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ON THE LINEARITY OF TESTING PLANARITY OF GRAPHS

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Abstract

In 1978, the author published a paper in which a characteristic theorem of planarity of a graph was provided as determining if another graph has a fundamental circuit with a certain property. However, the new graph is with, at worst, quadratic order of the vertex number of the original graph[2].

This paper presents a new criterion of testing planarity of a graph based on what the author obtained before. Fortunately, it is equivalent to finding a spanning tree in another graph with only linear order of the vertex number of the original one in the worst case.

§ 1. Introduction

In the seventies, W. Wu discovered that testing planarity of a graph can be transformed into solving linear equations on $GF(2)$ based on cohomology theory in algebraic topology[8]. Then, Y. Liu found a criterion of planarity which seemed to be much simpler[2]. In fact, the only thing that remained for testing planarity was to solve such linear equations on $GF(2)$ in each of which there were at most two variables. Furthermore, the problem was transformed into finding a circuit with a certain property or a tree in another graph H related to G, the original one.

In the 1979 Montreal Conference on Combinatorics, P. Rosenstiehl proved the result again in an algebraic way[4]. In a private communication, P. Rosenstiehl told Y. Liu that he and his colleague obtained an algorithm in linear time. However, he had not mentioned what method they used. Of course, the first linear time algorithm on this topic was due to J. E. Hopcroft and R. E. Tarjan whose paper was published in 1974[1]. The depth-first search tree technique they used plays a substantial role in the simplification.

This raper provides a new criterion which, in fact, is a simplified form of the one we obtained in [2]. Fortunately, from this criterion, a linear time algorithm for testing planarity of a graph and embedding a planar graph into the plane can be

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deduced. However, the procedure has been described for 3-regular graphs with a depth-first search tree being a path.

§ 2. A Criterion of Planarity

Let $G=(V, E)$ be a graph with V being the vertex set, E the edge set. Or G is treated as a 1-complex in Euclidean space with $G^0=V$ as 0-simplex set, $G^1=E$, 1-simplex set. T denotes a spanning tree of G, $T=(T^0, T^1)$, $T^0=G^0$. Here, only depth-first search trees are considered[1]. $\overline{T}=(\overline{T}^0, \overline{T}^1)=(G^0, G^1-T^1)$ is the cotree corresponding to T. Let $T_D(G)$ be the set of all the depth-first search trees of G. And, for $T\in T_D(G)$, let $\prec$ be the partial order on G^0 determined by T.

Proposition 2.1[5]. *For any $T\in T_D(G)$, there exists a unique orientation of the edges of G, e. g., $e=\langle u, v\rangle$ representing u to v such that*

(i) $u\prec v$, if $e\in T^1$;

(ii) $u\succ v$, if $e\in \overline{T}^1$.

Thus the vertices of G can be labelled so that $\prec$ becomes $<$. In what follows, all the vertices are treated as non-negative integers. For $e\in T^1$, there is a unique cocircuit $\overline{C}_e(T)$, called a fundamental cocircuit, with all the edges in $\overline{C}_e^1(T)$ being in $\overline{T}^1$ save only for one edge. And, for $\alpha\in \overline{T}^1$, there is a unique circuit $C_\alpha(T)$, called a fundamental circuit, with all the edges in $C_\alpha^1(T)$ being in T^1 except for α. A circuit C or path P with all edges in C^1 or P^1 having the same direction is said to be a dicircuit or dipath respectively.

Proposition 2.2. *For $T\in T_D(G)$, all the fundamental circuits of G are dicircuits and each fundamental cocircuit of G has all its edges with the same direction saving only for one edge which belongs to T^1.*

The vertex with the minimum label is said to be the root. The minimum label is always set to be 0,

Proposition 2.3. *For $\langle u, v\rangle\in T^1$, we always have*

(iii) $0\leqslant\nu\leqslant u$ and $v\in P^0\langle 0, u\rangle$, *for all $\langle \mu, \nu\rangle\in \overline{C}^1_{\langle u,v\rangle}(T)$. At each vertex, there is exactly one incoming edge in T^1 except for the root. And, for 2-connected graph G, the root has valency 1 on T.*

For $v\in V$, let $E_v=\{e|e\in E$ and e is incident to $v\}$. From Proposition 2.3, we have

$$E_v=e_v\cup E_v(T)\cup E_v(\overline{T}), \tag{2.1}$$

where e_v is the tree edge coming to v, $E_v(T)=E_v\cap T^1$, $E_v(\overline{T})=E_v\cap \overline{T}^1$. Now, let us introduce variables on $GF(2)$

$$x_{t,s}=x_{s,t},\quad t\neq s, \tag{2.2}$$

for s, or $t\in E_v(T)$ and the other in $E_v(T)\cup E_v(\overline{T})$, at each vertex $v\in V$. If both

$s, t \in E_v(T)$, then $x_{t,s}$ is said to be a tree variable; otherwise, a cotree variable.

For any two fundamental circuits $C_\alpha(T)$, $C_\beta(T)$, a variable $x_{t,s}$ with $s \in C^1_\alpha(T)$, $t \in C^1_\beta(T)$ or vice versa is said to be covariable of $C_\alpha(T)$ and $C_\beta(T)$.

Proposition 2.4. *For any two fundamental circuits $C_\alpha(T)$, $C_\beta(T)$ with $\alpha, \beta \in \overline{T}^1$ having no end in common, there are, if any, exactly two covariables*

Proof If $C^1_\alpha \cap C^1_\beta = \emptyset$, then no covariable exists; otherwise, there are only two possible cases.

Case 1. $C^1_\alpha \cap C^1_\beta = \{v\}$. By symmetry, we may suppose the tree edge coming to v to be in C_α. According to Proposition 2.2, only two configurations possibly appear as follows. In both configurations, $x_{s,t}$ and $x_{s,\beta}$ are the two covariables (in Fig. 2.1).

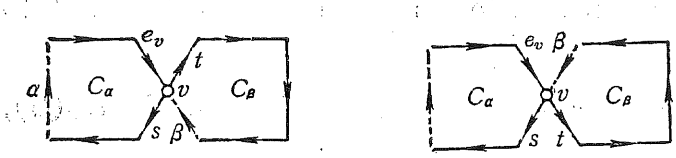


Fig. 2.1

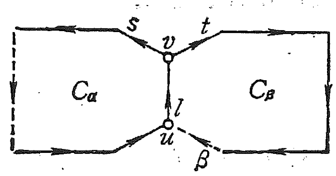


Fig. 2.2

Case 2. $C_\alpha \cap C_\beta = P\langle u, v\rangle$. Similarly to Case 1, we may also suppose the tree edge coming to u to be in C_α. From Proposirion 2.2, the only possible configuration is as in Fig. 2.2. In this configuration, only $x_{s,t}$ and $x_{l,\beta}$ are the covariables.

For $T \in T_D(G)$, we define a T-immersion of G as such a plane representation that two edges α, β cross only if $\alpha, \beta \in \overline{T}^1$ and have no end in common. According to Jordan Axiom, it does exist. Let $D = \{(\alpha, \beta) \mid \alpha, \beta \in \overline{T}^1$ and no end in common$\}$ and $w_{\alpha\beta}$ be the characteristic of α, β crossing, i.e., $w_{\alpha,\beta} = 1$, or 0 according as α, β cross, or do not for $(\alpha, \beta) \in D$.

Criterion I[2]. *A graph G is planar iff for any given $T \in T_D(G)$ and a T-immersion, the equation system on $GF(2)$*

$$x(\alpha, \beta) + y(\alpha, \beta) = w_{\alpha,\beta}, \text{ for } (\alpha, \beta) \in D, \tag{2.3}$$

has a solution, where $x(\alpha, \beta)$, $y(\alpha, \beta)$ are the covariables of C_α and C_β,

$x(\alpha, \beta) \in X = \{x \mid$ *corresponding to an angle with two edges having different directions*$\}$,

$y(\alpha, \beta) \in Y = \{y \mid$ *corresponding to an angle with two edges having the same direction*$\}$

are said to be a forward, backward variable, respectively.

Remarks. 1. Equation (2.3) is defined by Proposition 2.4.

2. The existence of a solution of (2.3) does not depend on the choice of $T \in T_D(G)$ and a T-immersion. Thus, a proper choice of T and a T-immersion are allowed to make the system simpler.

§ 3. Some Results Derived from the Criterion

Let Z be the set of all variables which occur in (2.3), i.e. $Z=X\cup Y$ is the vertex set, and two vertices are adjacent iff the two corresponding variables appear in one equation, or, say, are covariables. The resultant graph, denoted by $H_T^1(G)$, is said to be the first auxiliary graph of G for the T-immersion. Each edge of $H_T^1(G)$ is assigned a weight as the constant term of the corresponding equation.

A circuit in $H_T^1(G)$ is called a 1-circuit if the sum of the weights of all edges on it is 1 (mod 2).

Lemma 3.1. Equation (2.3) has a solution iff there is no 1-circuit in $H_T^1(G)$.

Proof Necessity. If not, suppose $C=x_1h_1x_2\cdots x_sh_sx_1$ to be a 1-circuit in $H_T^1(G)$, i. e., $\sum_{i=1}^{s} w_{h_i}=1$ (mod 2). However

$$0=\sum_{i=1}^{s}(z_i+z_{i+1})=\sum_{i=1}^{s}h_i=1 \pmod 2,$$

a contradiction appears.

Sufficiency. Let $\Delta(H)$ be a spanning tree of $H_T^1(G)$. Then since no 1-circuit appears in $H_T^1(G)$, any solution of the equations determined by $\Delta(H)$ can be extended into a solution of (2.3) determined by the whole $H_T^1(G)$.

Theorem 3.2[2]. *G is planar iff $H_T^1(G)$ has no fundamental 1-circuit.*

Proof Since the sum of 0-circuits does not contain a 1-circuit, from Lemma 3.1, it follows.

Lemma 3.3. *$H_T^1(G)$ has no 1-circuit iff the set of all the edges with weight 1 is a cocycle of $H_T^1(G)$.*

Proof Let W_1 be the set of all the edges with weight 1 in $H_T^1(G)$.

Necessity. Since no 1-circuit occurs, for any spanning tree $\Delta(H)$ of $H_T^1(G)$, there exists an edge $h\in\Delta^1$ and $w_h=1$ except for the trivial case of no edge with weight 1, in which $W_1=\emptyset$ is a cocycle. And we have

$$W_1=\sum_{h\in\Delta^1 \text{ and } w_h=1} C_h^1(\Delta).$$

Therefore, W_1 is a cocycle of $H_T^1(G)$.

Sufficiency. From W_1 being a cocycle, for any circuit C, we have

$$|W_1\cap C|=0 \pmod 2,$$

i.e., C is not a 1-circuit.

Theorem 3.4[4]. *G is planar iff W_1 is a cocycle of $H_T^1(G)$.*

Proof A direct conclusion of Lemma 3.3 and Theorem 3.2.

Generally, as it is only need to consider the number of edges of G not greater than $3n-6=0(n)$, both the numbers of vertices and edges in $H_T^1(G)$ are $0(n^2)$, where n is the vertex number of G. However, for a 3-regular graph, or a general one with T being a path, it is easy to see that the number of vertices in $H_T^1(G)$ becomes $O(n)$ while the number of the edges in $H_T^1(G)$ still remains $O(n^2)$.

For 3-regular graphs with T being a path, let $S(G)$ be the variable sequence in the order of occurrences along the direction from the root on the path T. Of course, the variables at the root can be omitted. And, for convenience, let d_α, y_α represent the two variables related to the cotree edge α, $x_\alpha \in X$, $y_\alpha \in Y$. A subsequence $x_\alpha x_\gamma x_\delta y_\beta y_\gamma y_\alpha$ of $S(G)$ for α, β, γ, and δ being cotree edges is said to be a forbidden subsequence if $x_\beta \bar{\in} [x_\gamma, y_\beta]$, $y_\delta \bar{\in} (x_\delta, y_\gamma]$, where $[x, y)$ (or, $(x, y]$) denotes the segment of $S(G)$ from x to y with y (or, x) being not included.

Lemma 3.5. *For 3-regular graph G with T being a path, G is planar iff there is no forbidden subsequence in $S(G)$.*

Proof The necessity is obvious since any forbidden subsequence leads to a subgraph of $H_T^1(G)$ in which there is a 1-circuit.

The sufficiency can be derived by a direct or an indirect way. However, the following seems much simpler. Since T is a path, for any subgraph J which is topologically equivalent to $K_{3,3}$, all the vertices are on T. And on account of the 3-regularity of G, J only has the configuration indicated as Fig. 3.1 in which x', y' are the two terminals of T, $x \in [x', b]-a$, $y \in (c, y']-d$, and both x and y are (or neither of x and y is) in (a, d). x, x'(y, y' as well) are allowed to be one vertex. The solid lines represent tree edges, and broken lines, cotree edges. All the solid lines consist of J. Hence, we can see that a forbidden sequence always happens, in spite of whatever happens to x, y, if any J in G exists.

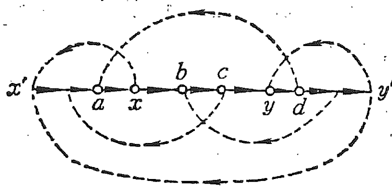


Fig. 3.1

Now we construct a subgraph H_s^1 of $H_T^1(G)$ as follows. The vertex set of H_s^1 is Z, the same as $H_T^1(G)$. And, for each $\alpha \in \overline{T}^1$, e. g., $\alpha=\langle y_\alpha, x_\alpha\rangle$, we may also use y_α, x_α here as vertices of G without confusion, if there exists $\beta \in \overline{T}^1$, $\beta=\langle y_\beta, x_\beta\rangle$ such that $x_\beta<x_\alpha$, $y_\alpha<y_\beta$, then (x_α, y_α) is an edge of H_s^1. Furthermore, let

$$x_\beta=\min\{x\,|\,x_\alpha<x_\xi<y_\alpha,\ y_\xi>y_\alpha\},$$

$$y_\gamma=\max\{y\,|\,x_\beta<y_\xi<y_\alpha,\ x_\xi<x_\alpha\},$$

then (x_α, y_γ), (x_β, y_α), and (x_β, y_γ) are taken as edges of H_s^1. The weights of the edges in H_s^1 are defined as the same as $H_T^1(G)$.

Theorem 3.6. *For 3-regular graph G with T being a path, G is planar iff there is no fundamental 1-circuit in H_s^1.*

Proof From the procedure of producing H_s^1, we can see that H_s^1 is a spanning subgraph of $H_T^1(G)$ and that H_s^1 has a forbidden subsequence iff so does $H_T^1(G)$. In consequence, the theorem follows by Lemma 3.5 and Theorem 3.2.

Whenever a noticing that both the vertex number and the edge number of H_s^1 are $O(n)$, for 3-regular graph with T being a path, an algorithm in linear time for testing planarity can be found according to Theorem 3.6.

Correspondingly, since the equation system on $GF(2)$ related can be solved in linear time, a linear algorithm for embedding such a graph into the plane can also be found by using the method similar to that given in [3].

§ 4. On the Linearity

In order to attach the linearity, we use an approach like that for 3-regular graph with T as a path. The first thing we should do is to reduce the order of $H_T^1(G)$.

Lemma 4.1. *If the equation system* (2.3) *has a solution, then it has at least two solutions.*

Proof By noticing that if (2.3) has a solution, then the complement of the solution on $GF(2)$ is also a solution of (2.3). the lemma thus follows.

Let $E_1(X, Y)$, $E_2(X)$ be two equation systems on $GF(2)$; $E_1(X, Y)$, $E_2(X)$ be the set of solutions of $E_1(X, Y)$, $E_2(X)$ respectively; and let

$$E_1(X; Y)/Y=\bigcup_{Y}\{X \mid (X; Y)\in E_1(X; Y)\}. \tag{4.1}$$

Two equation systems $E_1(X, Y)$, $E_2(X)$ are said to be interchangeable if

$$E_1(X; Y)/Y=E_2(X). \tag{4.2}$$

Lemma 4.2. *Equation systems* $E_1(X, Y; y)=\{x_i+y_i=w_i,\ i=1, 2, \cdots,$ and $y+y_j=c$ *for a fixed* $j\}$ *and* $E_2(X, Y)=\{x_i+y_i=w_i,\ i=1, 2, \cdots\}$ *are interchangeable.*

Proof Obviously, $E_1(X, Y; y)/y\subseteq E_2(X, Y)$.

Conversely, for any $(X, Y)\in E_2(X, Y)$, we have $(X, Y; y_j+c)\in E_1(X, Y; y)$. Therefore $(X, Y)\in E_1(X, Y; y)/y$, i. e., $E_2(X, Y)\subseteq E_1(X, Y; y)/y$.

Let D_0 be the set of pairs of cotree edges α, β with $C_\alpha\cap C_\beta=P\langle u, v\rangle$, $u\neq v$. Of course, $D_0\subseteq D$.

Theorem 4.3. *G is planar iff for $T\in T_D(G)$ and a T-immersion, the equation system on* $GF(2)$

$$x(\alpha, \beta)+y(\alpha, \beta)=w_{\alpha,\beta},\ (\alpha, \beta)\in D_0 \tag{4.3}$$

has a solution.

Proof By applying Lemma 4.2 to the equation system (2.3) one by one for $(\alpha, \beta)\in D$ satisfying $C_\alpha\cap C_\beta=\{v\}$, from the finiteness of such pairs of edges α, β, the final equation system is just (4.3).

This theorem makes $H_T^1(G)$ rather simpler by omitting all the edges corresponding to the equations related to the pairs of cotree edges α, β satisfing $C_\alpha\cap C_\beta=\{v\}$, and the vertices corresponding to the forward variables with angles each of which consists of one cotree edge Γ and one tree edge not on C_Γ.

Lemma 4.4. *Equation systems on* $GF(2)$ $E_1=\{y_i+x=w_i,\ i=1,\ 2,\ \cdots,\ n\}$ *and* $E_2=\{y_i+y_{i+1}=w_i+w_{i+1},\ i=1,\ 2,\ \cdots,\ n-1\}$ *are interchangeable.*

Proof For $n=2$, it is easy to check that $E_1(Y;\ x)=\{y_1+x=w_1,\ y_2+x=w_2\}$ is interchangeable with $E_2(Y)=\{y_1+y_2=w_1+w_2\}$.

In general, we have, by induction, that $E_1(Y;\ x)=\{y_1+y_2=w_1+w_2,\ y_i+x=w_i,\ i=2,\ 3,\ \cdots,\ n\}$ is interchangeable with $\{y_1+y_2=w_1+w_2,\ y_i+y_{i+1}=w_i+w_{i+1},\ i=2,\ 3,\ \cdots,\ n-1\}=E_2(Y)$.

For a tree variable x, let $B(Y;\ x)$ be the set of all the equations in (4.3) each of which involves x, it is just the form indicated in Lemma 4.4. Suppose the two tree edges related to x to be $\langle u,\ v\rangle$ and $\langle u,\ w\rangle$. We denote the branches with the respective root v, w by B_v, B_w. And all the cotree variables in $B(Y;\ x)$ are ordered in a fixed way, e. g., the order of occurrences of the cotree edges of the angles correspondiug to those cotree variables along the dipath $P\langle 0,\ u\rangle$ from the root and the rotation at each vertex.

Theorem 4.5. *G is planar iff, for $T\in T_D(G)$ and a T-immersion, the following equation system on* $GF(2)$

$$\begin{cases} y_i+y_{i+1}=w_i+w_{i+1}, & \text{for all } y_j \text{ related to any pair of branches;} \\ x(\alpha,\ \beta)+y(\alpha,\ \beta)=w_{\alpha,\beta}, & \text{for } \alpha,\ \beta\in D_0 \text{ and the four ends of } \alpha, \\ & \beta \text{ being on the same path of } T \text{ from the root,} \end{cases} \tag{4.4}$$

has a solution.

Proof By appling Lemma 4.4 one by one to the tree variables, (of course, each of which corresponds to a certain pair of branches), from the fiiniteness, the final equation system is just (4.4).

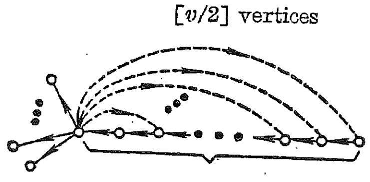


Fig. 4.1

The importance is that, in (4.4), no tree variable is involved. Therefore, the amount of variables are markedly reduced from (4.3). However, the number of variables is still $O(n^2)$, e. g., it is shown in Fig. 4.1. that there are $([n/2]-3)+([n/2]-2)\cdot(n-[n/2])=O(n^2)$ variables in the graph.

For a graph G, if, for $T\in T_D(G)$ and a T-immersion, each of cotree edges is subdivided by a new vertex, then the resultant graph G' is said to be an extended graph of G. And if the new edges incident to the tail ends of the original ones are set as tree edges, then T is extended into $T'\in T_D(G')$ and the T-immersion of G becomes a T'-immersion of G'.

Lemma 4.6. *G is planar iff G' is planar.*

Proof In fact, G' is topologically equivalent to G. Naturally, the lemma follows.

Theorem 4.7. *G is planar iff, for $T'\in T_D(G')$ and a T'-immersion, the equation system on $GF(2)$*

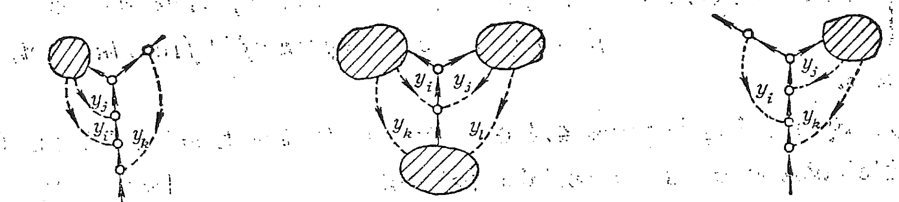
$$y_i+y_{i+1}=w_i+w_{i+1},\ \textit{for each pair of branches} \tag{4.5}$$

has a solution.

Proof From Lemma 4.6, this is a direct result of Theorem 4.5 since in G' all the variables are forward ones, no two cotree edges α, β can have all their ends on the same dipath of T'.

For convenience, let us have all the variables ordered as $y_0, y_1, y_2, \cdots$, say, in order of the occurrences of cotree edges along $T'\in T_D(G')$ from the root. Two variables y_i, y_j are said to be adjacent if they are successive in the sequence of variables related to a pair of branches. And, for a T'-immersion, y_i, y_j are said to be on the same side, or different sides according as so are the corresponding angles of the dipath on T' from the root passing through both the vertices of the angles.

Let $h(y_i)$, $b(y_i)$ be the head, tail ends or their labels of the corresponding cotree edge of y_i respectively. For any adjacent pair y_i, y_j with $b(y_i)$, $b(y_j)$ being non-comparable, (i. e., there is a pair of branches that $b(y_i)$ or $b(y_j)$ (but not both) is on each of them, branches we consider the following three types I, II, III as shown in Fig. 4.2.



Fih. 4.2

Type I $h(y_i)\prec h(y_j)$, and there exists y_k such that $h(y_k)\prec h(y_i)$ and $b(y_k)$, $b(y_j)$ are on the same branch.

Type II. $h(y_i)=h(y_j)$ and there exist y_k, y_l such that $\max(h(y_k), h(y_l))\prec h(y_i)$ and $b(y_k)$, $b(y_l)$ are on different branches related to $b(y_i)$, $b(y_j)$.

Type III. $h(y_i)\preccurlyeq h(y_j)$, $b(y_i)$, $b(y_j)$ on the same branch, and there exists a y_k such that $h(y_k)\prec h(y_i)$, $b(y_k)$ on the other branch.

Now, for a T'-immersion and an adjacent pair y_i, y_j, let us write

$$\lambda_{i,j}=\begin{cases}1, & \text{if } y_i, y_j \text{ are in Type I,}\\ & \text{or in Type II and on the same side;}\\ & \text{or in Type III on different sides;}\\ 0, & \text{otherwise}\end{cases} \tag{4.6}$$

which is called a crossing function. And, an adjacent pair y_i, y_j with $\lambda_{i,j}=1$ is said to be a causing cross pair for the T'-immersion.

Criterion II. *G is planar iff for any T'-immersion of G', $T'\in T_D(G')$, the following equation system*

$$y_i+y_j=\lambda_{i,j}, \text{ for all adjacent pairs } y_i, y_j \tag{4.7}$$

has a solution.

Proof. Since, for any T'-immersion, all the adjacent pairs possibly are in Type I, Type II, or Type III, by Jordan Axiom, it can be exhaustively checked that

$$w_i+w_j=\lambda_{i,j} \tag{4.8}$$

for all adjacent pairs y_i, y_j. The criterion follows from Theorem **4.7.**

If all the vertices represent the variables, two vertices are adjacent iff so are the corresponding variables, i. e., they appear in an equation of (4.7). The resultant graph, denoted by $H_T^2(G)$, is said to be a second auxiliary graph of G. Of course, the weight of an edge in $H_T^2(G)$ is taken as the constant term of the corresponding equation.

Theorem 4.8. *G is planar iff $H_T^2(G)$ has no fundamental 1-circuit, or, say, all the edges with the weight 1 form a cocycle in $H_T^2(G)$.*

Proof According to Lemma 3.1, Theorem 3.2, Lemma **3.3** and Theorem 3.4, the above theorem follows immediately.

Fortunately, since each cotree edge corresponds to at most one variable in (4.7), the number ef variables is not greater than $3n-6=O(n)$, here, n is the vertex number of G as mentioned above. In other words the vertex number of $H_T^2(G)$ is $O(n)$.

§ 5. Conclusion

As for how to determine the subgraph H_s^2 of $H_T^2(G)$ such that the number of edges of H_s^2 is also $O(n)$ and $H_T^2(G)$), can be replaced by H_s^2 (i. e., H_s^2 has a fundamental 1-circuit iff so does $H_T^2(G)$ one can make a series of simplifications through removing edges such that the existence of 1-circuits is invariant. The detail will be fund in a forthcoming paper.

The original derivation of Criterion I from which Criterion II has been found as above was in [2, 8]. There, a natural procedure from a theorem expressed in terms of cohomology theorey[7] to an equation system on $GF(2)$, and finally to (2.3) in Criterion I was shown through a series of transformations and simplifications of equations on $GF(2)$. Or, we take a theorem expressed in terms of group theory[6] as the starting point of the same procedure as just mentioned. However, proofs in view of general, or algebraic graph theory also exist[4, 9].

Since the depth-first search tree technique plays a substantial role in attaching the linearity to the computing complexity, this research suggests at sight that all the characteristic theorems of testing planarity of graphs, e. g., Kuratowski's, Whitney's, MacLane's, et al. can be simplified to the linearity.

Acknowledgement

The author would like to express his sincere thanks to Professor W. T. Tutte for his support and assistance on this research.

Postscript

After finishing the writing, the author finds the result obtained by H. de Fraysseix and P. Rosenstiehl in the paper: A depth-first-search characterization of planarity, Annals of Discrete Math. 13 (1982), pp. 75—80. However, the main theorem in their paper can be derived from Criterion II here, and seems still to be much greater in the order of the vertex number of H there.

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