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# PERTURBATIONS OF DEFINITIZABLE OPERATORS IN A SPACE WITH AN INDEFINITE METRIC

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**Abstract**

Perturbations of definitizable operators in Krein space are studied in this paper. First, the convergence of resolvents and spectral functions is discussed if a sequence of definitizable operators converges in a general sense. Second, for the operational calculus relating to continuous functions, various convergences of operator functions are studied. At last, the relation for the convergence of the sequence of resolvents and that of one-parameter unitary groups is studied. The main theorems of this paper can be regarded as the generalization of the results for self-adjoint operators in Hilbert space,

In recent years there has been an increasing amount of interest in perturbation theory for linear operators defined in a space with an indefinite metric. Several results have been obtained by H. Langer, P. Jonas, and B. Najman in this direction. But this theory is far from completion. In this paper, we are going to study perturbations of definitizable operators in Krein space and attempt to generalize the relevent results on self-adjoint operators defined in Hilbert space under appropriate conditions.

Throughout the paper, the letter $H$ will be used for a Krein space with an indefinite inner product $(\cdot, \cdot)$ and $J \neq P_+ - P_-$ for the metric operator, where $P_+$, $P_-$ are two projections which satisfy $P_+ + P_- = I$, $P_+P_- = 0$. According to the positive definite inner product $[\cdot, \cdot] = (J\cdot, \cdot)$, $H$ is a Hilbert space and $\|\cdot\|$ is the norm correspondingly.

The $J$-self-adjoint operator $A$ is called definitizable if $\rho(A) \neq \emptyset$ and there exists a polynomial $p$ such that

$$(p(A)x, x) \geqslant 0, \quad \forall x \in \mathscr{D}(A^k),$$

where $k = \deg p$. The above-mentioned $p$ is called the definitizing polynomial of $A$[1].

## § 1. Perturbations for the Resolvent and Spectral Function

In this section we shall be concerned with convergence of the resolvents and

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spectral functions if a sequence of definitizable operators $\{A_n\}$ converges to $A$ in some sense. First, we discuss the convergence of a sequence of operators in the generalized sense.

**Theorem 1.** *Suppose that* $\{A_n\}_{n=0}^{\infty}$ *is a sequence of definitizable operators in* $H$, *the degrees of all the definitizing polynomials do not exceed* $2k$, $\lambda_0$ *and* $\lambda$ *belong to* $\bigcap_{n=0}^{\infty}\rho(A_n)$, *and they can be connected by a continuous curve contained in the interior of* $\bigcap_{n=0}^{\infty}\rho(A_n)$. *If* $\lim_{n\to\infty}(\lambda_0-A_n)^{-1}=(\lambda_0-A_0)^{-1}$, *then*

$$\lim_{n\to\infty}(\lambda-A_n)^{-1}=(\lambda-A_0)^{-1},\quad \lim_{n\to\infty}(\bar{\lambda}-A_n)^{-1}=(\bar{\lambda}-A_0)^{-1},$$

*where the limit are all in the sense of norm topologies or of strong topologies.*

*Proof* Connect $\lambda$ and $\lambda_0$ by a continuous curve $\Gamma$ within the interior of $\bigcap_{n=0}^{\infty}\rho(A_n)$. Suppose that $z_0$ is in the interior of $\bigcap_{n=0}^{\infty}\rho(A_n)$ but does not belong to $\Gamma$. Since $\lim_{n\to\infty}(\lambda_0-A_n)^{-1}=(\lambda_0-A_0)^{-1}$, there exists $C'>0$ such that

$$\|(\lambda_0-A_n)^{-1}\|\leqslant C',\quad n=0,1,\cdots. \tag{1}$$

Suppose that the definitizing polynomial of $A_n$ is $p_n(z)=\prod_j(z-z_j^{(n)})^{k_j^{(n)}}$. Denote $r_n'(z)=\frac{p_n(z)}{(z-\lambda_0)^k(z-\bar{\lambda}_0)^k}$. From Lemma 2.2 in [2], there exist constants $\gamma_1'$, $\gamma_2'$, which are independent of $n$, such that for

$$\delta_n'(z_0)=\frac{1}{|r_n'(z_0)|}\left(\gamma_1'+\gamma_2'\frac{1}{\operatorname{dist}(z_0,\sigma(A_n)\cap\mathbf{R})}\right) \tag{2}$$

we have

$$\|(z_0-A_n)^{-1}\|\leqslant\delta_n'(z_0),\quad n=0,1,\cdots. \tag{3}$$

Since $\inf_n\operatorname{dist}(z_0,\sigma(A_n)\cap\mathbf{R})>0$, $\inf_{j,n}\operatorname{dist}(z_0,z_j^{(n)})>0$, it follows from (2) that there exists a constant $C$, which is independent of $n$, such that $\sup_n\delta_n'(z_0)\leqslant C$. From (3) we have

$$\|(z_0-A_n)^{-1}\|\leqslant C,\quad n=0,1,\cdots. \tag{4}$$

From (4), using Lemma 2.2 in [2] again, we conclude that there exist constants $\gamma_1$, $\gamma_2$, which are independent of $n$, such that for $r_n(z)=\frac{p_n(z)}{(z-z_0)^k(z-\bar{z}_0)^k}$ and

$$\delta_n(\lambda)=\frac{1}{|r_n(\lambda)|}\left(\gamma_1+\gamma_2\frac{1}{(\operatorname{dist}(\lambda,\sigma(A_n)\cap\mathbf{R})}\right)$$

we have

$$\|(\lambda-A_n)^{-1}\|\leqslant\delta_n(\lambda),\quad n=0,1,\cdots, \tag{5}$$

where $\lambda\in\Gamma$.

Similarly, it is easy to see that $\sup_n\delta_n(\lambda)<+\infty$ for $\lambda\in\Gamma$, and thus $\inf_n\frac{1}{\delta_n(\lambda)}>0$. By Borel's finite covering theorem, choose points $\lambda_0,\lambda_1,\cdots,\lambda_k=\lambda$ in $\Gamma$, and take

$r_i>0$ such that $\inf_n \frac{1}{\delta_n(\lambda_i)}>r_i$, $\bigcap_{i=0}^{k} O(\lambda_i, r_i)\subset\bigcap_{n=0}^{\infty}\rho(A_n)$, $\lambda_{i+1}\in O(\lambda_i, r_i)$. From (5) and the choice of $r_i$,

$$(\lambda-A_n)^{-1}=-\sum_{k=0}^{\infty}(\lambda-\lambda_i)^k(A_n-\lambda_i)^{-(k+1)}$$

holds for $\lambda\in O(\lambda_i, r_i)$. If $\lim_{n\to\infty}(A_n-\lambda_i)^{-1}=(A_0-\lambda_i)^{-1}$, it is then evident that

$$\lim_{n\to\infty}(A_n-\lambda_{i+1})^{-1}=(A_0-\lambda_{i+1})^{-1}.$$

Using recurrence relations, we have $\lim_{n\to\infty}(\lambda-A_n)^{-1}=(\lambda-A_0)^{-1}$.

Since $\sigma(A_n)$ is symmetric relating to the real axis, we have

$$(A_n-\bar\lambda_0)^{-1}-(A_0-\bar\lambda_0)^{-1}=(A_n-\lambda_0)(A_n-\bar\lambda_0)^{-1}[(A_n-\lambda_0)^{-1}-(A_0-\lambda_0)^{-1}]\cdot(A_0-\lambda_0)(A_0-\bar\lambda_0)^{-1}. \tag{6}$$

Since $\|(A_n-\lambda_0)^{-1}\|=\|(A_n-\bar\lambda_0)^{-1}\|$, it follows from (1) that

$$\|(A_n-\lambda_0)(A_n-\bar\lambda_0)^{-1}\|=\|I-(\lambda-\bar\lambda_0)(\bar\lambda_0-A_n)^{-1}\|\leqslant 1+|\lambda_0-\bar\lambda_0|C'. \tag{7}$$

Now, (6) and (7) imply that $\lim_{n\to\infty}(A_n-\bar\lambda_0)^{-1}=(A_0-\bar\lambda_0)^{-1}$. We note that $\bar\lambda$ and $\bar\lambda_0$ can be connected by a continuous curve contained in the interior of $\bigcap_{n=0}^{\infty}\rho(A_n)$. From the first part of this theorem, we have $\lim_{n\to\infty}(\bar\lambda-A_n)^{-1}=(\bar\lambda-A_0)^{-1}$. The theorem is proved.

For simplicity, we only discuss the definitizable operators with real spectrum in the following. Sppose that the definitizing polynomial of definitizable operator $A$ is $p$, $\deg p\leqslant 2k$, $z_0\in\rho(A)$. We call $r_n(z)=\frac{p_n(z)}{(z-z_0)^k(z-\bar z_0)^k}$ the definitizing factor of $A$.

**Theorem 2.** *Let $\{A_n\}_{n=0}^{\infty}$ be a family of definitizable operators with real spectrum, $\lim_{n\to\infty}\|(z-A_n)^{-1}-(z-A_0)^{-1}\|=0$ for $\operatorname{Im} z\neq 0$, $\mu\overline{\in}\sigma(A_0)$. Then for sufficiently large $n$, $\mu\overline{\in}\sigma(A_n)$ and*

$$\lim_{n\to\infty}\|(\mu-A_n)^{-1}-(\mu-A_0)^{-1}\|=0.$$

*Proof* First, we may assume that $\operatorname{Im}\mu=0$. For $\mu\in\rho(A_0)$, there exists $\delta>0$ such that $[\mu-\delta, \mu+\delta]\subset\rho(A_0)$. Suppose that the definitizing factor of $A_0$ is $r(z)$. With no loss of generality, we may assume that $z_0$, which appeared in the definition of $r(z)$, satisfies $|\operatorname{Re}z_0-\mu|>\delta$. As above, for any $a>0$, there exist constants $\gamma_1$, $\gamma_2$ such that

$$\|(z-A_0)^{-1}\|\leqslant\frac{1}{|r(z)|}\left(\gamma_1+\gamma_2\frac{1}{\operatorname{dist}(z, \sigma(A_0)\cap\mathbb{R})}\right)$$

for $\operatorname{Re}z\in[\mu-\delta, \mu+\delta]$, $|\operatorname{Im}z|\leqslant a$. Write $c=\frac{\gamma_1'\delta+\gamma_2'}{|\gamma(\mu)|\delta}$. Since $\operatorname{dist}(\mu+i\nu, \sigma(A_0)\cap\mathbb{R})>\delta$, we have

$$\|(\mu+i\nu-A_0)^{-1}\|\leqslant c \tag{8}$$

for real $\nu$, $|\nu|\leqslant a$. Take $\nu=\frac{1}{4c}$, and assume $\frac{1}{4c}\leqslant a$. In view of

$$\lim_{n\to\infty}\left\|\left(\mu+\frac{i}{4c}-A_n\right)^{-1}-\left(\mu+\frac{i}{4c}-A_0\right)^{-1}\right\|=0,$$

it follows from (8) that there exists $N$ so that $n\geqslant N$ implies

$$\left\|\left(\mu+\frac{i}{4c}-A_n\right)^{-1}\right\|\leqslant 2c. \tag{9}$$

Furthermore

$$(\lambda-A_n)^{-1}=-\sum_{k=0}^{n}\left(\lambda-\mu-\frac{i}{4c}\right)^k\left(A_n-\mu-\frac{i}{4c}\right)^{-(k+1)} \tag{10}$$

holds in a small neighborhood of $\mu+\frac{i}{4c}$. From (9) we find that the radius of convergence of the right-hand series of (10) is not less than $\frac{1}{2c}$. Hence $\mu\in\rho(A_n)$, and

$$\lim_{n\to\infty}\|(\mu-A_n)^{-1}-(\mu-A_0)^{-1}\|=0.$$

Thus Theorem 2 is proved.

**Theorem 3.** *Suppose that $\{A_n\}_{n=0}^{\infty}$ is a sequence of definitizable operators with real spectrum, $\mathscr{D}_A$ is the common domain, the spectral function of $A_n$ is $E_n$,*

$$\lim_{n\to\infty}\sup_{x}\frac{\|(A_n-A_0)x\|}{\|x\|+\|A_0x\|}=0,$$

*and real numbers $a$, $b\in\rho(A_0)$. Then for any non-negative integer $k$,*

$$\lim_{n\to\infty}\|A_n^kE_n((a,b))-A_0^kE_0((a,b))\|=0.$$

*Proof* Sppose that $\Gamma$ is a rectangular contour with vertexes $a\pm i$, $b\pm i$. By assumption, $\Gamma\subset\rho(A_0)$. It follows from Theorem VIII 1.1 in [3] that $(z-A_0)^{-1}$ are uniformly bounded for $z\in\Gamma$. Besides, since

$$A_0(z-A_0)^{-1}=z(z-A_0)^{-1}-I,$$

$A_0(z-A_0)^{-1}$ are uniformly bounded on $\Gamma$. Denote

$$M=\sup_{z\in\Gamma}(\|(z-A_0)^{-1}\|+\|A_0(z-A_0)^{-1}\|),$$

and $\delta_n=\sup\limits_{x\in\mathscr{D}_A}\frac{\|(A_n-A_0)x\|}{\|x\|+\|A_0x\|}$. Hence

$$\|(A_n-A_0)(z-A_0)^{-1}x\|\leqslant\delta_n(\|(z-A_0)^{-1}x\|+\|A_0(z-A_0)^{-1}x\|)\leqslant M\delta_n\|x\| \tag{11}$$

for $x\in H$. Since $z-A_n=[I+(A_0-A_n)(z-A_0)^{-1}](z-A_0)$, we see that for sufficiently large $n$ such that $M\delta_n<1$, $(z-A_n)^{-1}$ exist for $z\in\Gamma$, and

$$(z-A_n)^{-1}=(z-A_0)^{-1}\sum_{\nu=0}^{\infty}[(A_n-A_0)(z-A_0)^{-1}]^\nu. \tag{12}$$

From (11) and (12), we have

$$\|(z-A_0)^{-1}-(z-A_n)^{-1}\|=\left\|(z-A_0)^{-1}\sum_{\nu=1}^{\infty}[(A_n-A_0)(z-A_0)^{-1}]^\nu\right\|$$

$$\leqslant M\frac{M\delta_n}{1-M\delta_n}\to 0$$

for $z\in\Gamma$. Hence

$$\|A_n^k E_n((a, b)) - A_0^k E_0((a, b))\| = \left\| \frac{1}{2\pi i} \oint_\Gamma [A_n^k (z-A_n)^{-1} - A_0^k (z-A_0)^{-1}] dz \right\|$$
$$= \left\| \frac{1}{2\pi i} \oint_\Gamma [z^k (z-A_n)^{-1} - z^k (z-A_0)^{-1}] dz \right\|$$
$$\leq \frac{1}{2\pi} \int_\Gamma |z|^k \| (z-A_n)^{-1} - (z-A_0)^{-1} \| \, |dz| \to 0.$$

The proof is complete.

## § 2. Convergence of a Sequence of Operator Functions

In this section, for the operational calculus relating to continuous function, we study the convergence of a sequence of operator functions of definitizable operators.

Let $A$ be a definitizable operator defined in Krein space $H$. Its definitizing polynomial is $p(t) = \prod_{i=1}^{l} (t-t_i)^{k_i}$, $k = \max k_i$. The roots of $p$ are no other than the critical points of $A$. If closed interval $\Delta$ only contains one root of $p$, $t_i$, $g$ is a continuous function and $t_i$ is a zero point of $g$ with order $k_i$, then we define $\int_\Delta g(t) dE_t = w-\lim_{\varepsilon\to 0} \int_{\Delta\setminus(t_i-\varepsilon, t_i+\varepsilon)} g(t) dE_t$, where $E_t$ is the spectral function of $A$. From [1], it is easy to see that this definition is reasonable. Thus, if $f$ is a function defined on $(-\infty, \infty)$ and has continuous derivative of the $k+1$-th order, $\{\Delta_i\}$ is a system of finite intervals, the elements of $\{\Delta_i\}$ do not intersect each other, and $t_i$ is an interior point of $\Delta_i$, then we define

$$f(A) = \int \Big[ f(\lambda) - \sum_{i=1}^{l} \chi_{\Delta_i}(\lambda) \sum_{j=0}^{k_i} \frac{f^{(j)}(t_i)}{j!} (\lambda - t_i)^j \Big] dE_\lambda + \sum_{i=1}^{l} \sum_{j=0}^{k_i} \frac{f^{(j)}(t_i)}{j!} (A - t_i)^j E(\Delta_i), \tag{13.1}$$

$$\mathscr{D}(f(A)) = \left\{ x \int \left| f(\lambda) - \sum_{i=1}^{l} \chi_{\Delta_i}(\lambda) \sum_{j=0}^{k_i} \frac{f^{(j)}(t_i)}{j!} (\lambda - t_i)^j \right|^2 d(E_\lambda x, x) < \infty \right\}, \tag{13.2}$$

where $\chi_{\Delta_i}$ gre characteristic functions of $\Delta_i$. Apparently, the definition of $f(A)$ is independent of the choice of $\Delta_i$ and in (13.1), $k_i$ can be substituted by any integer which is larger than $k_i$.

**Theorem 4.** *Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of definitizable operators defined in separable Krein space $H$, definitizing polynomial of $A_n$ be $p_n(t) = \prod_{j=1}^{l} (t - t_j^{(n)})^{k_j^{(n)}}$, $\sum_j k_j^{(n)} \leq 2k$, $n = 1, 2, \cdots$. If $s\text{-}\lim_{n\to\infty} (z-A_n)^{-1} = (z-A_0)^{-1}$ for $\operatorname{Im} z \neq 0$, $f$ is a real-valued function defined on $(-\infty, \infty)$ with compact support and has continuous derivative of $(2k+1)$-th order, then*

$$w\text{-}\lim_{n\to\infty} f(A_n) = f(A_0).$$

*Proof* First, we may assume that $t_1^{(n)} < t_2^{(n)} < \cdots < t_l^{(n)}$. We claim that $\lim_{n\to\infty} t_j^{(n)} = t_j^{(0)}$ for fixed $j$. In fact, it is not hard to see that for any subsequence $\{n_k\}$ of the natural numbers sequence, $\{n\}$, there must be a subsequence $\{n_{k_r}\}$ such that $\lim_{r\to\infty} t_j^{(n_{k_r})}=\bar{t}_j$, where $\bar{t}_j$ is a certain real number or infinite. Otherwise, one $t_j^{(0)}$ does not equal to any $\bar{t}_i$. Take a closed interval $\Delta$ such that $t_j^{(0)}$ is an interior point of $\Delta$ but $\Delta$ does not contain any $\bar{t}_i$ and the endpoints of $\Delta$ do not belong to $\sigma_p(A_0)$. These requirments can be satisfied since $H$ is separable. For sufficiently large $n_{k_r}$, $t_j^{(n_{k_r})}$ do not belong to $\Delta$. Hence $(E^{(n_{k_r})}(\Delta)x, x)\geqslant 0$ for any $x\in H$. By assumption, $s\text{-}\lim_{n\to\infty}(z-A_n)^{-1}=(z-A_0)^{-1}$ where $\operatorname{Im} z\neq 0$. It follows from [2] that $s\text{-}\lim_{n\to\infty} E^{(n)}(\Delta)=E^{(0)}(\Delta)$, so that $(E^{(0)}(\Delta)x, x)\geqslant 0$, which contradicts the fact that $t_j^{(0)}$ is an interior point of $\Delta$. Therefore $\lim_{n\to\infty} t_j^{(n)}=t_j^{(0)}\,(j=1, \cdots, l)$.

Take a closed interval $\Delta_j$ such that $t_j^{(0)}$ is its interior point and its endpoints do not belong to $\sigma_p(A_0)$, $t_i^{(0)}\overline{\in}\Delta(i\neq j)$. Since $s\text{-}\lim_{n\to\infty} E^{(n)}(\Delta_j)=E^{(0)}(\Delta_j)$, we see that

$$s\text{-}\lim_{n\to\infty}(z-A_n)^{-1}E^{(n)}(\Delta_j)=(z-A_0)^{-1}E^{(0)}(\Delta_j) \tag{14}$$

holds for $\operatorname{Im} z\neq 0$. Moreover, it is evident that

$$s\text{-}\lim_{n\to\infty} z^{-1}(I-E^{(n)}(\Delta_j))=z^{-1}(I-E^{(0)}(\Delta_j)). \tag{15}$$

From (14) and (15), we have

$$s\text{-}\lim_{n\to\infty}(z-A_nE^{(n)}(\Delta_j))^{-1}=(z-A_0E^{(0)}(\Delta_j))^{-1}. \tag{16}$$

Apparently, to complete the proof it will suffice to show that

$$w\text{-}\lim_{n\to\infty} f(A_nE^{(n)}(\Delta_j))=f(A_0E^{(0)}(\Delta_j)),\ j=1, \cdots, l.$$

Hence, we may assume that the definitizing polynomial of $A_n$ is $p_n(t)=(t-t_n)^{k_n}$, $k_n\leqslant 2k$. Now, let us explain that we may assume that all $k_n$ equal to $\bar{k}$, $\bar{k}\leqslant 2k$. In fact, from $k_n\leqslant 2k$, it can be seen that we may take $(t-t_n)^{2k}$ or $(t-t_n)^{2k-1}$ as the definitizing polynomial of $A_n$. If there are infinite number of $A_n$ with definitizing polynomials $(t-t_n)^{\bar{k}}$, then from Theorem 3.1 in [2] $(t-t_0)^{\bar{k}}$ is a definitizing polynomial of $A_0$. For our purposes, it will be suffice to consider the relevant subsequence of $\{A_n\}$. In the followings, we assume $p_n(t)=(t-t_n)^{\bar{k}}$. For brevity, assume $t_0=0$. By what has been proved above, $\lim_{n\to\infty} t_n=0$.

Suppose $\varepsilon>0$. We should prove that for $x\in H$, there exists $N$ such that

$$|(f(A_n)x, x)-(f(A_0)x, x)|<\varepsilon$$

for $n>N$.

Take $z_0$, $\operatorname{Im} z_0\neq 0$. Put $r_n(z)=\dfrac{(z-t_n)^{\bar{k}}}{(z-z_0)^k(z-\bar{z}_0)^k}$. Then for $z\neq z_0, \bar{z}_0$, we have $\lim_{n\to\infty} r_n(z)=r_0(z)$. Denote $\Delta_0=(\alpha_0, \beta_0)$, where $-\infty<\alpha_0<0<\beta_0<+\infty$, $\alpha_0, \beta_0\overline{\in}\sigma_p(A_0)$. Since $s\text{-}\lim_{n\to\infty} E^{(n)}(\Delta_0)=E^{(0)}(\Delta_0)$, in a way similar to (16), we have

$$s\text{-}\lim_{n\to\infty}(z-A_nE^{(n)}(\Delta_0))^{-1}=(z-A_0E^{(0)}(\Delta_0))^{-1},$$

which implies

$$\begin{aligned}\text{s-}\lim_{n\to\infty} A_nE^{(n)}(\Delta_0)(z-A_nE^{(n)}(\Delta_0))^{-1} &= \text{s-}\lim_{n\to\infty}[z(z-A_nE^{(n)}(\Delta_0))^{-1}-I]\\ &= z(z-A_0E^{(0)}(\Delta_0))^{-1}-I\\ &= A_0E^{(0)}(\Delta_0)(z-A_0E^{(0)}(\Delta_0))^{-1}. \end{aligned}\tag{17}$$

Suppose that $r_n(z)=\sum_{i=0}^{k} a_i^{(n)} z^i (z-z_0)^{-k}(z-\bar{z}_0)^{-k}$. Then $\lim_{n\to\infty} a_i^{(n)}=a_i^{(0)}$. In view of (17), we obtain

$$\begin{aligned}\text{s-}\lim_{n\to\infty} r_n(A_nE^{(n)}(\Delta_0)) &= \text{s-}\lim_{n\to\infty}\sum_{i=0}^{k} a_i^{(n)}(A_nE^{(n)}(\Delta_0))^i(A_nE^{(n)}(\Delta_0)-z_0)^{-k}\\ &\quad\cdot(A_nE^{(n)}(\Delta_0)-\bar{z}_0)^{-k}\\ &= \sum_{i=0}^{k} a_i^{(0)}(A_0E^{(0)}(\Delta_0))(A_0E^{(0)}(\Delta_0)-z_0)^{-k}(A_0E^{(0)}(\Delta_0)-\bar{z}_0)^{-k}\\ &= r_0(A_0E^{(0)}(\Delta_0)). \end{aligned}\tag{18}$$

From [1], there exists a monotone increasing function $\sigma_x^{(n)}(t)$ such that

$$(E^{(n)}(\Delta)x,x)=\int_\Delta \frac{d\sigma_x^{(n)}(t)}{r_n(t)}$$

holds if $t_n$ is not contained in $\Delta$. Suppose that $n$ is sufficiently large so that $t_n\in(\alpha_0,\beta_0)$. Denote $\Delta_1=(-\infty,\alpha_0]$, $\Delta_2=(\beta_0,+\infty)$. Then

$$\begin{aligned}\sigma_x^{(n)}(\alpha_0)-\sigma_x^{(n)}(-\infty) &= \int_{-\infty}^{\alpha_0} r_n(t)\,d\int_{-\infty}^{t}\frac{d\sigma_x^{(n)}(\tau)}{r_n(\tau)}\\ &= \int_{-\infty}^{\alpha_0} r_n(t)\,d(E^{(n)}(-\infty,t)x,x)\\ &= (r_n(A_nE^{(n)}(\Delta_1)x,x)). \end{aligned}\tag{19}$$

In a similar fashion, we have

$$\sigma_x^{(n)}(+\infty)-\sigma_x^{(n)}(\beta_0)=(r_n(A_nE^{(n)}(\Delta_2))x,x).\tag{20}$$

Besides from [1], $\sigma_x^{(n)}(+\infty)-\sigma_x^{(n)}(-\infty)=(r_n(A_n)x,x)$. Using (19), (20) then we have

$$\sigma_x^{(n)}(\beta_0)-\sigma_x^{(n)}(\alpha_0)=(r_n(A_nE^{(n)}(\Delta_0))x,x).\tag{21}$$

From (18) and (20), it follows that there exists a constant $K>0$ such that

$$\sigma_x^{(n)}(\beta_0)-\sigma_x^{(n)}(\alpha_0)\leqslant K,\quad n=0,1,\cdots.\tag{22}$$

Set

$$\begin{aligned} g_n(\lambda) &= \left|\frac{f(\lambda)-\sum_{i=0}^{2k}\frac{f^{(i)}(t_n)}{i!}(\lambda-t_n)^i}{r_n(\lambda)}\right|\\ &= \left|\frac{f^{(2k+1)}(t_n+\theta_n(\lambda-t_n))}{(2k+1)!}(\lambda-t_n)^l(\lambda-z_0)^k(\lambda-\bar{z}_0)^k\right|, \end{aligned}$$

where $0\leqslant\theta_n\leqslant 1$, $l\geqslant 1$. Since $f^{(2k+1)}(t)$ is bounded for $t\in[\alpha_0,\beta_0]$, there exists a constant $M>0$ such that

$$|g_n(\lambda)|\leqslant M|\lambda-t_n|,\ \lambda\in[\alpha_0,\beta_0].$$

Take $\Delta\subset\Delta_0$, $\Delta=(\alpha,\beta)$, $\alpha$, $\beta\bar{\in}\sigma_p(\Delta_0)$ such that $\alpha<0<\beta$, $\beta-\alpha<\frac{\varepsilon}{6MK}$. By assumption, this can be done. Thus, there exists $N_1>0$ such that

$$|g_n(\lambda)|\leqslant\frac{\varepsilon}{6K},\quad\forall\lambda\in[\alpha,\beta]$$

for $n=0$ or $n>N_1$. From (22), we have

$$\left|\int_\Delta g_n(\lambda)d\sigma_x^{(n)}(\lambda)\right|\leqslant\frac{\varepsilon}{6}.$$

Hence

$$\left|\left(\int_\Delta\left[f(\lambda)-\sum_{i=0}^{2k}\frac{f^{(i)}(t_n)}{i!}(\lambda-t_n)^i dE_\lambda^{(n)}x-\int_\Delta\left[f(\lambda)-\sum_{i=0}^{2k}\frac{f^{(i)}(0)}{i!}\lambda^i\right]dE_\lambda^{(0)}x,\ x\right)\right|$$
$$\leqslant\left|\int_\Delta g_n(\lambda)d\sigma_x^{(n)}(\lambda)\right|+\left|\int_\Delta g_0(\lambda)d\sigma_x^{(0)}(\lambda)\right|<\frac{\varepsilon}{3}\tag{23}$$

holds for $n>N_1$.

Now, consider the sequence of definitizable operators $\{A_nE^{(n)}(\Delta)\}$. It is evident that $\sigma(A_nE^{(n)}(\Delta))\subset\Delta$. From [1], we see that $A_nE^{(n)}(\Delta)$ is bounded. Take $\delta>0$ arbitrarily. By $\Gamma$ we denote the rectangular contour with vertex $\alpha-\delta\pm\delta i$, $\beta+\delta\pm\delta i$.Then $\Gamma\subset\rho(A_nE^{(n)}(\Delta))$. By Theorem 1, we find that

$$s\text{-}\lim_{n\to\infty}(z-A_nE^{(n)}(\Delta))^{-1}=(z-A_0E^{(0)}(\Delta))^{-1}$$

still holds for $z=\alpha-\delta$, $\beta+\delta$. Thus from Theorem VIII 1.2 in [3], $(z-A_nE^{(n)}(\Delta))^{-1}$ uniformly converges to $(z-A_0E^{(0)}(\Delta))^{-1}$ for $z\in\Gamma$ in the sense of strong operator topology, and hence

$$s\text{-}\lim_{n\to\infty}A_n^jE^{(n)}(\Delta)=s\text{-}\lim_{n\to\infty}\frac{1}{2\pi i}\oint_\Gamma z^j(z-A_nE^{(n)}(\Delta))^{-1}dz$$
$$=\frac{1}{2\pi i}\oint_\Gamma z^j(z-A_0E^{(0)}(\Delta))^{-1}dz=A_0^jE^{(0)}(\Delta)\tag{24}$$

holds for any natural number $j$.

Since $f^{(j)}(t_n)\to f^{(j)}(0)$ $(0\leqslant j\leqslant 2k)$ as $n\to\infty$, form (24) we have

$$s\text{-}\lim_{n\to\infty}\sum_{j=0}^{2k}\frac{f^{(j)}(t_n)}{j!}(A_n-t_n)^jE^{(n)}(\Delta)=\sum_{j=0}^{2k}\frac{f^{(j)}(0)}{j!}A_0^jE^{(0)}(\Delta).$$

Heneo there exists $N_2>0$ such that

$$\left|\left(\sum_{j=0}^{2k}\frac{f^{(j)}(t_n)}{j!}(A_n-t_n)^jE^{(n)}(\Delta)x-\sum_{j=0}^{2k}\frac{f^{(j)}(0)}{j!}A_0^jE^{(0)}(\Delta)x,\ x\right)\right|<\frac{\varepsilon}{3}\tag{25}$$

for $n>N_2$.

At last, let us prove $w\text{-}\lim_{n\to\infty}f(A_n)E^{(n)}(\Delta^c)=f(A_0)E^{(0)}(\Delta^c)$. Suppose that supp $f\subset(a,b)$. With no loss of generality, we may assume that $a<\alpha$, $b>\beta$, $a,b\bar{\in}\sigma_p(A_0)$. We note that

$$\sigma_x^{(n)}(\lambda)-\sigma_x^{(n)}(a)=(r_n(A_nE^{(n)}(a,\lambda))x,\ x),$$

and therefore

$$\lim_{n\to\infty}[\sigma_x^{(n)}(\lambda)-\sigma_x^{(n)}(a)]=\sigma_x^{(0)}(\lambda)-\sigma_x^{(0)}(a),$$

if $\lambda\bar{\in}\sigma_p(A_0)$. Moreover, we note that but $\sigma_p(A_0)$ is at most a denumerable set. It is not hard to generalize the ordinary

$$\lim_{n\to\infty}\max_{\lambda\in[a,\alpha]}\left|\frac{f(\lambda)}{r_n(\lambda)}-\frac{f(\lambda)}{r_0(\lambda)}\right|=0,$$

limit theorem of Riemann-Stieltjes integral, which proves that

$$\lim_{n\to\infty}\int_a^\alpha \frac{f(\lambda)}{r_n(\lambda)}\,d\sigma_x^{(n)}(\lambda)=\lim_{n\to\infty}\int_a^\alpha \frac{f(\lambda)}{r_n(\lambda)}\,d(\sigma_x^{(n)}(\lambda)-\sigma_x^{(n)}(a))$$
$$=\int_a^\alpha \frac{f(\lambda)}{r_0(\lambda)}\,d(\sigma_x^{(0)}(\lambda)-\sigma_x^{(0)}(a))=\int_a^\alpha \frac{f(\lambda)}{r_0(\lambda)}\,d\sigma_x^{(0)}(\lambda). \tag{26}$$

In the same way

$$\lim_{n\to\infty}\int_\beta^b \frac{f(\lambda)}{r_n(\lambda)}\,d\sigma_x^{(n)}(\lambda)=\int_\beta^b \frac{f(\lambda)}{r_0(\lambda)}\,d\sigma_x^{(0)}(\lambda). \tag{27}$$

From (26), (27), we have

$$\lim_{n\to\infty}(f(A_n)E^{(n)}(\Delta^c)x,\,x)=\lim_{n\to\infty}\int_{\Delta^c}\frac{f(\lambda)}{r_n(\lambda)}\,d\sigma_x^{(n)}(\lambda)=\int_{\Delta^c}\frac{f(\lambda)}{r_0(\lambda)}\,d\sigma_x^{(0)}(\lambda)$$
$$=(f(A_0)E^{(0)}(\Delta^c)x,\,x). \tag{28}$$

Suppose that

$$|(f(A_n)E^{(n)}(\Delta^c)x,\,x)-(f(A_0)E^{(0)}(\Delta^c)x,\,x)|<\frac{\varepsilon}{3} \tag{29}$$

for $n>N_3$. Consequently, it follows from (23), (25), (29) that

$$|(f(A_n)x,\,x)-(f(A_0)x,\,x)|<\varepsilon \tag{30}$$

for $n>\max(N_1, N_2, N_3)$, which means $w\text{-}\lim_{n\to\infty} f(A_n)=f(A_0)$, so that the theorem is verified.

Suppose that $A$ is a definitizable operator with real spectrum, and the definitizing factor of $A$ is $r(z)$. For $x, y\in H$, it follows from [1] that there exists a bounded variation function $\sigma_{x,y}(t)$ defined on $(-\infty, \infty)$ such that

$$(r(z)(z-A)^{-1}x,\,y)=\int\frac{d\sigma_{x,y}(t)}{z-t}.$$

Suppose that $f$ is a continuous function defined on $(-\infty, \infty)$, and $\lim_{t\to\infty} f(t)=0$. Define $(rf)(A)$ by

$$((rf)(A)x,\,y)=\int f(t)\,d\sigma_{x,y}(t). \tag{31}$$

It is easily seen that if $f$ has a bounded support and has continuous derivative of an appropriate order, the definition of $(rf)(A)$ agrees with the operational function given before Theorem 4.

**Theorem 5.** *Suppose that* $\{A_n\}_{n=0}^\infty$ *is a sequence of definitizable operators with real spectrum, the definitizing polynomial of* $A_n$ *is* $p_n$, $\deg p_n\leqslant 2k$, $\lim_{n\to\infty}\max_{|z|\leqslant 1}|p_n(z)-p^0(z)|=0$, $z_0\in\bigcap_{n=0}^\infty\rho(A_n)$, $\operatorname{Im} z_0\neq 0$, $r_n(z)=\dfrac{p_n(z)}{(z-z_0)^k(z-\bar z_0)^k}$, *and* $f$ *is a continuous function defined on real axis, satisfying* $\lim_{t\to\infty} f(t)=0$.

(i) *If* $\lim_{n\to\infty}\|(z-A_n)^{-1}-(z-A_0)^{-1}\|=0$ *for* $\operatorname{Im} z\neq 0$, *then*

$$\lim_{n\to\infty}\|(r_nf)(A_n)-(r_0f)(A_0)\|=0.$$

(ii) *If* $s\text{-}\lim_{n\to\infty}(z-A_n)^{-1}=(z-A_0)^{-1}$ *for* $\operatorname{Im} z\neq 0$, *then*

$$s\text{-}\lim_{n\to\infty}(r_nf)(A_n)=(r_0f)(A_0).$$

*Proof* (i) Since $A_n(z-A_n)^{-1}=I-z(z-A_n)^{-1}$ if $\operatorname{Im} z\neq 0$, we see that $A_n(z-A_n)^{-1}$ are bounded operators, and then it is easy to see that $r_n(A_n)$ are bounded operators. Besides, since $\lim_{n\to\infty}\|(z-A_n)^{-1}-(z-A_0)^{-1}\|=0$, we have

$$\lim_{n\to\infty}\|A_n(z-A_n)^{-1}-A_0(z-A_0)^{-1}\|=\lim_{n\to\infty}\|z(z-A_n)^{-1}-z(z-A_0)^{-1}\|=0.$$

Since $\lim_{n\to\infty}\max_{|z|\le 1}|p_n(z)-p_0(z)|=0$, by an argument like that given in (18), it now follows that

$$\lim_{n\to\infty}\|r_n(A_n)-r_0(A_0)\|=0. \tag{32}$$

Hence, there exists a positive number $M$ such that

$$\|r_n(A_n)\|\le M,\quad n=0,1,2,\cdots.$$

For any fixed $\varepsilon>0$, from Stone-Weierstrass theorem, there exists a polynomial of two variables, $P(u, v)$, such that

$$\max_{-\infty<t<+\infty}|f(t)-P((t+i)^{-1},(t-i)^{-1})|<\frac{\varepsilon}{3M}.$$

For $x\in H$, $\|x\|=1$, denote $y_n=J[(r_nf)(A_n)-(r_nP)(A_n)]x$, where $P(x)=P((x+i)^{-1},(x-i)^{-1})$. From [1], it follows that there exists a bounded variation function $\sigma_{x,y_n}(t)$ on $(-\infty,\infty)$ such that

$$\begin{aligned}
&\|[(r_nf)(A_n)-(r_nP)(A_n)]x\|^2\\
&=([(r_nf)(A_n)-(r_nP)(A_n)]x, y_n)\\
&=\int[f(t)-P(t)]d\sigma_{x,y_n}(t)\\
&\le\frac{\varepsilon}{3M}\int|d\sigma_{x,y_n}(t)|\le\frac{\varepsilon}{3M}(r_n(A_n)x,x)^{\frac12}(r_n(A_n)y_n,y_n)^{\frac12}\\
&\le\frac{\varepsilon}{3M}\|r_n(A_n)\|^{\frac12}\|r_n(A_n)\|^{\frac12}\|[(r_nf)(A_n)-(r_nP)(A_n)]x\|\\
&<\frac{\varepsilon}{3}\|[(r_nf)(A_n)-(r_nP)(A_n)]x\|.
\end{aligned}$$

Therefore

$$\|(r_nf)(A_n)-(r_nP)(A_n)\|<\frac{\varepsilon}{3},\quad n=0,1,2,\cdots. \tag{33}$$

It is easily verified that

$$(r_nP)(A_n)=r_n(A_n)P((A_n+i)^{-1},(A_n-i)^{-1}).$$

By assumption and (32), we have

$$\|r_n(A_n\|-r_0(A_0)\|\to 0,\quad \|(A_n\pm i)^{-1}-(A_0\pm i)^{-1}\|\to 0.$$

Thus, thare exists $N$ such that

$$\|(r_n P)(A_n) - (r_0 P)(A_0)\| < \frac{\varepsilon}{3} \tag{34}$$

for $n > N$. From (33) and (34)

$$\|(r_n f)(A_n) - (r_0 f)(A_0)\| < \varepsilon$$

holds for $n > N$.

(ii) can be proved in a fashion analogous to (i), so we omit these statements. The theorem is proved.

In order to discuss the convergence of a sequence of operator functions, we introduce a new concept which is called the condition for uniformly equivalence of induced norms. Suppose that $\{A_n\}_{n=0}^{\infty}$ is a sequence of definitizable operators with real spectrum, the definitizing polynomial of $A_n$ is $p_n$, and the spectral function of $A_n$ is $E^{(n)}$. All the roots of $p_n$ are contained in $(a, b)$. Denote $\Delta = (a, b)$. Then $E_{\Delta^c}^{(n)} H$ is an orthogonal complemented subspace and

$$I = E_{\Delta}^{(n)} + E_{\Delta^c}^{(n)}.$$

Define an inner product on $H$:

$$[x, x]_n = [E_{\Delta}^{(n)} x, E_{\Delta}^{(n)} x] + (E_{\Delta^c}^{(n)} x, E_{\Delta^c}^{(n)} x).$$

Then $H$ is a Hilbert space, and the topology induced by $[\cdot, \cdot]_n$ coincides with that induced by $[\cdot, \cdot]$. Denoting the norms by $\|\cdot\|_n$, $\|\cdot\|$ respectively, we see that there exist $m_{\Delta}^{(n)}$, $M_{\Delta}^{(n)} > 0$ such that

$$m_{\Delta}^{(n)} \|\cdot\|_n \leqslant \|\cdot\| \leqslant M_{\Delta}^{(n)} \|\cdot\|_n.$$

If there exist $m_\Delta$, $M_\Delta$, $0 < m_\Delta < M_\Delta < +\infty$, which are independent of $n$, such that

$$\text{(U)} \qquad m_\Delta \|\cdot\|_n \leqslant \|\cdot\| \leqslant M_\Delta \|\cdot\|_n, \quad n = 0, 1, 2, \cdots,$$

then we say the norms induced by $E^{(n)}$ satisfy the condition of uniformly equivalence, or, simply, the condition (U).

**Proposition.** *The condition* (U) *is independent of the choice of* $\Delta$.

*Proof* Suppose that $\Delta_2 \supset \Delta_1$. Write $[x, x]_n^{(i)} = [E_{\Delta_i}^{(n)} x, E_{\Delta_i}^{(n)} x] + (E_{\Delta_i^c}^{(n)} x, E_{\Delta_i^c}^{(n)} x)$, and denote the related norm by $\|\cdot\|_n^{(i)}$. Suppose that for $\Delta_1$ there exist $m_{\Delta_1}$, $M_{\Delta_1}$ satisfying the condition (U), i. e.

$$m_{\Delta_1} \|\cdot\|_n^{(1)} \leqslant \|\cdot\| \leqslant M_{\Delta_1} \|\cdot\|_n^{(1)}, \quad n = 0, 1, 2, \cdots.$$

For brevity, we omit indexes $n$ for the moment. Denote $\Delta = \Delta_2 \backslash \Delta_1$. For $x \in H$,

$$\begin{aligned}
[x, x]^{(2)} &= [E_{\Delta_2} x, E_{\Delta_2} x] + (E_{\Delta_2^c} x, E_{\Delta_2^c} x) \leqslant M_{\Delta_1}^2 [E_{\Delta_2} x, E_{\Delta_2} x]^{(1)} + (E_{\Delta_2^c} x, E_{\Delta_2^c} x) \\
&= M_{\Delta_1}^2 \{[E_{\Delta_1} x, E_{\Delta_1} x] + (E_\Delta x, E_\Delta x)\} + (E_{\Delta_2^c} x, E_{\Delta_2^c} x) \\
&\leqslant \max(M_{\Delta_1}^2, 1) \{[E_{\Delta_1} x, E_{\Delta_1} x] + (E_{\Delta_1^c} x, E_{\Delta_1^c} x)\} \\
&= \max(M_{\Delta_1}^2, 1) [x, x]^{(1)}.
\end{aligned}$$

In the same way, we have

$$[x, x]^{(1)} \leqslant \max\left(\frac{1}{m_{\Delta_1}^2}, 1\right) [x, x]^{(2)}.$$

From the above estimation and the condition (U) relating to $\Delta_1$, it follows that

$$\frac{m_{\Delta_1}}{\max(M_{\Delta_1}, 1)}\|\cdot\|_n^{(2)}\leqslant\|\cdot\|\leqslant M_{\Delta_1}\max\left(\frac{1}{m_{\Delta_1}}, 1\right)\|\cdot\|_n^{(2)}, \quad n=0, 1, \cdots.$$

The same argument can be applied to other cases for $\Delta_1$ and $\Delta_2$. We omit these statements. Our theorem is proved.

**Theorem 6.** *Let $\{A_n\}$ be a sequence of definitizable operators with real spectrum, which are defined in a separable Krein space $H$, the definitizing polynomial of $A_n$ be $p_n$,* $\deg p_n\leqslant 2k$, $\lim\limits_{n\to\infty}\max\limits_{|z|\leqslant 1}|p_n(z)-p_0(z)|=0$, $s\text{-}\lim\limits_{n\to\infty}(z-A_n)^{-1}=(z-A_0)^{-1}$, *where* $\operatorname{Im} z\neq 0$, $r_n(z)=\dfrac{p_n(z)}{(z-z_0)^k(z-\bar{z}_0)^k}$, *where* $\operatorname{Im} z_0\neq 0$. *If $\{A_n\}$ satisfies the condition* (U) *and $f$ is a bounded continuous function in $(-\infty, \infty)$, then* $s\text{-}\lim\limits_{n\to\infty}(r_nf)(A_n)=(r_0f)(A_0)$.

*Proof* Take a sequence of positive numbers $a_k$, $a_k\nearrow\infty$, $a_k\bar{\in}\sigma_p(A_0)$. Define functions $\varphi_k$ on $(-\infty, \infty)$:

$$\varphi_k(\lambda)=\begin{cases}1, & |\lambda|\leqslant a_k,\\ \text{linear}, & a_k\leqslant|\lambda|\leqslant a_{k+1},\\ 0, & |\lambda|\geqslant a_{k+1}.\end{cases}$$

With no loss of generality, we may assume that the roots of all $p_n$ are contained in $(-a_1, a_1)$. Denote $\Delta=(-a_1, a_1)$ and

$$[x, x]_n=[E_\Delta^{(n)}x, E_\Delta^{(n)}x]+(E_{\Delta^c}^{(n)}x, E_{\Delta^c}^{(n)}x).$$

By assumption, there exist positive numbers $m_\Delta$, $M_\Delta$ such that

$$m_\Delta^2[x, x]_n\leqslant[x, x]\leqslant M_\Delta^2[x, x]_n, \quad \forall x\in H,\ n=0, 1, \cdots.$$

Since $s\text{-}\lim\limits_{n\to\infty}(z-A_n)^{-1}=(z-A_0)^{-1}$ and $\lim\limits_{n\to\infty}\max\limits_{|z|\leqslant 1}|p_n(z)-p_0(z)|=0$, we have $s\text{-}\lim\limits_{n\to\infty}r_n(A_n)=r_0(A_0)$ and hence $\sup\limits_n\|r_n(A_n)\|<+\infty$. For $x\in H$, there exist monotone increasing functions $\sigma_x^{(n)}(t)$ such that

$$|((r_nf)(A_n)x, x)|=\left|\int f(t)d\sigma_x^{(n)}(t)\right|\leqslant\sup_x|f(x)|\int|d\sigma_x^{(n)}(t)|$$
$$=\sup_x|f(x)|(r_n(A_n)x, x).$$

Therefore $\sup|((r_nf)(A_n)x, x)|<+\infty$. Consequently, $\sup\|(r_nf)(A_n)\|\leqslant M<+\infty$, where $M$ is a certain constant.

Fo rfixed $x\in H$, estimate $\varphi\|_k(A_n)x-x\|$ as follows. It is easily seen that

$$\|\varphi_k(A_n)x-x\|\leqslant\|E^{(n)}(|\lambda|>a_{k+1})x\|+\left\|\int_{a_k<|\lambda|<a_{k+1}}(\varphi_k(\lambda)-1)dE_\lambda^{(n)}x\right\|$$
$$\leqslant M_\Delta\left\{\|E^{(n)}(|\lambda|>a_{k+1})x\|_n+\left\|\int_{a_k<|\lambda|<a_{k+1}}(\varphi_k(\lambda)-1)dE_\lambda^{(n)}x\right\|_n\right\}$$
$$\leqslant M_\Delta\{\|E^{(n)}(|\lambda|>a_{k+1})x\|_n+\|E^{(n)}(a_k<|\lambda|<a_{k+1})x\|_n\}$$
$$\leqslant\sqrt{2}M_\Delta\|E^{(n)}(|\lambda|>a_k)x\|_n\leqslant\frac{\sqrt{1}M_\Delta}{m_\Delta}\|E^{(n)}(|\lambda|>a_k)x\|.$$

Since $s\text{-}\lim\limits_{k\to\infty}E^{(0)}(|\lambda|<a_k)=I$, there exists a positive integer $k$ such that

$$\|E^{(0)}(|\lambda|>a_k)x\|<\frac{m_\Delta\varepsilon}{3\sqrt{2}M_\Delta M}.$$

Besides, since $s\text{-}\lim_{n\to\infty} E^{(n)}(|\lambda|>a_k)=E^{(0)}(|\lambda|>a_k)$, there exists $N_1$ such that

$$\|E^{(n)}(|\lambda|>a_k)x\|<\frac{m_\Delta\varepsilon}{3\sqrt{2}M_\Delta M}$$

for $n>N_1$. We note that $f\varphi_k$ is a continuous function defined in $(-\infty, \infty)$ with compact support. From Theorem 5, it follows that there exists $N_2$ such that

$$\|(r_n f\varphi_k)(A_n)x-(r_0 f\varphi_k)(A_0)x\|<\frac{\varepsilon}{3}$$

for $n>N_2$. Summarizing what have been proved, we have

$$\begin{aligned}&\|(r_n f)(A_n)x-(r_0 f)(A_0)x\|\\ &\quad\leqslant\|((r_n f)(A_n)x-(r_n f\varphi_k)(A_n)x\|+\|(r_0 f)(A_0)x-(r_0 f\varphi_k)(A_0)x\|\\ &\qquad+\|(r_n f\varphi_k)(A_n)x-(r_0 f\varphi_k)(A_0)x\|\\ &\quad\leqslant\|(r_n f(A_n)\|\,\|x-\varphi_k(A_n)x\|+\|(r_0 f)(A_0)\|\,\|x-\varphi_k(A_0)x\|+\frac{\varepsilon}{3}<\varepsilon.\end{aligned}$$

In the preceding estimation, we have used the relation $(r_n f\varphi_k)(A_n)=(r_n f)(A_n)\varphi_k(A_n)$, which can be verified easily. The theorem is proved.

**Remark.** In fact, the condition $s\text{-}\lim_{n\to\infty}(z-A_n)^{-1}=(z-A_0)^{-1}$, $\forall$ Im $z\neq0$, has already ensured the existence of $m_\Delta$ appeared in the condition (U). The reason is as follows. Denote $B_n=E_\Delta^{(n)}JE_\Delta^{(n)}+E_{\Delta'}^{(n)}$. Then

$$[x, x]_n=(B_n x, x).$$

Since $s\text{-}\lim_{n\to\infty} B_n$ exists, there must be $M>0$ such that $\sup_n\|B_n\|<M<+\infty$. Therefore

$$[x, x]_n\leqslant M[x, x].$$

So we can take $m_\Delta=M^{-\frac{1}{2}}$.

## § 3. One-Parameter Unitary Group

Let $A$ be a bounded definitizable operator, and $r(A)$ a definitizing factor of $A$. As before, for $x\in H$ there exists a monotone increasing function $\sigma_x(t)$ such that

$$(r(A)(z-A)^{-1}x, x)=\int_{-\infty}^{\infty}\frac{d\sigma_x(t)}{z-t}.$$

If $f$ is analytic in a neighborhood of $\sigma(A)$, $\Gamma$ is a Jordan contour within the analytic region of $f$ and $\sigma(A)$ is surrounded by $\Gamma$, then

$$\begin{aligned}(r(A)f(A)x, x)&=\frac{1}{2\pi i}\oint_\Gamma(r(A)f(z)(z-A)^{-1}x, x)dz\\ &=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\oint_\Gamma\frac{f(z)dz}{z-t}d\sigma_x(t)=\int_{-\infty}^{\infty}f(t)d\sigma_x(t).\end{aligned}$$

**Theorem 7.** *Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of bounded definitizable operators with real spectrum, $\sup_n\|A_n\|<+\infty$, and the definitizing factor of $A_n$ be $r_n$. Then for any $z\in$ (Im $z\neq0$), $w\text{-}\lim_{n\to\infty} r_n(A_n)(z-A_n)^{-1}=r_0(A_0)(z-A_0)^{-1}$ holds if and only if $\sup_n\|r_n(A_n)\|$*

$<+\infty$ *and* $w\text{-}\lim\limits_{n\to\infty} r_n(A_n)e^{itA_n}=r_0(A_0)e^{itA_0}$ *for* $t\in(0,\infty)$.

*Proof* First, suppose that $w\text{-}\lim\limits_{n\to\infty} r_n(A_n)(z-A_n)^{-1}=r_0(A_0)(z-A_0)^{-1}$, $\forall z\in$ (Im $z\neq 0$). For a fixed complex number $z$ with nonzero imaginary part, by Banach-Steinhaus principle, $\sup\limits_n\|r_n(A_n)(z-A_n)^{-1}\|<+\infty$, so that

$$\sup_n\|r_n(A_n)\|\leqslant\sup_n\|r_n(A_n)(z-A_n)^{-1}\|(|z|+\sup_n\|A_n\|)<+\infty.$$

Take a circle $\Gamma$ with centre 0 and radius $r=\sup\limits_n\|A_n\|+1$. Then we have

$$r_n(A_n)e^{itA_n}=\frac{1}{2\pi i}\oint_\Gamma e^{itz}r_n(A_n)(z-A_n)^{-1}dz.$$

Therefore

$$|(r_n(A_n)e^{itA_n}x,x)-(r_0(A_0)e^{itA_0}x,x)|$$
$$\leqslant\frac{1}{2\pi}\int_\Gamma|(r_n(A_n)(z-A_n)^{-1}x,x)-(r_0(A_0)(z-A_0)^{-1}x,x)||dz|\to 0$$

holds for $x\in H$, i. e. $w\text{-}\lim\limits_{n\to\infty} r_n(A_n)e^{itA_n}=r_0(A_0)e^{itA_0}$.

Secondly, suppose that $w\text{-}\lim\limits_{n\to\infty} r_n(A_n)e^{itA_n}=r_0(A_0)e^{itA_0}$, $\sup\limits_n\|r_n(A_n)\|<+\infty$, Im $z<0$. Write $Q_n(z,A_n)=((A_n-z)^{-1}(r_n(A_n)-r_n(z)))$. For $x\in H$, from [1] and the illustration given before the present theorem, there exists a monotone increasing function $\sigma_x^{(n)}(t)$ defined on real axis such that

$$\begin{aligned}((z-A_n)^{-1}x,x)&=\frac{1}{r_n(z)}\int\frac{d\sigma_x^{(n)}(t)}{z-t}+\frac{1}{r_n(z)}(Q_n(z,A_n)x,x)\\&=\frac{1}{r_n(z)}\int_{-\infty}^{\infty}\Big(i\int_0^\infty e^{-i\lambda z}e^{i\lambda t}\,d\lambda\Big)d\sigma_x^{(n)}(t)+\frac{1}{r_n(z)}(Q_n(z,A_n)x,x)\\&=\frac{i}{r_n(z)}\int_0^\infty e^{-i\lambda z}\Big(\int_{-\infty}^{\infty}e^{i\lambda t}d\sigma_x^{(n)}(t)\Big)d\lambda+\frac{1}{r_n(z)}(Q_n(z,A_n)x,x)\\&=\frac{i}{r_n(z)}\int_0^\infty e^{-i\lambda z}(r_n(A_n)e^{i\lambda A_n}x,x)d\lambda\\&\quad+\frac{1}{r_n(z)}((r_n(A_n)-r_n(z))(A_n-z)^{-1}x,x).\end{aligned}$$

Hence

$$(r_n(A_n)(z-A_n)^{-1}x,x)=i\int_0^\infty e^{-i\lambda z}(r_n(A_n)e^{i\lambda A_n}x,x)d\lambda.\tag{35}$$

Thus, we have the estimation

$$|(r_n(A_n)(z-A_n)^{-1}x,x)-(r_0(A_0)(z-A_0)^{-1}x,x)|$$
$$\leqslant\int_0^\infty e^{\operatorname{Im}z\cdot t}|(r_n(A_n)e^{itA_n}x,x)-(r_0(A_0)e^{itA_0}x,x)|dt.$$

Moreover, we note that

$$|(r_n(A_n)e^{itA_n}x,x)|=\Big|\int_{-\infty}^{\infty}e^{it\lambda}d\sigma_x^{(n)}(\lambda)\Big|\leqslant\int_{-\infty}^{\infty}d\sigma_x^{(n)}(\lambda)=\sigma_x^{(n)}(+\infty)-\sigma_x^{(n)}(-\infty)$$
$$=(r_n(A_n)x,x)\leqslant\|r_n(A_n)\|\|x\|^2.$$

Since $\sup\limits_n\|r_n(A_n)\|<+\infty$, it follows from Lebesgue dominated convergence theorem that

$$w\text{-}\lim_{n\to\infty} r_n(A_n)(z-A_n)^{-1}=r_0(A_0)(z-A_0)^{-1}.$$

The same argument can be applied to show that $r_n(A_n)(z-A_n)^{-1}$ converges to $r_0(A_0)(z-A_0)^{-1}$ in the sense of weak operator topology for Im $z>0$. The theorem is proved.

In Pontrjagin space $\Pi_k$, all the selfadjoint operators and unitary operators are definitizable operators. The following theorem is a simple generalization of the classical Von Neumann theorem [6].

**Theorem 8.** *Let $U(t)$ be a weakly measurable one-parameter unitary group in separable Pontrjagin space $\Pi_k$. Then $U(t)$ is strongly continuous if and only if there exists $a>0$ such that $\sup\limits_{t\in[0,a]}\|U(t)\|<+\infty$.*

*Proof* The necessity is evident. Let us prove the sufficiency. First, for any $\xi>0$, we claim that $\sup\limits_{t\in[-\xi,\xi]}\|U(t)\|<+\infty$. In fact, if $\xi\leqslant na$, where $n$ is a natural number, then

$$\sup_{\in[0,\xi]}\|U(t)\|\leqslant\sup_{t\in[0,a]}\|U(t)\|\cdot\max_{1\leqslant i\leqslant nt}\|U(ia)\|<+\infty$$

holds. Since $U(-t)=U(t)^*$, we have

$$\sup_{t\in[-\xi,\xi]}\|U(t)\|<+\infty.$$

For $x, y\in\Pi_k$, we notice that

$$\left|\int_0^\xi(U(t)x, y)dt\right|\leqslant\int_0^\xi\|U(t)\|dt\|x\|\|y\|.$$

Thus there exists $y_\xi\in\Pi_k$, for any $x\in\Pi_k$, such that

$$(y_\xi, x)=\int_0^\xi(U(t)y, x)dt.$$

Now, we estimate the difference $(U(t)y_\xi, x)-(y_\xi, x)$. From direct calculus

$$\begin{aligned}|(U(t)y_\xi, x)-(y_\xi, x)|&=\left|\int_0^\xi((U(t+\tau)-U(\tau))y, x)d\tau\right|\\&\leqslant\left|\int_\xi^{\xi+t}(U(\tau)y, x)d\tau\right|+\left|\int_0^t(U(\tau)y, x)d\tau\right|\\&\leqslant(\|U(\xi)\|+1)\left|\int_0^t\|U(\tau)\|d\tau\right|\|x\|\|y\|.\end{aligned}$$

Since $\lim\limits_{t\to0}\int_0^t\|U(\tau)\|d\tau=0$, we conclude that

$$\lim_{t\to0}(U(t)y_\xi, x)=(y_\xi, x). \tag{36}$$

Picking an orthonormal system $\{e_i\}_{i=1}^k\subset\Pi_k$ such that

$$\|\cdot\|^2=(\cdot, \cdot)+2\sum_{i=1}^k|(\cdot, e_i)|^2 \text{ [4]},$$

we have

that

$$w\text{-}\lim_{n\to\infty} r_n(A_n)(z-A_n)^{-1}=r_0(A_0)(z-A_0)^{-1}.$$

The same argument can be applied to show that $r_n(A_n)(z-A_n)^{-1}$ converges to $r_0(A_0)(z-A_0)^{-1}$ in the sense of weak operator topology for Im $z>0$. The theorem is proved.

In Pontrjagin space $\Pi_k$, all the selfadjoint operators and unitary operators are definitizable operators. The following theorem is a simple generalization of the classical Von Neumann theorem [6].

**Theorem 8.** *Let $U(t)$ be a weakly measurable one-parameter unitary group in separable Pontrjagin space $\Pi_k$. Then $U(t)$ is strongly continuous if and only if there exists $a>0$ such that $\sup_{t\in[0,a]}\|U(t)\|<+\infty$.*

*Proof* The necessity is evident. Let us prove the sufficiency. First, for any $\xi>0$, we claim that $\sup_{t\in[-\xi,\xi]}\|U(t)\|<+\infty$. In fact, if $\xi\leqslant na$, where $n$ is a natural number, then

$$\sup_{\in[0,\xi]}\|U(t)\|\leqslant\sup_{t\in[0,a]}\|U(t)\|\cdot\max_{1\wedge i\leqslant nt}\|U(ia)\|<+\infty$$

holds. Since $U(-t)=U(t)^*$, we have

$$\sup_{t\in[-\xi,\xi]}\|U(t)\|<+\infty.$$

For $x, y\in\Pi_k$, we notice that

$$\left|\int_0^\xi (U(t)x, y)dt\right|\leqslant\int_0^\xi\|U(t)\|dt\|x\|\|y\|.$$

Thus there exists $y_\xi\in\Pi_k$, for any $x\in\Pi_k$, such that

$$(y_\xi, x)=\int_0^\xi (U(t)y, x)dt.$$

Now, we estimate the difference $(U(t)y_\xi, x)-(y_\xi, x)$. From direct calculus

$$\begin{aligned}|(U(t)y_\xi, x)-(y_\xi, x)| &= \left|\int_0^\xi ((U(t+\tau)-U(\tau))y, x)d\tau\right|\\ &\leqslant\left|\int_\xi^{\xi+t}(U(\tau)y, x)d\tau\right|+\left|\int_0^t (U(\tau)y, x)d\tau\right|\\ &\leqslant(\|U(\xi)\|+1)\left|\int_0^t\|U(\tau)\|d\tau\right|\|x\|\|y\|.\end{aligned}$$

Since $\lim_{t\to 0}\int_0^t\|U(\tau)\|d\tau=0$, we conclude that

$$\lim_{t\to 0}(U(t)y_\xi, x)=(y_\xi, x). \tag{36}$$

Picking an orthonormal system $\{e_i\}_{i=1}^k\subset\Pi_k$ such that

$$\|\cdot\|^2=(\cdot,\cdot)+2\sum_{i=1}^k|(\cdot, e_i)|^2 \text{ [4]},$$

we have

$$\|U(t)y_\xi-y_\xi\|^2=(U(t)y_\xi-y_\xi,\ U(t)y_\xi-y_\xi)+2\sum_{i=1}^{k}|(U(t)y_\xi-y_\xi,\ e_i)|^2$$
$$=2(y_\xi,\ y_\xi)-(U(t)y_\xi,\ y_\xi)-(y_\xi,\ U(t)y_\xi)$$
$$+2\sum_{i=1}^{k}|(U(t)y_\xi-y_\xi,\ e_i)|^2. \tag{37}$$

Using (36) and taking limit in the both sides of (37), we have

$$\lim_{t\to 0}\|U(t)y_\xi-y_\xi\|=0. \tag{38}$$

Write $\mathscr{L}=\{y_\xi|y\in\Pi_k,\ \xi>0\}$. By an argument like that given for unitary group in Hilbert space[6], one can prove $\overline{\mathscr{L}}=\Pi_k$. Now, applying $\sup_{t\in[0,a]}\|U(t)\|<+\infty$ and (38), we have

$$s\text{-}\lim_{t\to 0}U(t)=I.$$

The theorem is proved.

**Corollary.** *If $U(t)$ is a weakly continuous one-parameter unitary group, then $U(t)$ is strongly continuous.*

*Proof* Since (36) holds for any $x,\ y\in\Pi_k$, by the same way as (37) we have

$$\lim_{t\to 0}\|U(t)y-y\|=0,\quad \forall y\in\Pi_k.$$

**Remark.** For this corollary, the space need not be separable.

By the way, we discuss the self-adjointness of symmetric operators in $\Pi_k$.

**Theorem 9.** *Let $A$ be a symmetric operator in $\Pi_k$, $z$ be a complex number, $\operatorname{Im} z\neq 0$, both $z$ and $\bar z$ do not belong to the approximate point spectrum. The following propositions are equivalence.*

(i) *$A$ is self-adjoint.*

(ii) *$A$ is a closed operator and* $\operatorname{Ker}(A^*-z)=\operatorname{Ker}(A^*-\bar z)=\{0\}$.

(iii) $\operatorname{ran}(A-z)=\operatorname{ran}(A-\bar z)=\Pi_k$.

*Proof* Suppose that $z=x+iy$, $x$ and $y$ are real numbers, $y\neq 0$.

(i)⇒(ii) Certainly, self-adjoint operator $A$ is a closed operator, and

$$\operatorname{Ker}(A^*-z)=\operatorname{Ker}(A-z)=\{0\},\ \operatorname{Ker}(A^*-\bar z)=\operatorname{Ker}(A-\bar z)=\{0\}.$$

(ii)⇒(iii) Suppose that condition (ii) is satisfied. Then

$$\overline{\operatorname{ran}(A-z)}=\overline{\operatorname{ran}(A-\bar z)}=\Pi_k.$$

So to complete the proof, it will suffice to show that both $\operatorname{ran}(A-z)$ and $\operatorname{ran}(A-\bar z)$ are closed subspaces. Suppose that $(A-z)u_n\to v$. We have

$$((A-z)u_n,\ (A-z)u_n)=((A-x)u_n,\ (A-x)u_n)+|y|^2(u_n,\ u_n). \tag{39}$$

Assume that $\{e_i\}_{i=1}^k\subset\Pi_k$, $\|\cdot\|^2=(\cdot,\ \cdot)+2\sum_{i=1}^{k}|(\cdot,\ e_i)|^2$. From (39), it follows that

$$\|(A-z)u_n\|^2-2\sum_{i=1}^{k}|((A-z)u_n,\ e_i)|^2$$
$$=\|(A-x)u_n\|^2+|y|^2\|u_n\|^2-2\sum_{i=1}^{k}\{|((A-x)u_n,\ e_i)|^2+|y|^2|(u_n,\ e_i)|^2\}. \tag{40}$$

By assumption, $\{(A-z)u_n\}$ converges. Since $z\bar{\in}\sigma_a(A)$, $\{\|u_n\|\}$ is bounded. Besides, it follows from $\|(A-x)u_n\|\leqslant\|(A-z)u_n\|+|y|\,\|u_n\|$ that $\{\|(A-x)u_n\|\}$ is bounded. Thus, it is not hard to see that for our purpose we may assume $\{((A-z)u_n, e_i)\}_{n=1}^{\infty}$, $\{((A-x)u_n, e_i)\}_{i=1}^{\infty}$, $\{(u_n, e_i)\}_{n=1}^{\infty}$ $(1\leqslant i\leqslant k)$ all have finite limits. Substitute $u_n$ by $u_n-u_m$ in (40). We find that both $\{(A-x)u_n\}$ and $\{u_n\}$ are Cauchy sequences in $\Pi_k$ space. Since $A$ is a closed operator, we know that there exists $u_0\in\Pi_k$, $u_n\to u_0$ and $(A-z)u_n\to(A-z)\,u_0$. Hence $(A-z)\;u_0=v$. In the same way, one can prove $\operatorname{ran}(A-\bar{z})$ is closed. Thus (iii) is proved.

(iii)⇒(i) Since $\operatorname{ran}(A-z)=\operatorname{ran}(A-\bar{z})=\Pi_k$, for any $u\in\mathscr{D}(A^*)$, there exists $v\in\mathscr{D}(A)$ such that $(A-z)v=(A^*-z)u$, so that $(A^*-z)(u-v)=0$. But $\operatorname{Ker}(A^*-z)=\operatorname{ran}(A-\bar{z})^{\perp}=\{0\}$. Hence $u=v\in\mathscr{D}(A)$ and therefore $\mathscr{D}(A)=\mathscr{D}(A^*)$, i. e. $A$ is a self-adjoint operator. The theorem is proved.

Let $A$ be a symmetric operator in $\Pi_k$ space. If its closure is self-adjoint, then we call $A$ essential self-adjoint. Besides, it is obvious that $z$ belongs to the approximate point spectrum of $A$ if and only if $z$ belongs to the approximate point spectrum of $\bar{A}$. Thus, we have the following result, which is similar to that for symmetric operators defined on Hilbert space.

**Corollary.** *Let $A$ be a symmetric operator in $\Pi_k$, $z$ be a complex number,* $\operatorname{Im} z\neq 0$, *both $z$ nd $\bar{z}$ do not belong to the approximate point spectrum of $A$. Then $A$ is essential self-adjoint if and only if* $\operatorname{Ker}(A^*-z)=\operatorname{Ker}(A^*-\bar{z})=\{0\}$.

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