions. In section 3, we prove the a priori estimate of higher order, and in sections 4 and 5, the existence of differentiable solution is proved with Ω being a halfspace and a bounded domain, respectively.

§ 2. Notations and the Result of Strikwdrda

In this section, unless otherwise stated, Ω is always assumed to be the halfspace:

$$\Omega=\{(x, y); x>0\}, \text{ where } y=(y_1, y_2, \cdots, y_{n-1}).$$

Rewrite (1.1) as follows:

$$\begin{cases} u_t-P_0u_{xx}-\sum_j P_{1j}u_{xy_j}-\sum_{i,j} P_{2ij}u_{y_iy_j}-A_0v_x-\sum_j A_jv_{y_j} \\ \quad -C_0u_x-\sum_j C_{0j}u_{y_j}-C_{11}u-C_{12}v=F_1, \\ v_t-B_0u_x-\sum_j B_ju_{y_j}-Q_0v_x-\sum_j Q_jv_{y_j}-C_{21}u-C_{22}v=F_2. \end{cases} \tag{2.1}$$

Here, one can assume, without loss of generality, $Q_0=\text{diag }(Q_0^-, Q_0^+)$, $Q_0^-<0$, $Q_0^+>0$ are diagonal matrices of order q^-, q^+ respectively. It is evident that the terms C_0u_x, $C_{0j}u_y$, $C_{11}u$, $C_{12}v$, $C_{21}u$, $C_{22}v$ in (2.1) are of lower order, so we will omit them in the following discussion.

On the boundary $x=0$, we give the boundary conditions:

$$\begin{cases} T_1u_x+\sum_j T_{2j}u_{y_i}+S_1v+T_0u=g_1, \\ Tu+Sv=g_2, \end{cases} \tag{2.2}$$

where T_1, T_{2j}, S_1, T_0 are matrices with b_1 rows, T, S are matrices with b_2 rows. In (2.2), we may assume $b_1+b_2=p+q^-$ and the former b_1 equatiions to be linearly independent with respect to the terms containing derivatives. It is easily seen that those assumptions are necessary for the well-posedness of the problem.

At $t=0$, the zero initial condition is imposed:

$$u|_{t=0}=0, \; v|_{t=0}=0. \tag{2.3}$$

The general case with nonzero initial conditions can be easily changed into the zero one if a little stronger regularity is imposed on the initial values.

For $\eta>1$, we introduce the usual hyperbolic η-weighted norm

$$\|\varphi\|^2_{k,\eta}=\sum_{|k_1+k_2|\le k}\int_{\Omega\times\mathbb{R}^+_t}\eta^{2k_1}|D^{k_2}\varphi|^2e^{-2\eta t}\,dx\,dy\,dt. \tag{2.4}$$

$$|\varphi|^2_{k,\eta}=\sum_{|k_1+k_2|\le k}\int_{\partial\Omega\times\mathbb{R}^+_t}\eta^{2k_1}|D^{k_2}\varphi|^2e^{-2\eta t}\,dy\,dt. \tag{2.5}$$

The corresponding inner products with $k=0$ are denoted by $(\;,\;)_\eta$ and $\langle \;,\; \rangle_\eta$ respectively.

Denote the dual variables of (y, t) by (ω, τ), $s=i\tau+\eta$, $\sigma=(\omega^2+s)^{\frac{1}{2}}$, and $\mathscr{E}$, $\mathscr{E}^{-1}$ are the pseudo-differetial operators with symbols σ, σ^{-1}, $|S|$ is the pseudo-differential operator with symbol $|s|$ and ∇ is the gradient with respect to the space variables (x, y).

For the linear problem (2.1)—(2.3), following [11], we set the following definitions:

Definition 2.1. *The problem (2.1)—(2.3) is called stable if: there are constants η_0, $C_0>0$ such that for $\eta\ge\eta_0$, the strong solutions $w=(u, v)$ of (2.1)—(2.3) satisfy the following à priori estimate*

$$\mathrm{Re}(u, \mathscr{E}_u)_\eta + \mathrm{Re}(u_x, \mathscr{E}^{-1}u_x)_\eta + \eta\|v\|^2_{0,\eta} + |\mathscr{E}^{-1}u_x|^2_{0,\eta} + |w|^2_{0,\eta}$$
$$\leqslant C_0(|\mathscr{E}^{-1}g_1|^2_{0,\eta} + |g_2|^2_{0,\eta} + \|\mathscr{E}^{-1}F_1\|^2_{0,\eta} + \|F_2\|^2_{0,\eta}). \quad (2.6)$$

Definition 2.2. *The problem* (2.1)—(2.3) *is called strongly stable if: there are positive constants* η_1, C_1 *such that for* $\eta \geqslant \eta_1$, *the strong solutions* $w=(u, v)$ *satisfy the following a priori estimate:*

$$\mathrm{Re}\ (u, \mathscr{E}|S|u)_\eta + \mathrm{Re}\ (\nabla u, \mathscr{E}\nabla u)_\eta + \eta\|v\|^2_{0,\eta} + ||S|^{\frac{1}{2}}u|^2_{0,\eta} + |w|^2_{0,\eta}$$
$$\leqslant C_1(|g|^2_{0,\eta} + \|F\|^2_{0,\eta}). \quad (2.7)$$

Here we have made the abbreviation $g=(g_1, g_2)$, $F=(F_1, F_2)$.

Let $v=(v^-, v^+)^T$, with v^- the first q^- components of v, v^+ the last q^+ components. For the time being, assume that the problem (2.1)—(2.3) have constant coefficients. Let $P_1\cdot\omega=\sum_j P_{1j}\omega_j$, $P_2(\omega)=\sum_{i,j} P_{2ij}\omega_i\omega_j$, $B\cdot\omega=\sum_j B_j\omega_j$, $A\cdot\omega=\sum_j A_j\omega_j$. We introduce the following definitions:

Definition 2.3. (u, v_0) *will be called the parabolic eigenvector of* (2.1)—(2.3) *at* $(\omega, s)\neq 0$ Re $s\geqslant 0$, *if:*

1) $v_0^+=0$, $v_0\in C^q$, $u\in L^2(R^1_+)$;
2) $su=P_0u_{xx}+iP_1\cdot\omega u_x-P_2(\omega)u$;
3) $T_1u_x+iT_2\cdot\omega u=0$, $Tu+Sv=0$ *at* $x=0$.

Definition 2.4. $(u, v)\neq 0$ *will be called the hyperbolic eigenvector of* (2.1)—(2.3) at (ω, s), $\omega\neq 0$, Re $s\geqslant 0$, *if:*

1) $0=P_0u_{xx}+iP_1\cdot\omega u_x-P_2(\omega)u$;
2) $sv=B_0u_x+iB\cdot\omega u+Q_0v_x+iQ\cdot\omega v$;
3) $T_1u_x+iT_2\cdot\omega u=0$, $Tu+Sv_0=0$, *at* $x=0$.
4) $u\in L^2(R^1_+)$, *and* $v\in L^2(R^1_+)$ *when* Re $s>0$, *and if* Re $s=0$, *v is the limit of a series* (v_n) *with each* $v_n\in L^2(R^1_+)$, v_n *satisfying* 2) *with* Re $s_n>0$, $s_n\to s$.

As in [11], one can define the strongly parabolic eigenvector and hyperbolic eigenvector. This paper will be confined to the study of the stable problem, while for the strongly stable problem, similar theorems are valid and more easy to prove.

First let us state the main results in [11].

The case of constant coefficients:

Theorem 2.5. (Strikwerda) 1) *The constant coefficients problem* (2.1)—(2.3) *is stable (or strongly stable) if and only if:* $b_1<p$ (or $b_1\geqslant p$) *and there is no (or strongly) parabolic and hyperbolic eigenvector.*

2) *When* $b_1=p$, *the stability is equivalent to the strong stability, and is equivalent to the well-posedness of two seperate parbolic and hyperbolic problems obtained by setting* A, B, S_1, T *all equal to zero.*

Transition to the case of variable coefficients:

Theorem 2.6 (Strikwerda). *In the variable coefficients problem (2.1)—(2.3), suppose the coefficients are homogeneous of zero order with respect to* $|x|$, $|y|$, $|t|$ *for* $|x|+|y|+|t|$ *sufficiently large. If, at every point on* $\partial\Omega\times R^1_+$, *the frozen coefficient problem is stable, then, the variable coefficients problem (2.1)—(2.3) is stable.*

When Ω is a bounded domain, from the stability of frozen coefficient problem at every point on the boundary, we can deduce the following estimate

$$\sqrt{\eta}\,\|u\|^2_{0,\eta}+\eta\|v\|^2_{0,\eta}+|w|^2_{0,\eta}\leqslant C\Big(\frac{1}{\eta}\,|g_1|^2_{0,\eta}+|g_2|^2_{0,\eta}+\frac{1}{\eta}\|F_1\|^2_{0,\eta}+\|F_2\|^2_{0,\eta}\Big), \quad (2.8)$$

and for strongly stable problem, we have

$$\eta\|w\|^2_{0,\eta}+\|\nabla u\|^2_{0,\eta}+|w|^2_{0,\eta}\leqslant C(|g|^2_{0,\eta}+\|F\|^2_{0,\eta}). \quad (2.9)$$

Now, we are going to analyse the stable boundary conditions and to rewrite them into the typical form.

In (2.2), rewrite $Tu+Sv=Tu+\tilde{S}v^-+\tilde{\tilde{S}}v^+$, where $\tilde{S}$ is a $b_2\times q^-$ matrix, $\tilde{\tilde{S}}$ is a $b_2\times q^+$ matrix. From the definition of stability it follows easily that $q^-\leqslant b_2$ and the rank of $\tilde{S}$ is q^-, otherwise we would have $(0, v_0)\neq 0$ which is a parabolic eigenvector. Therefore we can solve v^- from the boundary condition and rewrite (2.2) as

$$\begin{cases} T_1u_x+\sum T_{2j}u_{y_j}+S_1v+T_0u=g_1, \\ T_3u+S_3v^+=g_{21}, \\ v^-+S_4v^++T_4u=g_{22}, \end{cases} \quad (2.10)$$

where T_3 is a $(p-b_1)\times p$ matrix.

Suppose rank $T_3<p-b_1$, then we have less than p boundary conditions from $T_1u_x+iT_2\cdot\omega u=0$ and $T_3u=0$, and the system of p second order ordinary differential equations

$$su=P_0u_{xx}+iP_1\cdot\omega u_x-P_2(\omega)u$$

must have nonzero $L^2(R^1_+)$ solution, i. e., there exists a parabolic eigenvector. Thus from the stability it follows that rank $T_1=p-b_1$.

Similarly, rank $T_1=b_1$. Else, at $\omega=0$, the system

$$su=P_0u_{xx}$$

has only less than p boundary conditions, and this implies the existence of parabolic eigenvector.

Also, T_1u, T_3u must be linearly independent. Otherwise, one may perform an invertible linear transformation of u, so that $T_1u=(u_1, \cdots, u_{b_1}, 0, \cdots, 0)$, $T_3u=(0, \cdots, 0, u_{b_1-1}, \cdots, u_{b_1-1+p-b_1}, 0, \cdots, 0)$. At $\omega=0$, take $u_1=\cdots=u_{p-1}=0$. Then it is easily seen that there is a $u_p\neq 0$, $u_p\in L^2(R^1_+)$, satisfying $su=P_0u_{xx}$, i. e., there exists a parabolic eigenvector.

From the above discussion, without loss of generality, the boundary conditions for the stable problem (2.1)—(2.3) may always be written as

$$\begin{cases} u'_x+\sum T_{2j}u_{y_j}+S_1v+T_0u=g_1, \\ u''+S_3v^+=g_{21}, \\ v^-+S_4v^++T_4u=g_{22}, \end{cases} \tag{2.2$'$}$$

where $u=(u', u'')^T$, u' the first b_1 components, u'' the last $p-b_1$ components.

To prove the existence of solution, we should consider the adjoint problem of (2.1)—(2.3).

Proposition 2.6. *The stability of the problem* (2.1), (2.2)$'$, (2.3) *is equivalent to the stability of its adjoint problem if and only if*: $S_3=0$.

Proof Without loss of generality, we may assume $S_1=0$ and $T_0=0$. To deduce the form of the adjoint problem, for (u, v), $(\varphi, \psi)\in C_0^\infty(\overline{\Omega}\times \mathbf{R}_+^1)$, integrating by parts, one gets

$$\begin{aligned}(u_t-Pu-&Av, \varphi)+(v_t-Bu-Qv, \psi)\\ &=(u, -\varphi_t-P_0^*\varphi_{xx}-\sum_j P_{1j}^*\varphi_{xy_j}-\sum_{i,j} P_{2ij}^*\varphi_{y_iy_j}+B_0^*\psi_x+\sum_j B_j^*\varphi_{y_j})\\ &\quad+(v, -\psi_t+Q_0^*\psi_x+\sum_j Q_j^*\psi_{y_j}+A_0^*\varphi_x+\sum A_j^*\varphi_{y_j})\\ &\quad+\text{the terms of lower order } + \text{ the bouneary term on } x=0,\end{aligned}$$

where

$$\begin{aligned}\text{the boundary term}=&\langle u_x, P_0^*\varphi\rangle-\langle u, (P_0^*\varphi)_x\rangle\\ &-\langle u, \sum_j (P_{1j}^*\varphi)_{y_j}\rangle+\langle u, B_0^*\psi\rangle+\langle v, Q_0^*\psi+A_0^*\varphi\rangle.\end{aligned} \tag{2.11}$$

By the homogeneous boundary condition satisfied by (u, v), substituting the expressions of u'_y, u'', v^- into the above equation, and noticing the arbitrariness of u''_x, u', v^+, we can find the form of the adjoin equations and the adjoint boundary conditions.

Omitting the terms of lower order, the adjoint equations are

$$\begin{cases} -\varphi_t-P_0^*\varphi_{xx}-\sum_j P_{1j}^*\varphi_{xy_j}-\sum_{i,j} P_{2ij}^*\varphi_{y_iy_j}+B_0^*\psi_x+B_j^*\psi_{y_j}=0, \\ -\psi_t+Q_0^*\psi_x+\sum_j Q_j^*\psi_{y_j}+A_0^*\varphi_x+\sum_j A_j^*\varphi_{y_j}=0. \end{cases} \tag{2.12}$$

The adjoint boundary conditions on $x=0$ are

$$\begin{cases} (P_0^*\varphi)''=0, \\ -(P_0^*\varphi)'_x+\sum_j (T_{2j}^*(P_0\varphi)'_{y_j}-(P_{1j}^*\varphi)_{y_j})'=0, \\ (Q_0^*\psi)^++(A_0^*\varphi)^+-(S_4^*(Q_0^*\psi)^-+S_4^*(A_0^*\varphi)^-)^+-S_3^*[(T_{2j}^*(P_0^*\varphi)'_{y_j}-(P_{1j}^*\varphi)_{y_j})'' \\ \quad -(P_0^*\varphi)''_x+(B_0^*\psi)''-(T_4^*(Q_0^*\psi)^-+T_4^*(A_0^*\varphi)^-)'']=0. \end{cases} \tag{2.13}$$

The anti-initial condition on $t=t_0>0$ is

$$\varphi|_{t=t_0}=0, \quad \psi|_{t=t_0}=0. \tag{2.14}$$

If $S_3\neq 0$, the third equation in (2.13) must contain the components of $(P_0^*\varphi)''_x$, which are linearly independent to $(P_0^*\varphi)'_x$ in the second equation. Noting that the first equation of (2.13) contains no terms of ψ, so in the equations of (2.13) which contain no derivatives of φ, the equations containing the terms $(Q_0^*\psi)^+=Q_0^+\psi^+$ must be less than q^+. From the above analysisis of the stable boundary condition, (2.12)—(2.14) could not be stable.

On the other hand, if $S_3\neq 0$, (2.12)—(2.14) could not be strongly stable. In fact, the strongly stable problem has nothing to do with T in the boundary condition $Tu+Sv=0$, i. e., the first relation in (2.13) plays no role in deciding the strong stability. In order to have $p+q^-$ boundary conditions, the dimension of φ'' must be zero, i. e., $p=b_1$, then we have $S_3=0$, and the original problem is also strongly stable.

On the contrary, if $S_3=0$, from the definitions of parabolic and hyperbolic eigenvectors, it is easily deduced that the stability of the original problem is reduced to the well-posedness of following two independent problems:

$$\begin{cases} u_t-P_0u_{xx}-\sum_j P_{1j}u_{xy_j}-\sum_{i,j} P_{2ij}u_{y_iy_j}=F_1, \\ u'_x+\sum_j T_{2j}u_{y_j}=g_1,\ u''=g_{21}, \text{ on } x=0, \\ u|_{t=0}=0; \end{cases} \tag{2.15}$$

$$\begin{cases} v_t-Q_0v_x-\sum_j Q_jv_{y_j}=F_2, \\ v^-+S_4v^+|_{x=0}=g_{22},\ v|_{t=0}=0. \end{cases} \tag{2.16}$$

Notice that we now have $S_3=0$ also in the adjoint problem, and that the corresponding principal parts in the adjoint problem are the adjoint equations and boundary conditions when the hyperbolic and parabolic problems are separated. From [1, 4, 9], the adjoint problems of uniformly Lopatinsky well-posed hyperbolic or parabolic initial-boundary value problems are themselves uniformly Lopatinsky well-posed. Therefore, (2.12)—(2.14) is stable when $S_3=0$.

In what follows, we will always assume $S_3=0$ in the problem (2.1), (2.2)', (2.3).

§ 3. A Priori Estimate of Higher Order

Let Ω be an open set in R^n, with boundary $\partial\Omega$ sufficiently smooth. Consider the following initial-boundary value problem:

$$\begin{cases} u_t-Pu-Av=F_1, \\ v_t-Qv-Bu=F_2, \end{cases} \quad \text{in } t>0,\ x\in\Omega, \tag{3.1}$$

$$\begin{cases} \partial_n u'+T_2(\partial)u+S_1v+T_0u=g_1, \\ u''=g_{21}, \\ S_5v+T_5u=g_{22}, \end{cases} \quad \text{on } t>0,\ x\in\partial\Omega, \tag{3.2}$$

$$w|_{t=0}=0, \tag{3.3}$$

where P, A, Q, B are the operators described in § 1, ∂_n denotes the differentiation in the direction normal to $\partial\Omega$, and $T(\partial)$ denotes the first order aifferential operator tangential to $\partial\Omega$.

First, we want to point out that by Theorem 2.5, under the assumption of Theorem 2.6, we can have stronger estimate than that in Theorem 2.6, even when Ω is a bounded domain. For this, we introduce again the operators $\mathscr{E}$ and $\mathscr{E}^{-1}$. When $\overline{\Omega}$ is homeomorphic to half space, the operators $\mathscr{E}$ and $\mathscr{E}^{-1}$ can be defined globally. But for bounded domain Ω, they have to be defined locally by the decomposition of unity. In the boundary patch, $\mathscr{E}$, $\mathscr{E}^{-1}$ can be defined naturally, while in the overlap of boundary patches we take the same coordinates. In inner patches, we can simply take $\mathscr{E}=\eta$. So if $\sum\limits_{\alpha}\varphi_\alpha=1$ is the corresponding decomposition of unity on $\overline{\Omega}$, then $\mathscr{E}$ is defined by $\mathscr{E}u=\sum\limits_{\alpha}\mathscr{E}(\varphi_\alpha u)$, and define

$$\|\mathscr{E}^{-1}u\|_\eta^2=\sum_\alpha\|\mathscr{E}^{-1}\varphi_\alpha u\|_\eta^2.$$

From the proof of Theorem 2.6 (cf. [11], Theorem 6.1), the energy inequality is proved by localization. In every inner local patch, the errors invoked by localization can be estimated by $\|u\|_\eta$, while in every boundary local patch, the errors invoked by localization can be controlled by $\sum\limits_{\alpha}\|\mathscr{E}^{-1}(\varphi_\alpha u)_x\|_\eta$.

So in fact, we have

Theorem 3.1. *Suppose that the problem* (3.1)—(3.3) *satisfies the condition in Theorem* 2.6. *Then, we have the following a priori estimate*

$$\operatorname{Re}(u,\mathscr{E}u)+\operatorname{Re}(u_x,\mathscr{E}^{-1}u_x)+\eta\|v\|_\eta^2+|\mathscr{E}^{-1}u_x|_\eta^2+|w|_\eta^2$$
$$\leqslant C(\|\mathscr{E}^{-1}F_1\|_\eta^2+\|F_2\|_\eta^2+|\mathscr{E}^{-1}g_1|_\eta^2+|g_2|_\eta^2). \tag{3.4}$$

According to Friedrichs[2], we construct the tangential differential enlarged systems of (3.1)—(3.3). Using Theorem 3.1, we are going to prove the following estimate of higher order:

Theorem 3.2. *Suppose the coefficients in* (3.1)—(3.3) *are sufficiently smooth and are zero degree homogeneous in of* (t, x) *for large* $|t|+|x|$. *At each point on the boundary* $\partial\Omega\times R_+^1$, *the frozen coefficients problem with flattened boundary is stable. Then its smooth solution* $w=(u, v)\in C_0^\infty(\overline{\Omega}\times R_+^1)$ *satisfies the estimate:*

$$\eta^{\frac{1}{2}}\|w\|_{k,\eta}^2+\eta^{\frac{1}{2}}\|\mathscr{E}^{-1}u_x\|_{k,\eta}^2+|\mathscr{E}^{-1}u_x|_{k,\eta}^2+|w|_{k,\eta}^2$$
$$\leqslant C_k(\|\mathscr{E}^{-1}F_1\|_{k,\eta}^2+\|F_2\|_{k,\eta}^2+|\mathscr{E}^{-1}g_1|_{k,\eta}^2+|g_2|_{k,\eta}^2). \tag{3.5}$$

Proof According to [2], let $\{D_\sigma\}$ be the complete system of first order tangential operators in Ω. Add $D_t=\partial_t$ into the system $\{D_\sigma\}$, again denoted by $\{D_\sigma\}$, it becomes the complete system in $\Omega\times R^1$. Using $\{D_\sigma\}$, we can define the corresponding Sobolev spaces $H_k(\Omega\times R_+^1)$ and $H_k(\partial\Omega\times R_+^1)$. Denote the corresponding hyperbolic η-weighted norms by $\|D^k\varphi\|_\eta$ and $|D^k\varphi|_\eta$ respectively. $H_k(\Omega\times R_+^1)$ is the same as the usual $H^k(\Omega\times R_+^1)$ except that the function in $H_k(\Omega\times R_+^1)$ may not have the L^2 normal derivatives near the boundary. Evidently, $H_k(\partial\Omega\times R_+^1)=H^k(\partial\Omega\times R_+^1)$.

Consider the first order enlarged system of (3.1)—(3.3) with regard to $\{D_\sigma\}$. In local coordinates, omitting the terms of lower order, (3.1)—(3.3) can be rewritten as

$$\begin{cases} u_t-P_0u_{xx}-\sum_j P_{1j}u_{xy_j}-\sum_{i,j} P_{2ij}u_{y_iy_j}-A_0v_x-\sum_j A_jv_{y_j}=F_1, \\ v_t-B_0u_x-\sum_j B_ju_{y_j}-Q_0v_x-\sum_j Q_jv_{y_j}=F_2, \end{cases} \tag{3.6}$$

$$\begin{cases} u'_x+\sum_j T_{2j}u_{y_j}+S_1v=g_1, \\ u''=g_{21}, \\ v^-+S_4v^++T_4u=g_{22}, \end{cases} \tag{3.7}$$

$$w|_{t=0}=0. \tag{3.8}$$

Locally, $D_y=a(x, y)\partial_y$, $D_x=x\cdot a(x; y)\partial_x$, here $a(x, y)$ is the unity decomposition function. D_σ acts on the first equation of (3.6):

$$(D_\sigma u)_t-P_0(D_\sigma u)_{xx}-\sum_j P_{1j}(D_\sigma u)_{xy_j}-\sum_{i,j} P_{2ij}(D_\sigma u)_{y_iy_j}$$
$$-A_0(D_\sigma v)_x-\sum_j A_j(D_\sigma v)_{y_j}+\mathscr{M}_1(w)=D_\sigma F_1, \tag{3.9}$$

where $\mathscr{M}_1(w)$ is the linear combination of w, $D_\sigma w$, $(D_\sigma u)_y$, $(D_\sigma u)_x$, u_{xx} and v_x. The first four terms are evidently of lower order in the first order enlarged system. Since P_0, Q_0 are invertible, u_{xx} can be expressed by the linear combination of th first four terms and v_x can be expressed by the linear combination of w, $D_\sigma w$, u_x, F_2. To sum up,

$$\mathscr{M}_1(w)=\text{the terms of lower order}+h_1F_1+h_2F_2. \tag{3.10}$$

D_σ acts on the second equation of (3.6):

$$(D_\sigma v)_t-Q_0(D_\sigma v)_x-\sum_j Q_j(D_\sigma v)_{y_j}-B_0(D_\sigma u)_x-\sum_j B_j(D_\sigma u)_{y_j}+\mathscr{M}_2(w)=D_\sigma F_2, \tag{3.11}$$

where $\mathscr{M}_2(w)$ is the linear combination of w, $D_\sigma w$, u_x, v_x. Since the first two terms are of lower order, the stability of (3.6)—(3.8) is independent of u_x, and v_x can be expressed by the linear combination of the first three terms and F_2 by invertibility of Q_0, so we have

$$\mathscr{M}_2(w)=\text{terms of lower order }+hF_2. \tag{3.12}$$

From (3.9)—(3.12), the principal part of equations for (w, Dw) in the first order enlarged system is a block diagonal matrix with every block exactly equal to the principrl part of the original problem. So the assumptions on the equations in section 1 are satisfied for the enlarged system.

Now turn to the boundary condition. D_σ acts on the last two equations of (3.7). It is easily seen that the principal part of the boundary condition with regard to $D_\sigma w$ remains unchanged. In order to make sure the stability after uniting with the conditions for w, one needs only to take η sufficiently large (equivalent to the introduction of a small factor in D_σ, cf. [2]). Since the set of all coefficients deciding stable boundary conditions is open, we can thus guarantee the stability for the first order enlarged system.

D_σ acts on the first equation of (3.7), we can easily see that the form of the principal part for Dow is unchanged,and other terms are of lower order, notaffecting the stability of the system.

From all the above discussion, we know that for (w, Dw), the estimate similar to (3.4) is valid. Repeating the same procedure and constructing the k-th order enlarged system, we can prove

$$\operatorname{Re}(D^k u, \mathscr{E} D^k u)_\eta + \operatorname{Re}((D^k u)_x, \mathscr{E}^{-1}(D^k u)_x)_\eta + \eta\|D^k v\|_\eta^2 + |D^k w|_\eta^2 + |\mathscr{E}^{-1}(D^k u)_x|_\eta^2$$
$$\leqslant C_k'(\|\mathscr{E}^{-1}D^k F_1\|_\eta^2 + \|D^k F_2\|_\eta^2 + |\mathscr{E}^{-1}D^k g_1|_\eta^2 + |D^k g_2|_\eta^2). \tag{3.13}$$

Consequently

$$\eta^{\frac{1}{2}}\|D^k u\|_\eta^2 + \eta^{\frac{1}{2}}\|\mathscr{E}^{-1}D^k u_x\|_\eta^2 + \eta\|D^k v\|_\eta^2 + |\mathscr{E}^{-1}D^k u_x|_\eta^2 + |D^k w|_\eta^2$$
$$\leqslant C_k'(\|\mathscr{E}^{-1}D^k F_1\|_\eta^2 + \|D^k F_2\|_\eta^2 + |\mathscr{E}^{-1}D^k g_1|_\eta^2 + |D^k g_2|_\eta^2). \tag{3.14}$$

That Q_0 is invertible implies that v_x in the first equation of (3.6) can be expressed by Dw, u_x and F_2, and that P_0 is invertible implies that u_{xx} can be expressed by Dw, Du_x, u_x and F_1. To act with ∂_x successively, one gets the estimate for $\|\mathscr{E}^{-1}u_x\|_{k,\eta}$, consequently the estimate for $\|u\|_{k,\eta}$. Using the second equation in (3.6), $\|v\|_{k,\eta}$ can be estimated by $\|u\|_{k,\eta}$, $\|D^k v\|_\eta$ and $\|F_2\|_{k-1,\eta}$.

Adding the above norms involving normal derivatives into (3.14), noting that we have differentiated (3.6) $k-1$ times to get the expression of the k-order normal derivatives, i. e., we have used the norm $\|F_1\|_{k-1,\eta} \leqslant C\|\mathscr{E}^{-1}F_1\|_{k,\eta}$ and $\|F_2\|_{k-1,\eta}$, we get immediately the energy estimate (3.5).

From the definition of the strong solution and the operator $\mathscr{E}^{-1}$, the following two corollaries are straight forward:

Corollary 3.3. *Assuming the conditions in Theorem 3.2,* $(F, g) \in H^k(\Omega \times R_+^1) \times H^k(\partial\Omega \times R_+^1)$, *and their traces up to order* $(k-1)$ *are zero at* $t=0$, *then the differentiable strong solution* w *of* (3.1)—(3.3) *satisfies the energy inequality* (3.5).

Corollary 3.4. *Assuming the conditions in Theorem 3.2, the solution* $w \in C_0^\infty((\bar{\Omega} \times R_+^1)$ *of* (3.1)—(3.3) *satisfies*

$$\sqrt{\eta}\,\|u\|_{k,\eta}^2 + \sqrt{\eta}\,\|v\|_{k,\eta}^2 + |w|_{k,\eta}^2$$
$$\leqslant C_k''\Big(\frac{1}{\eta}\,\|F_1\|_{k,\eta}^2 + \|F_2\|_{k,\eta}^2 + \frac{1}{\eta}\,|g_1|_{k,\eta}^2 + |g_2|_{k,\eta}^2\Big). \tag{3.15}$$

In the estimates (3.5) and (3.15), the space derivatives of u is one order lower than in the usual energy estimate (cf. [14]). To raise the space derivative for one order, we must raise the regularity of the boundary value g. In fact, we have

Theorem 3.5. *Under the assumption of Theorem 3.2, the solution* $w \in C_0^\infty(\bar{\Omega} \times R_+^1)$ *of* (3.1)—(3.3) *satisfies the follow estimate:*

$$\|\nabla^2 u\|^2_{k-1,\eta}+\eta^{\frac{1}{2}}\|w\|^2_{k,\eta}+|w|^2_{k,\eta}$$
$$\leqslant C'''_k\left(\frac{1}{\eta}\|F_1\|^2_{k,\eta}+\|F_2\|^2_{k,\eta}+\frac{1}{\eta}|g_1|^2_{k,\eta}+|g_{21}|^2_{k+\frac{1}{2},\eta}+|g_{22}|^2_{k,\eta}\right). \qquad (3.16)$$

Proof For the $k-1$ order enlarged system, from the well-posedness of parabolic problem (cf. e. g. [1])

$$\|\nabla^2 D^{k-1}u\|^2_\eta+\eta\|D^{k-1}u\|_\eta$$
$$\leqslant C\left(\|D^k v\|^2_\eta+\|D^{k-1}F_1\|^2_\eta+\|D^{k-1}F_2\|^2_\eta+\frac{1}{\eta}|g_1|^2_{k,\eta}+\frac{1}{\eta}|v|^2_{k,\eta}+|g_{21}|^2_{k+\frac{1}{2},\eta}\right). \qquad (3.17)$$

Here we have employed the inequality $|\varphi|^2_{\mu-\frac{1}{2},\eta}\leqslant\frac{C}{\eta}|\varphi|^2_{k,\eta}$.

From the interpolation formula:

$$\eta^{\frac{1}{2}}\|\nabla D^{k-1}u\|^2_\eta\leqslant(\|\nabla^2 D^{k-1}u\|^2_\eta+\eta\|D^{k-1}u\|^2_\eta),$$

the left side of (3.17) can be added by $\eta^{\frac{1}{2}}\|\nabla D^{k-1}u\|^2_\eta$. Differentiate normally the parablic equation $k-1$ times, because of the noncharacteristics of the boundary, one gets the estimate:

$$\eta^{\frac{1}{2}}\|\nabla^2 u\|^2_{k-1,\eta}\leqslant C\left(\|v\|^2_{k,\eta}+\|F_1\|^2_{k-1,\eta}+\|F_2\|^2_{k-1,\eta}+\frac{1}{\eta}|g_1|^2_{k,\eta}+\frac{1}{\eta}|v|^2_{k,\eta}+|g_{21}|^2_{k+\frac{1}{2},\eta}\right). \qquad (3.18)$$

Uniting (3.18) with (3.15) gives (3.16).

The result of this Theorem will not be used in the remaining part of our paper.

Remark. In proving (3.5) and (3.16), we have made use of the already obtained tangential estimate (3.14). If in (3.2), the boundary condition for u is simply $u=0$ or the Neumann condition corresponding to the operator $P-\partial_t$, then we can proceed as in [14] to treat the hyperbolic part and parabolic part separately to deduce the desired estimate similar to (3.16). But for the general Lopatinsky conditions of parabolic variable (e. g. the mixed Dirichlet and Neumann condition), we can't deduce (3.16) by treating the parabolic and hyperbolic part separately, even if the parabolic and hyperbolic variables are separated in boundary condition. Then, the uniform treatment of the parabolic and hyperbolic part as a whole is necessary.

§ 4. The Existence of Differentiable Solutions The Case when Ω is Halfspace

Let

$$\Omega=\{(x, y);\ x>0,\ y\in R^{n-1}\},$$
$$H^m_\eta(\Omega\times R^1)=\{u;\ e^{-\eta t}u\in H^m(\Omega\times R^1)\},$$

$H^m_\eta(\partial\Omega\times R^1)$ is defined similarly.

Consider the boundary-value problem

$$\begin{cases} u_t - Pu - Av = F_1, \\ v_t - Qv - Bu = F_2, \end{cases} \text{in } \Omega\times R^1, \tag{4.1}$$

$$\begin{cases} u'_x + \sum_j T_{2j} u_{y_j} + S_1 v = g_1, \\ u'' = g_{21}, \\ v^- + S_4 v^+ + T_4 u = g_{22}, \end{cases} \quad \text{on } x=0. \tag{4.2}$$

Theorem 4.1. *Suppose the problem* (4.1), (4.2) *satisfies the conditions in Theorem* 2.6, $F\in H^k_\eta(\Omega\times R^1)$, $g\in H^k_\eta(\partial\Omega\times R^1)$, $(F, g)=0$ *in* $t<0$. *Then* (4.1), (4.2) *has a unique strong solution* $w\in H^k_\eta(\Omega\times R^1)$, *satisfying the estimate* (3.5), *and identical to zero in* $t<0$.

If $g_{21}\in H^{k+\frac{1}{2}}_\eta(\partial\Omega\times R^1)$, *then the strong solution satisfies the estimate* (3.16).

Proof By the results of § 3, we need only to prove the existence of solutions of (4.1), (4.2) for smooth (F_j, g_j). Let the sequence of smooth (F_j, g_j) converge to the given (F, g). Then the corresponding solution sequence $\{w_j\}$ converges to the desired solution w.

Our proof follows Sakamoto[10] in proving the existence of hyperbolic systems of higher order.

Denote the adjoint problem of (4.1), (4.2) by

$$\begin{cases} \mathscr{L}^*_1 w^* = -u^*_t - P^* u^* - A^* v^* = F^*_1, \\ \mathscr{L}^*_2 w^* = -v^*_t - Q^* v^* - B^* u^* = F^*_2. \end{cases} \quad \text{in } x>0, \tag{4.3}$$

$$\begin{cases} \Gamma^*_1 w^* = u^{*\prime}_x + \sum_j T^*_{2j} u^*_{y_j} + S^*_1 v^* = g^*_1, \\ \Gamma^*_2 w^* = T^* u^* + S^* v^* = g^*_2, \end{cases} \quad \text{on } x=0, \tag{4.4}$$

Corresponding the adjoint problem, introduce the spaces $H^k_{-\eta}(\Omega\times R^1)$ and $H^k_{-\eta}(\partial\Omega\times R^1)$. Let Λ be the pseudo-differetial operator with symbol $(\tau^2+\eta^2+|\omega|^2)^{\frac{1}{2}}$. Then $\forall s\in R^1$, if $\Lambda^s w^*\in H^2(\Omega\times R^1)$, by the definition of stablity, because of the equivalence of the stability of the original and adjoint problem, we have

$$\operatorname{Re}(\Lambda^s u^*, \mathscr{E}\Lambda^s u^*)_{-\eta} + \operatorname{Re}(\Lambda^s u^*_x, \mathscr{E}^{-1}\Lambda^s u^*_x)_- + \eta\|\Lambda^s v^*\|^2_{-\eta} + |\Lambda^s w^*|^2_{-\eta} + |\mathscr{E}^{-1}\Lambda^s u^*_x|^2_{-\eta}$$
$$\leqslant C(\|\mathscr{E}^{-1}\mathscr{L}^*_1(\Lambda^s w^*)\|^2_{-\eta} + \|\mathscr{L}^*_2(\Lambda^s w^*)\|^2_{-\eta} + |\mathscr{E}^{-1}\Gamma^*_1(\Lambda^s w^*)|^2_{-\eta} + |\Gamma^*_2(\Lambda^s w^*)|^2_{-\eta}). \tag{4.5}$$

Now we examine the following commutators.

I) $[\mathscr{L}^*_1, \Lambda^s]$

1) The commutators of the first order tangential operators in $\mathscr{L}^*_1$ with Λ^s are the s-order tangential operators, which are evidently controlled by the left side of (4.5).

2) The normal derivative of v^* in $\mathscr{L}^*_1$ can be substituted by the tangential derivatives, the normal derivatives of u^* and $\mathscr{L}^*_2 w^*$, on account of the hyperbolic equation. So the commutator consists of the s-order tangential differentiation of w^*

(same as in 1)), the (s-1)-order tangential differentiation of u_x^* and $\mathscr{L}_2^* w^*$. $\|\mathscr{E}^{-1}\Lambda^{s-1}u_x^*\|_{-\eta}^2$ is controlled by Re $(\Lambda^s u_x^*, \mathscr{E}^{-1}\Lambda^s u_x^*)_{-\eta}$ of the left side, while the (s-1)-order tangential differentiation of $\mathscr{L}_2^* w^*$ can be absorbed by the term $\|\Lambda^s \mathscr{L}_2^* w^*\|_{-\eta}^2$.

3) $[P_2^* \partial_{y_i y_j}, \Lambda^s]$ is an $(s+1)$-order tangential operator, and it has at most (s-1) order in ∂_t. So $\mathscr{E}^{-1}$ $[P_2^* \partial_{y_i y_j}, \Lambda^s]$ is an s-order tangential operator, which can be estimated by Re $(\Lambda^s u^*, \mathscr{E}\Lambda^s u^*)_{-\eta}$ on the left side of (4.5).

4) $[P_1^* \partial_{x y_j}, \Lambda^s]$ includes s-order tangential, 1-order normal differentiation, so can be estimated by Re $(\Lambda^s u_x^*, \mathscr{E}^{-1}\Lambda^s u_x^*)_{-\eta}$.

5) Using the parabolic equation, $[P_0^* \partial_{xx}, \Lambda^s]$ can be expressed by the terms already treated above and the $(s-1)$-order tangential differentiation of $\mathscr{L}_1^* w^*$. The latter can evidently absorbed by $\|\mathscr{E}^{-1}\Lambda^s \mathscr{L}_1^* w^*\|_{-\eta}$.

II) $[\mathscr{L}_2^*, \Lambda^s]$

1) The 1-order tangential operators in $\mathscr{L}_2^*$ are estimated as in I), 1).

2) The 1-order normal operator in $\mathscr{L}_2^*$ is estimated as in I), 2), by use of the equation. The term $\|\Lambda^{s-1}u_x^*\|_{-\eta}$ is evidently dominated by Re $(\Lambda^s u_x^*, \mathscr{E}^{-1}\Lambda^s u_x^*)_{-\eta}$, though there is no action of operator $\mathscr{E}^{-1}$ here.

III) $[\Gamma_1^*, \Lambda^s]$

1) The commutators resulted from the tangential operators in Γ_1^* are s-order tangential operators. After the action of $\mathscr{E}^{-1}$, they can evidently estimated by $|\Lambda^s w^*|_{-\eta}^2$.

2) The commutator of ∂_x and Λ^s is an $(s-1)$-order tangential operator. After the action of $\mathscr{E}^{-1}$, it is controlled by $|\mathscr{E}^{-1}\Lambda^s u_x^*|_{-\eta}^2$.

IV) $[\Gamma_2^*, \Lambda^s]$ is an (s-1)-order tangential operator, dominated by $|\Lambda^s w^*|_{-\eta}^2$.

From I)-IV), denoting the norms on the left side of (4.5) by $|||\Lambda^s w^*|||_{-\eta}^2$, we have

$$|||\Lambda^s w^*|||_{-\eta}^2 \leq C(\|\Lambda^s \mathscr{L}_1^* w^*\|_{-\eta}^2 + \|\Lambda^s \mathscr{L}_2^* w^*\|_{-\eta}^2 + |\Lambda^s \Gamma_1^* w^*|_{-\eta}^2 + |\Lambda^s \Gamma_2^* w^*|_{-\eta}^2). \quad (4.6)$$

So for w^*: $\Lambda^s w^* \in H_{-\eta}^2(\Omega \times R^1)$, the right side of (4.6) is a norm for w^*, which will be denoted by $\langle\langle w^* \rangle\rangle_{s,-\eta}$. The complete space in this norm is denoted by $\mathscr{H}_{s,-\eta}$.

Taking $s=-k$, we can prove the existence of differentiable solution. Since

$$|(F_1, w^*)| \leq |(\Lambda^k F_1, \Lambda^{-k} w^*)| \leq \|F_1\|_{k,\eta} \|\Lambda^{-k} w^*\|_{-\eta} \quad (4.7)$$

$$|(F_2, w^*)| \leq \|F_2\|_{k,\eta} \cdot C\langle\langle w^* \rangle\rangle_{-k,-\eta}, \quad (4.8)$$

$$|\langle g \cdot w^* \rangle| \leq |g|_{k,\eta} \cdot |\Lambda^{-k} w|_{-\eta} \leq C|g|_{k,\eta} \cdot \langle\langle w^* \rangle\rangle_{-k,-\eta}, \quad (4.9)$$

so, $(F, w^*) + \langle g, w^* \rangle$ is a continuous linear functional of w^* on the space $\mathscr{H}_{-k,-\eta}$, consequently it is a continuous linear functional of $(\mathscr{L}^* w^*, \Gamma^* w^*)$ on $H_{-\eta}^{-k}(\Omega \times R_1) \times H_{-\eta}^{-k}(\partial\Omega \times R^1)$. From Riesz Theorem, $\forall (\mathscr{L}^* w^*, \Gamma^* w^*) \in H_{-k}^{-\eta} \times H_{-\eta}^{-k}$, $\exists (w, \bar{w}) \in H_\eta^k \times H_\eta^k$ such that

$$(F, w^*) + \langle g, w^* \rangle = (w, \mathscr{L}^* w^*) + \langle \bar{w}, \Gamma^* w^* \rangle. \quad (4.10)$$

Taking $w^*\in C_0^\infty(\Omega\times R^1)$, it is easy to see that w satisfies equation (4.1); taking $w^*\in C_0^\infty(\overline{\Omega}\times R^1)$ such that $\Gamma^*w^*|_{x=0}=0$, we can deduce that w satisfies the boundary condition (4.2). $\overline{w}=w|_{\partial\Omega}$ follows from the arbitrariness of w^*.

The solution w thus obtained has only the tangential regularity. Following the procedure in the proof of Theorem 3.5 and making use of the well-posedness of parabolic problem for u and hyperbolic problem for v, we can see that w has certain normal regularity. Normally differentiate the equations successively, one gets $w\in H_\eta^k(\Omega\times R^1)$.

To finish the proof of Theorem 4.1, it remains to verify $w=0$ at $t<0$.

Let $\varphi_\varepsilon\in C^\infty(R^1)$ with $\varphi_\varepsilon=0$ when $t\geqslant\varepsilon$, $\varphi_\varepsilon=1$ when $t\leqslant 0$. Substituting $w_\varepsilon=\varphi_\varepsilon w$ into (2.8), we get (4.11)

$$\begin{aligned}\sqrt{\eta}\int_{-\infty}^{\varepsilon}\int_\Omega|w_\varepsilon|^2e^{-2\eta t}d\Omega\,\mathrm{nt}+\int_{-\infty}^{\varepsilon}\int_{\partial\Omega}|w_\varepsilon|^2e^{-2\eta t}dS\,dt\\ \leqslant C\Big(\frac{1}{\eta}\int_0^\varepsilon\int_\Omega|\varphi_\varepsilon F_1|^2e^{-2\eta t}\,d\Omega\,dt+\int_0^\varepsilon\int_\Omega|\varphi_\varepsilon F_2|^2e^{-2\eta t}\,d\Omega\,dt\\ +\frac{1}{\eta}\int_0^\varepsilon\int_{\partial\Omega}|\varphi_\varepsilon g_1|^2e^{-2\eta t}\,d\Omega\,dt+\int_0^\varepsilon\int_{\partial\Omega}|\varphi_\varepsilon g_2|^2e^{-2\eta t}\,dS\,dt\\ +\int_0^\varepsilon\int_\Omega|(\partial_t\varphi_\varepsilon)w|^{2\ -2\eta t}\,d\Omega\,dt\Big)\\ \leqslant\text{constant (independent of }\eta).\end{aligned}\tag{4.11}$$

Consequently

$$\sqrt{\eta}\int_{-\infty}^0\int_\Omega|w_\varepsilon|^2\,d\Omega\,dt=\sqrt{\eta}\int_{-\infty}^0\int_\Omega|w|^2\,d\Omega\,dt\leqslant C.$$

Let $\eta\to\infty$, and we have $w=0$ in $t<0$.

§ 5. The Existence of Differentiable Solutions the Case when Ω is Bounded

Here we will give the outline for the proof of the existence when Ω is a bounded domain.

First we define the operator Λ^{2k} and give some of its properties.

According to Friedrichs[2], let $\{\Omega_\alpha\}$ be the finite covering of Ω, $\{\varphi_\alpha\}$ be the corresponding decomposition of unity. We construct the complete system of tangential operators $\{D_\sigma\}$, including $D_t=\partial_t$, and consequently the spaces $H_{k,\eta}(\Omega\times R^1)$ and $H_{k,\eta}(\partial\Omega\times R^1)$, the norms $\|\cdot\|_{k,\eta}$, $|\cdot|_{k,\eta}$.

Define $\Lambda^{2k}=\sum\limits_{\substack{k\ +k_2=k\\ i_1,\cdots,i_{k2}}}\eta^{2k_1}D^*_{\sigma_{i_1}}\cdots D^*_{\sigma_{i_{k2}}}D_{\sigma_{i_{k2}}}\cdots D_{\sigma_{i_1}}$. It is a degenerate self-adjoint elliptic operator of $2k$ order. We list some of its properties in the following:

1) Λ^{2k} is an isomorphism from $H_{2k,\pm\eta}$ to $H_{0,\pm\eta}$, its inverse denoted by Λ^{-2k}.

2) Define the negative norm

$$\|w^*\|_{-2k,-\eta}=\sup_{w}\frac{|(w^*, w)|}{\|w\|_{2k,\eta}} \text{ (similarly is defined } |w^*|_{-2k,-\eta}).$$

Then Λ^{2k} is an isomorphism from $H_{0,\pm\eta}$ to $H_{-2k,\pm\eta}$.

3) Denote by $H^r_{2k+r_2}$ the space of functions which are $2k+r_2$ times tangentially differentiable and r times normally differentiable. Then Λ^{2k} is an isomorphism from $H^r_{2k+r_2,\pm\eta}$ to $H^r_{r_2,\pm\eta}$.

4) Let $\mathscr{L}^*$ be an 1-order tangential operator, then

$$\|\Lambda^{-2k}[\Lambda^{2k}, \mathscr{L}^*]w^*\|_{0,-\eta}\leqslant C\|w^*\|_{0,-\eta}.$$

The properties 1)–4) are only what we need in the following. The proof of 1) is similar to Friedrichs in [2]. In fact, if taking $\Lambda^{2k}=(\eta^2+\sum_{\sigma} D^*_\sigma D_\sigma)^k$, we can simply quote the results of [2]. Here, to make the notation simpler, we take Λ^{2k} to be self-adjoint. The property 2) is straightforward from the definition. By the isomorphic property of 1), 2), it is not difficult to duduce 3), 4) (cf. e. g., Xu Zhengfan[18] for details).

Using the operator Λ^{2k}, we are now going to prove the existence in bounded domain Ω.

As indicated in the proof of Theorem 4.1, we need to prove the existence for smooth (F, g).

First, we prove the localized dual negative norm inequality.

For smooth w, let $\hat{w}=\Lambda^{-2k}w=\sum_{\alpha}\varphi_\alpha\hat{w}=\sum_{\alpha}\hat{w}_\alpha$. Construct $\psi_\alpha\in C_0^\infty(\Omega_\alpha)$, $\psi_\alpha=1$ on supp φ_α. Applying to $\hat{w}_\alpha$ the inequality for the dual problem, similar to (2 6). Since $\hat{w}_\alpha$ is defined locally, all the terms in (2.6) make sense. We have

$$\begin{aligned}&\mathrm{Re}\,(\hat{u}_\alpha, \mathscr{E}\hat{u}_\alpha)_{-\eta}+\mathrm{Re}(\hat{u}_{\alpha x}, \mathscr{E}^{-1}\hat{u}_{\alpha x})_{-\eta}+\eta\|\hat{v}_\eta\|^2_{-\eta}+|\mathscr{E}^{-1}u_{\alpha x}|^2_{-\eta}+|\hat{w}_\alpha|_{-\eta}\\&\leqslant C_0(|\mathscr{E}^{-1}\Gamma_1^*\hat{w}_\alpha|^2_{-\eta}+|\Gamma_2^*\hat{w}_\alpha|^2_{-\eta}+\|\mathscr{E}^{-1}\mathscr{L}_1^*\hat{w}_\alpha\|^2_{-\eta}+\|\mathscr{L}_2^*\hat{w}_\alpha\|^2_{-\eta}).\end{aligned}\tag{5.1}$$

Denote the left side of (5.1) by $\|\hat{w}_\alpha\|^2_{-\eta}$, the right side of (5.1) by $\|\mathscr{M}^*\hat{w}_\alpha\|_R$. From [11], we have

$$\begin{aligned}\|\mathscr{M}^*\hat{w}_\alpha\|^2_R&\leqslant C\Big(T\|\varphi_\alpha\mathscr{M}^*\hat{w}\|^2_R+\|\psi_\alpha\hat{w}\|^2_{-\eta}+\frac{1}{\eta}|\psi_\alpha\hat{w}|^2_{-\eta}\Big)\\&\leqslant C\Big(\|\varphi_\alpha\Lambda^{-2k}\psi_\alpha\mathscr{M}^*w\|^2_R+\|\psi_\alpha\hat{w}\|^2_{-\eta}+\frac{1}{\eta}|\psi_\alpha\hat{w}|^2_{-\eta}\\&\quad+\|\varphi_\alpha\Lambda^{-2k}[\Lambda^{2k},\psi_2\mathscr{M}^*]\hat{w}\|^2_R\Big).\end{aligned}\tag{5.2}$$

The last term on the right side is estimated as follows:

$$\begin{aligned}\|\varphi_\alpha\Lambda^{-2k}[\Lambda^{2k}, \psi_\alpha\mathscr{M}^*]\hat{w}\|^2_R\leqslant&\|\psi_\alpha\Lambda^{-2k}[\Lambda^{2k}, \psi_\alpha\mathscr{M}^*]\hat{w}_\alpha\|^2_R\\&+\|\psi_\alpha[\varphi_\alpha, \Lambda^{-2k}[\Lambda^{2k}, \psi_\alpha\mathscr{M}^*]]\hat{w}\|^2_R.\end{aligned}\tag{5.3}$$

Consider the two terms on the right side of (5.3) separately.

I) $\|\varphi_\alpha\Lambda^{-2k}[\Lambda^{2k}, \psi_\alpha\mathscr{M}^*]\hat{w}_\alpha\|^2_R=\|\psi_\alpha\Lambda^{-2k}[\Lambda^{2k}, \mathscr{M}^*]\hat{w}_\alpha\|^2_R$.

1) $[\Lambda^{2k}, \Gamma_2^*]$ is a $(2k-1)$-order tangential operator, so

$$|\psi_\alpha \Lambda^{-2k}[\Lambda^{2k}, \Gamma_2^*]\hat{w}_\alpha|_{-\eta}^2 \leqslant C\frac{1}{\eta^2}|\hat{w}_\alpha|_{-\eta}^2;$$

2) $[\Lambda^{2k}, \Gamma_1^*]$ is a $2k$-order operator which has at most one order normal derivative, therefore

$$|\mathscr{E}^{-1}\psi_\alpha \Lambda^{-2k}[\Lambda^{2k}, \Gamma_1^*]\hat{w}_\alpha|_{-\eta}^2 \leqslant C\Big(\frac{1}{\eta}|\hat{w}_\alpha|_{-\eta}^2 + \frac{1}{\eta^2}|\mathscr{E}^{-1}\hat{u}_{\alpha x}|_{-\eta}^2\Big);$$

3) In $[\Lambda^{2k}, \mathscr{L}_1^*]$:

$$\|\mathscr{E}^{-1}\psi_\alpha \Lambda^{-2k}[\Lambda^{2k}, A^*]\hat{v}_\alpha\|_{-\eta}^2 \leqslant C\Big(\|\hat{w}_\alpha\|_{-\eta}^2 + \|\mathscr{E}^{-1}\hat{u}_{\alpha x}\|_{-\eta}^2 + \frac{1}{\eta^2}\|\mathscr{L}_2^*\hat{w}_\alpha\|_{-\eta}^2\Big).$$

Noticing that $[\Lambda^{2k}, P_1^*\partial_{xy}]$ is $(2k+1)$-order, where $2k$ order is tangential, with ∂_t at most of $2k-1$ order. So

$$\|\mathscr{E}^{-1}\psi_\alpha \Lambda^{-2k}[\Lambda^{2k}, P_1^*\partial_{xy}]\hat{u}_\alpha\|_{-\eta}^2 \leqslant C(\|\mathscr{E}^{-1}\hat{u}_{\alpha x}\|_{-\eta}^2 + \|\hat{u}_\alpha\|_{-\eta}^2).$$

And $[\Lambda^{2k}, P_2^*\partial_y^2]$ is a $(2k+1)$-order tangential operator, which has at most $2k-1$ order of ∂_t, thus

$$\|\mathscr{E}^{-1}\psi_\alpha \Lambda^{-2k}[\Lambda^{2k}, P_2^*\partial_y^2]\hat{u}_\alpha\|_{-\eta}^2 \leqslant C\|\hat{u}_\alpha\|_{-\eta}^2.$$

Using the equation, we have

$$\|\mathscr{E}^{-1}\psi_\alpha \Lambda^{-2k}[\Lambda^{2k}, P_0^*\partial_x^2]\hat{u}_\alpha\|_{-\eta}^2$$
$$\leqslant C\Big(\|\mathscr{E}^{-1}\hat{u}_{\alpha x}\|_{-\eta}^2 + \|\hat{w}_\alpha\|_{-\eta}^2 + \frac{1}{\eta^2}\|\mathscr{L}_2^*\hat{w}_\alpha\|_{-\eta}^2 + \frac{1}{\eta^2}\|\mathscr{E}^{-1}\mathscr{L}_1^*\hat{w}_\alpha\|_{-\eta}^2\Big).$$

4) $\|\psi_\alpha \Lambda^{-2k}[\Lambda^{2k}, \mathscr{L}_2^*]\hat{w}_\alpha\|_{-\eta}^2 \leqslant C\Big(\|\mathscr{E}^{-1}\hat{u}_{\alpha x}\|_{-\eta}^2 + \|\hat{w}_\alpha\|_{-\eta}^2 + \frac{1}{\eta^2}\|\mathscr{L}_2^*\hat{w}\|_{-\eta}^2\Big).$

Summing up 1)–4), we get

$$|\!|\!|\psi_\alpha \Lambda^{-2k}[\Lambda^{2k}, \psi_\alpha \mathscr{M}^*]\hat{w}_\alpha|\!|\!|_R^2 \leqslant C\Big(\frac{1}{\sqrt{\eta}}|\!|\!|\hat{w}_\alpha|\!|\!|_{-\eta}^2 + \frac{1}{\eta}|\!|\!|\mathscr{M}^*\hat{w}_\alpha|\!|\!|_R^2\Big). \tag{5.4}$$

II) $|\!|\!|\psi_\alpha[\varphi_\alpha, \Lambda^{-2k}[\Lambda^{2k}, \psi_\alpha \mathscr{M}^*]]\hat{w}|\!|\!|_R^2$.

For inner local coordinates patches, $\psi_\alpha \mathscr{M}^*$ is a tangential operator, $[\Lambda^{2k}, \psi_\alpha \mathscr{M}^*]$ is a $(2k+1)$ order tangential, so it is easy to see that $[\varphi_\alpha, \Lambda^{-2k}[\Lambda^{2k}, \psi_\alpha \mathscr{M}^*]]$ is of zero order, therefore

$$|\!|\!|\psi_\alpha[\varphi_\alpha, \Lambda^{-2k}[\Lambda^{2k}, \psi_\alpha \mathscr{M}^*]]\hat{w}|\!|\!|_R^2 C\|\hat{w}\|_{-\eta}^2.$$

For the boundary patches:

1) $|\psi_\alpha[\varphi_\alpha, \Lambda^{-2k}[\Lambda^{2k}, \psi_\alpha \Gamma_2^*]]\hat{w}|_{-\eta}^2 \leqslant \frac{C}{\eta^2}|\hat{w}|_{-\eta}^2;$

2) $|\mathscr{E}^{-1}\psi_\alpha[\varphi_\alpha, \Lambda^{-2k}[\Lambda^{2k}, \psi_\alpha \Gamma_1^*]]\hat{w}|_{-\eta}^2 \leqslant C\Big(\frac{1}{\eta^2}|\hat{w}|_{-\eta}^2 + \frac{1}{\eta^2}\sum_\beta |\mathscr{E}^{-1}\hat{u}_{\beta x}|_{-\eta}^2\Big);$

3) $\|\psi_\alpha[\varphi_\alpha, \Lambda^{-2k}[\Lambda^{2k}, \psi_\alpha \mathscr{L}_2^*]]\hat{w}\|_{-\eta}^2 \leqslant C\Big(\|\hat{w}\|_{-\eta}^2 + \frac{1}{\eta^2}\sum_\beta \|\mathscr{E}^{-1}\hat{u}_{\beta x}\|_{-\eta}^2 + \frac{1}{\eta^2}\|\mathscr{L}_2^*\hat{w}\|_{-\eta}^2\Big);$

4) $\|\mathscr{E}^{-1}\psi_\alpha[\varphi_\alpha, \Lambda^{-2k}[\Lambda^{2k}, \psi_\alpha \mathscr{L}_1^*]]\hat{w}\|_{-\eta}^2 \leqslant C\Big(\|\hat{w}\|_{-\eta}^2 + \frac{1}{\eta^2}\|\mathscr{L}_2^*\hat{w}\|_{-\eta}^2$

$$+\frac{1}{\eta^2}\sum_\beta \|\mathscr{E}^{-1}\hat{u}_{\beta x}\|_{-\eta}^2 + \frac{1}{\eta^2}\sum_\beta \|\mathscr{E}^{-1}\varphi_\beta \mathscr{L}_1^*\hat{w}\|_{-\eta}^2\Big).$$

Here 1), 2) is evident, the estimates of 3), 4) is similar to I), but in the norms involving operator $\mathscr{E}^{-1}$, the estimates in Ω_α should be combined with the estimates of the neighboring patches to have the global estimate (This is exactly why the way we have defined the operator $\mathscr{E}$ globally).

Combining 1)–4), we have

$$\|\psi_\alpha[\varphi_\alpha, \Lambda^{-2k}[\Lambda^{2k}, \psi_\alpha\mathscr{M}^*]]\hat{w}\|_R^2$$
$$\leqslant C\Big(\frac{1}{\sqrt{\eta}}\|\hat{w}\|_{-\eta}^2+\frac{1}{\eta^2}\|\mathscr{L}_2^*\hat{w}\|_{-\eta}^2+\frac{1}{\eta^2}\sum_\beta\|\mathscr{E}^{-1}\varphi_\beta\mathscr{L}_1^*\hat{w}\|_{-\eta}^2\Big). \qquad (5.5)$$

Substituting (5.3)—(5.5) into (5.2), making summation of the indices α, for $\eta\gg 1$, we get

$$\sum_\alpha\|\mathscr{M}^*\hat{w}_\alpha\|_R^2\leqslant C\Big(\sum_\alpha\|\varphi_\alpha\Lambda^{-2k}\psi_\alpha\mathscr{M}^*w\|_R^2+\frac{1}{\sqrt{\eta}}\|\hat{w}\|_{-\eta}^2+\frac{1}{\eta}\sum_\alpha\|\mathscr{M}^*\hat{w}_\alpha\|_R^2\Big). \qquad (5.6)$$

In (5.1), making summation of α and substituting (5.6) into (5.1), we have

$$\sqrt{\eta}\,\|\hat{w}\|_{-\eta}^2+|\hat{w}|_{-\eta}^2\leqslant C(\|\Lambda^{-2k}\mathscr{L}_1^*w\|_{-\eta}^2+\|\Lambda^{-2k}\mathscr{L}_2^*w\|_{-\eta}^2+|\Lambda^{-2k}\Gamma_1^*w|_{-\eta}^2$$
$$+\|\Lambda^{-2k}\Gamma_2^*w|_{-\eta}^2). \qquad (4.7)$$

Noticing that $\hat{w}=\Lambda^{-2k}w$, we know that (5.7) is exactly the global dual inequality we seek, which is similar to (4.7) in section 4. Using (5.7), as in § 4, we can prove the existence of differentiable solution. We omit the details.

References

[1] Agranovich, M. S. & Visik, M. I., Elliptic Problems with Parameter and Parabolic Problems of General Type (in Russian), *У. M. H.*, **19**: 3 (1964), 53—161.

[2] Friedrichs, K. O., Symmetric Positive Linear Differential Equations, *Commu. Pure Appl. Math.*, **11** (1958), 333—418.

[3] Kaeiss, H—O., Initial Boundary Value Problems for Hyperbolic Systems, *Commu. Pure Appl. Math.*, **23** (1970), 277—298.

[4] Lions, J. L. & Magenes, E., Non-homogeneous Boundary Value Problems and Applications, Springer-Vealag, 1972.

[5] Li Daqian, Yu Wenci & Shen Weixi, Boundary Value Problems and Free Boundary Problems for Quasilinear Hyperbolic-Parabolic Coupled Systems, *Univ.* Wisconsin-Madison, MSR 2273 (1981).

[6] Li Dening, The Nonlinear Boundary Value Problems for Quasilinear Hyperbolic Systems.

[7] Li Dening, The General Nonlinear Initial-Boundary Value Problems for Quasilinear Hyperbolic-Parabolic Coupled System (*submitted to Chin. Ann. Math*).

[8] Matsumura, A. & Nishida, T., The Initial Boundary Value Problems for the Equations of Motion of Compressible Viscous and Heat-Conductive Fluid, Univ. Wisconson-Madiso, MRC, TSR, #2237 (1981), #2430 (1982).

[9] Rause, J. B., $\mathscr{L}_2$ is a Contnuable Condition for Kreiss' Mixed Problems, *Commu. Pure Appl. Math.*, **25** (1972), 265—285.

[10] Sakamoto, R., Mixed Problems for Hyperbolic Equations I, II, *J. Math. Kyoto Univ.*, **10** (1970), 349—373, 403—417.

[11] Strikwerda, J. C., Intial Boundary Value Problems for Incompletely Parabolic Systems, *Commu. Pure Appl. Math.*, **30** (1977), 797—822.

[12] Tani, A., On the Eirst Initial-Boundary Value Problem of Compressible Viscous Fluid Motion, *Publ. RIMS, Kyoto Univ.*, **13** (1977), 193—254.

[13] Xu Zhegfan, Uniformity of Strong and Weak Solutions and their Differentiabilities for Linear PDE of Higher Order, Proc. Beijing Symposium, DD_2 (1980), 1639—1653.

[14] Zheng Songmu, Initial Boundary Value Problems for Quasilinear Hyperbolic Parabolic Coupled Systems in Higher Dimensional Spaces, *Chin. Ann. Math.*, **4B**: 4 (1983), 443—462.