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OPTIMAL CONTROL PROBLEMS WITH HYBRID QUADRATIC CRITERIA

YOU YUNCHENG (尤云程)*

Abstract

In this paper, the author investigates optimal control problems of continuous time linear system with hybrid quadratic criteria such as $J(u)=\sum_{i=1}^{N}\langle Q_i x(t_i), x(t_i)\rangle+\int_0^{t_f}\langle Ru(t), u(t)\rangle dt$. Closed-loop results are obtained in which the optimal control can be expressed by means of state values measured only at these discrete-time points. Formulae are given for determination of feedback operators.

§ 1. Introduction

The linear-quadratic optimal control of continuous-time type or discrete-time type has a well-known theory[1,2] and extensive applications. Here we consider a continuous-time system to which there are requirements for optimization of its state at several discrete time points as well as its total energy of control, and for restriction of state measurement which can be done only at these discrete time points.

Such a class of problems involves hybrid criteria consisting of both the discrete state terms and the integral control term, and requires a closed-loop optimal control expressed merely by means of those non-advanced state values measured at the discrete time points.

Along this line, three types of problems will be investigated for a linear distributed parameter system. All results are applicable to the finite dimensional case.

Assume that the state and control value spaces, X and U respectively, are real Hilbert spaces. $t_f>0$ is finite. Denote $\mathscr{U}=L^2(0, t_f; U)$. The linear system which we consider is

$$x(t)=T(t)x_0+\int_0^t T(t-\sigma)Bu(\sigma)d\sigma,\quad t\in[0, t_f], \tag{1.1}$$

where $T(t)\ (t\geqslant 0)$ is a C_0-semigroup of linear operators in $\mathscr{L}(X)$, its generator is

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* Institute of Mathematics, Fudan University, Shanghai, China.

A: $\mathscr{D}(A)$ $(\subset X)\to X$; $B\in\mathscr{L}(U;\ X)$; intial state $x_0\in X$ is arbitrarily fixed, and $u\in\mathscr{U}$. The integral involved is a Bochner integral. $x(\cdot)\in C([0,\ t_f];\ X)$.

We make a convention that $T(t)$ and its adjoint operator $T^*(t)$ are identically zero when $t<0$.

Problem (I) Find an optimal control $u(t)$ and its feedback expression by real-time state measurement $x(t)$ of system (1.1) with quadratic criterion

$$\inf_{u\in\mathscr{U}}\left\{ J(u)=\sum_{i=1}^{N}\langle Q_i x(t_i),\ x(t_i)\rangle+\int_0^{t_f}\langle Ru(t),\ u(t)\rangle\, dt\right\}. \tag{1.2}$$

Here, $0=t_0<t_1<\cdots<t_N=t_f$.

Problem (II) Find a feedback expression of the optimal control $u(t)$ of system (1.1) with quadratic criterion (1.2), by means of state measurement $x(t_i)$ at discrete time, $t_i\leqslant t<t_{i+1}$.

Problem (III) Find an optimal control $u(t)$ of system (1.1) with more general hybrid quadratic criterion

$$\inf_{u\in\mathscr{U}}\left\{ J(u)=\sum_{i=1}^{N}\langle Q_i x(t_i),\ x(t_i)\rangle+\int_0^{t_f}[\langle Qx(t),\ x(t)\rangle+\langle Ru(t),\ u(t)\rangle]\,dt\right\}, \tag{1.3}$$

and its feedback expression by means of state measurement $x(t_i)$ at discrete time, $t_i\leqslant t<t_{i+1}$.

In (1.2) and (1.3), $Q_i(i=1,\ \cdots\ N)$ and Q are assumed to be nonnegative self-adjoint operators in $\mathscr{L}(X)$, $R\in\mathscr{L}(U)$ coercively positive so that R^{-1} exists and bounded. In this paper, all the integrals of operator-valued functions will be Bochner integrals in strong sense.

Closed-loop solution of the problems mentioned above will be obtained via different approaches. According to their respective inherences, the feed-back operators of Problems (I) and (II) will be determined by explicit recursion formulae, while the feedback operator function of Problem (III) will be characterized by solution of a system of Fredholm linear integral equations.

§ 2. Closed-Loop Solution of Problem (I)

Adopting the argument similar to [3], we obtain first the existence, uniqueness and open-loop relation of the optimal control of Problem (I).

Theorem 1. *For each given $x_0\in X$, there exists a unique optimal control of Problem (I). $u(\cdot)$ is the optimal control if and only if*

$$u(t)=-R^{-1}B^*\sum_{k=i}^{N}T^*(t_k-t)Q_k x(t_k),\quad t\in[t_{i-1},\ t_i),\ i=1,\ \cdots,\ N, \tag{2.1}$$

where $x(\cdot)$ is the corresponding state trajectory.

Proof Denote $\Gamma_i u=\int_0^{t_i} T(t_i-\sigma)Bu(\sigma)d\sigma$, $i=1, \cdots, N$. $\Gamma_i \in \mathscr{L}(\mathscr{U}; X)$. Substitute $x(t_i)=T(t_i)x_0+\Gamma_i u$ into (1.2), then

$$J(u)=\langle \Phi u, u\rangle_{\mathscr{U}}+2\sum_{i=1}^{N}\langle \Gamma_i^* Q_i T(t_i)x_0, u\rangle_{\mathscr{U}}+\sum_{i=1}^{N}\langle Q_i T(t_i)x_0, T(t_i)x_0\rangle. \tag{2.2}$$

Here $\Phi=RI_{\mathscr{U}}+\sum_{i=1}^{N}\Gamma_i^* Q_i \Gamma_i$ is coercively positive on $\mathscr{U}$. Therefore, the problem of minimizing $J(u)$ has a unique solution $u(\cdot)$ characterized by

$$u(\cdot)=-\Phi^{-1}\sum_{i=1}^{N}\Gamma_i^* Q_i T(t_i)x_0. \tag{2.3}$$

Noting that $[\Gamma_i^* y](t)=\chi_{[0,t_i]}B^*T^*(t_i-t)y$, $t\in[0, t_f]$, $\forall y\in X$, $i=1, \cdots, N$, We obtain that (2.3) is equivalent to (2.1).

We make use of the method of Dynamic Programming to find closed-loop solution of Problem (I). Denote $\mathscr{U}_i=L^2(t_{i-1}, t_i; U)$, $i=1, \cdots, N$. According to the optimality principle, the following relation is true:

$$\begin{aligned}\inf_{u\in\mathscr{U}} J(u)=\inf_{u_1\in\mathscr{U}_1}\Big\{&\langle Q_1x(t_1), x(t_1)\rangle+\int_0^{t_1}\langle Ru_1(t), u_1(t)\rangle dt+\inf_{u_2\in\mathscr{U}_2}[\langle Q_2x(t_2), x(t_2)\rangle\\&+\int_{t_1}^{t_2}\langle Ru_2(t), u_2(t)\rangle dt+\cdots+\inf_{u_N\in\mathscr{U}_N}(\langle Q_Nx(t_N), x(t_N)\rangle\\&+\int_{t_{N-1}}^{t_N}\langle Ru_N(t), u_N(t)\rangle dt)\cdots]\Big\}.\end{aligned} \tag{2.4}$$

Moreover, u is optimal solution of the left-side of (2.4) if and only if $\{u_i=u|_{[t_{i-1},t_i)}; i=1, \cdots, N\}$ is an optimal solution of the right-side of (2.4).

Based on (2.4), we can prove the closed-loop result of Problem (I).

Theorem 2. *For each given $x_0\in X$, $u(\cdot)$ is the optimal control of Problem (I) if and only if it is the state feedback as follows,*

$$u(t)=\begin{cases}-R^{-1}B^*P_1(t)x(t), & t\in[0, t_1),\\ \cdots & \cdots\\ -R^{-1}B^*P_N(t)x(t), & t\in[t_{N-1}, t_N],\end{cases} \tag{2.5}$$

where $x(\cdot)$ is the corresponding state and $P_i(\cdot)$ are determined by the following

$$P_N(t)=T^*(t_f-t)\sqrt{Q_N}[I+\sqrt{Q_N}\Lambda_N(t)\sqrt{Q_N}]^{-1}\sqrt{Q_N}T(t_f-t),\ t\in[t_{N-1}, t_f], \tag{2.6}_N$$

$$\begin{aligned}P_i(t)=T^*(t_i-t)\sqrt{Q_i+P_{i+1}(t_i)}\,&[I+\sqrt{Q_i+P_{i+1}(t_i)}\,\Lambda_i(t)\sqrt{Q_i+P_{i+1}(t_i)}]^{-1}\\&\sqrt{Q_i+P_{i+1}(t_i)}T(t_i-t),\quad t\in[t_{i-1}, t_i],\ i=1,\cdots,N-1,\end{aligned} \tag{2.6}_i$$

where $\Lambda_i(t)=\int_t^{t_i}T(t_i-\sigma)BR^{-1}B^*T^*(t_i-\sigma)d\sigma$, $t\in[t_{i-1}, t_i]$, $i=1, \cdots, N$, and,

$$\inf_{u\in\mathscr{U}} J(u)=\langle P_1(0)x_0, x_0\rangle. \tag{2.7}$$

Proof In view of (2.4), we first deal with the optimal control subproblem of system (1.1) on the time interval $[t_{N-1}, t_f]$ with criterion as follows

$$\begin{cases} x(t)=T(t-t_{N-1})x(t_{N-1})+\int_{t_{N-1}}^{t} T(t-\sigma)\, Bu_N(\sigma)d\sigma, \quad t\in[t_{N-1}, t_f], & (2.8)\\ \inf_{u_N\in\mathscr{U}_N}\left\{J_N(u_N;\, x(t_{N-1}))=\langle Q_N x(t_N), x(t_N)\rangle+\int_{t_{N-1}}^{t_N}\langle Ru_N(t), u_N(t)\rangle dt\right\}. & (2.9)\end{cases}$$

From the standard result[2], this optimal control is given by

$$u_N(t)=-R^{-1}B^*P_N(t)x(t), \quad t\in(t_{N-1}, t_f], \tag{2.10}$$

where $P_N(\cdot)$ is the unique strongly continuous and self-adjoint solution of the following operator Riccati integral equation[3]

$$P_N(t)=T^*(t_f-t)Q_N T(t_f-t) - \int_t^{t_f} T^*(\sigma-t)P_N(\sigma)BR^{-1}B^*P_N(\sigma)T(\sigma-t)d\sigma, \quad t\in[t_{N-1}, t_f]. \tag{2.11}$$

Moreover, similar to Theorem 3.2 of [4], the solution of (2.11) is given by $(2.6)_N$. Besides, $\inf_{u_N\in\mathscr{U}_N} J_N(u_N;\, x(t_{N-1}))=\langle P_N(t_{N-1})x(t_{N-1}),\, x(t_{N-1})\rangle$, and we know $P_N(t_{N-1})\geqslant 0$.

Inductively we deal with a sequence of optimal control subproblems of system (1.1) on $[t_{i-1}, t_i]$ with the following criterion

$$\inf_{u_i\in\mathscr{U}_i}\left\{J_i(u_i;\, x(t_{i-1}))=\langle (Q_i+P_{i+1}(t_i))x(t_i),\, x(t_i)\rangle+\int_{t_{i-1}}^{t_i}\langle Ru_i(t),\, u_i(t)\rangle dt\right\}, \tag{2.12}$$

$i=N-1, \cdots, 1$. By the same reason as before, recursively, we obtain the conclusions of this theorem.

Remark. As $u(t_i)-u(t_i-0)=R^{-1}B^*Q_i x(t_i)$, in this case, the optimal control function $u(\cdot)$ is piecewise strongly continuous with at most finite number of discontinuous points at $t_1, \cdots, t_{N-1}$, and is bounded on $[0, t_f]$.

Theorem 3. *The optimal state trajectory $x(\cdot)$ of problem (I) is given by*

$$x(t)=G(t, s)x(s)=G(t, 0)x_0, \quad 0\leqslant s\leqslant t\leqslant t_f, \tag{2.13}$$

where $G(t,s)$ is the mild evolution operator determined by the mild solution of $\frac{dx}{dt}=[A-BR^{-1}B^*P(t)]x(t)$, here $P(t)=P_i(t)$, $t\in[t_{i-1}, t_i)$, $i=1, \cdots, N-1$, $P(t)=P_N(t)$, $t\in[t_{N-1}, t_f]$. And the feedback operators shown by (2.6) satisfy

$$P_i(t)=\sum_{k=i}^{N} T^*(t_k-t)Q_k G(t_k, t), \quad t\in[t_{i-1}, t_i),\ i=1,\cdots, N. \tag{2.14}$$

Proof By (2.6), the feedback operator function $P(\cdot)$ is piecewise strongly continuous and its norm is uniformly bounded on $[0, t_f]$. Thus we have $-BR^{-1}B^*P(\cdot)\in L^\infty(0, t_f;\, \mathscr{L}(X))$. According to [2] Theorem 2.33 and Definition 2.32, mild evolution operator $G(t, s)$ with $A\text{-}BR^{-1}B^*P(t)$ as its generator family is uniquely determined by the operator integral equation

$$G(t, s)=T(t-s)-\int_s^t T(t-\sigma)BR^{-1}B^*P(\sigma)G(\sigma, s)d\sigma$$

$$=T(t-s)-\int_s^t G(t, \sigma)BR^{-1}B^*P(\sigma)T(\sigma-s)d\sigma, \quad 0\leqslant s\leqslant t\leqslant t_f, \quad (2.15)$$

and from (2.5) it follows that the optimal state trajectory must be given by (2.13). Besides, similar to the proof of [3], Lemma 4, we get

$$P_N(t)=T^*(t_N-t)Q_NG(t_N, t), \quad t\in[t_{N-1}, t_N].$$

Inductively, for $i=1, \cdots, N-1$, we get

$$P_i(t)=T^*(t_i-t)[Q_i+P_{i+1}(t_i)]G(t_i, t)=\sum_{k=i}^N T^*(t_k-t)Q_kG(t_k, t), \quad t\in[t_{i-1}, t_i].$$

Thus (2.14) holds.

§ 3. Closed-Loop Solution of Problem (II)

Now that Problem (II) restricts us to express the optimal control by means of the feedback of non-advanced state values measured at discrete-time points, we take another point of view to find its closed-loop solution.

Let $\delta=\min\limits_{1\leqslant i\leqslant N}\{t_i-t_{i-1}\}$. In this section we make a hypothesis:

$$W\equiv\int_0^\delta T(s)BR^{-1}B^*T^*(s)ds \text{ is coercively positive.} \quad (3.1)$$

Lemma 1. *For each given $x_0\in X$, the following relation is true,*

$$\inf_{u\in\mathscr{U}} J(u)=\inf_{\{x_1,\cdots,x_N\}\subset X}\{\inf_{u\in\mathscr{U}}\{J(u)\,|\,x(t_1)=x_1, \cdots, x(t_N)=x_N\}\}. \quad (3.2)$$

Moreover, u is optimal solution of the left-side of (3.2) if and only if u with corresponding $\{x(t_i; u)\,|\,i=1, \cdots, N\}$ is optimal solution of the right-side of (3.2).

This lemma can be proved by the argument of opposite inequalities.

For any given set of points $\{x_1, \cdots, x_N\}$ in X, we denote

$$\hat{J}(x_1, \cdots, x_N)=\inf_{u\in\mathscr{U}}\{J(u)\,|\,x(t_i)=x_i, i=0, 1, \cdots, N\}. \quad (3.3)$$

Lemma 2. *For any given set of points $\{x_1, \cdots, x_N\}\subset X$, the minimization problem* (1.1)—(3.3) *has a unique solution:*

$$u(t)=R^{-1}B^*T^*(t_i-t)W_i^{-1}[x_i-T(t_i-t_{i-1})x_{i-1}], \; t\in[t_{i-1}, t_i), \; i=1, \cdots, N. \quad (3.4)$$

where

$$W_i=\int_{t_{i-1}}^{t_i} T(t_i-\sigma)BR^{-1}B^*T^*(t_i-\sigma)d\sigma, \quad i=1, \cdots, N. \quad (3.5)$$

$$\hat{J}(x_1, \cdots, x_N)=\sum_{i=1}^N[\langle Q_ix_i, x_i\rangle+\langle W_i^{-1}(x_i-T(t_i-t_{i-1})x_{i-1}), x_i-T(t_i-t_{i-1})x_{i-1}\rangle]. \quad (3.6)$$

Proof The hypothesis (3.1) implies that W_i^{-1} exists and is bounded, $i=1, \cdots, N$. Denote

$$y_i=x_i-T(t_i-t_{i-1})x_{i-1}, \quad i=1,\cdots, N. \quad (3.7)$$

We see, for any given set $\{x_1, \cdots, x_N\}\subset X$,

$$\hat{J}(x_1, \cdots, x_N) = \sum_{i=1}^{N} \langle Q_i x_i, x_i \rangle + \sum_{i=1}^{N} \inf_{u_i \in \mathscr{U}_i} \left\{ \int_{t_{i-1}}^{t_i} \langle Ru_i(t), u_i(t) \rangle dt \,\middle|\, \int_{t_{i-1}}^{t_i} T(t_i - \sigma) Bu_i(\sigma) d\sigma = y_i \right\}. \tag{3.8}$$

Because we have the following inequality for any $u_i \in \mathscr{U}_i$ satisfying the constraint

$$\int_{t_{i-1}}^{t_i} T(t_i - \sigma) Bu_i(\sigma) d\sigma = y_i,$$

$$\int_{t_{i-1}}^{t^i} \langle Ru_i(t), u_i(t) \rangle$$
$$= \int_{t_{i-1}}^{t_i} \langle R[u_i(t) - R^{-1}B^*T^*(t_i - t)W_i^{-1}y_i], u_i(t) - R^{-1}B^*T^*(t_i - t)W_i^{-1}y_i \rangle dt + \langle W_i^{-1}y_i, y_i \rangle \geqslant \langle W_i^{-1}y_i, y_i \rangle, \quad i = 1, \cdots, N, \tag{3.9}$$

the minimization problem (1.1)—(3.3) has a unique solution given by (3.4), and (3.6) holds.

Lemma 3. *For $\hat{J}(x_1, \cdots, x_N)$: $X^N \to \mathbf{R}$ given by (3.6), the following problem*

$$\inf_{\{x_1, \cdots, x_N\} \subset X} \hat{J}(x_1, \cdots, x_N) \tag{3.10}$$

has a unique optimal solution given by

$$x_i = \Pi_i^{-1} W_i^{-1} T(t_i - t_{i-1}) x_{i-1}, \quad i = 1, \cdots, N, \tag{3.11}$$

where operators Π_i are defined to be

$$\Pi_N = Q_N + W_N^{-1},$$
$$\Pi_i = Q_i + W_i^{-1} + T^*(t_{i+1} - t_i) H_{i+1} T(t_{i+1} - t_i), \quad i = N-1, \cdots, 1, \tag{3.12}$$

and operators H_i are defined to be

$$H_i = W_i^{-1} - W_i^{-1} \Pi_i^{-1} W_i^{-1}, \quad i = N, N-1, \cdots, 1. \tag{3.13}$$

Proof Hypothesis (3.1) implies W_i^{-1} coercively positive. $\Pi_N \geqslant W_N^{-1}$ so that Π_N is invertible. $H_N = W_N^{-1} - (W_N Q_N W_N + W_N)^{-1} \geqslant 0$. And we obtain by induction that Π_i is invertible and $H_i \geqslant 0$, $i = N-1, \cdots, 1$. Thus Π_i and H_i in (3.12) and (3.13) are well-defined.

The solution of the problem (3.10) must make the first-order Fréchet derivatives of the quadratic functional (3.6) vanish, i. e.,

$$Q_i x_i + W_i^{-1}[x_i - T(t_i - t_{i-1}) x_{i-1}] - T^*(t_{i+1} - t_i) W_{i+1}^{-1}[x_{i+1} - T(t_{i+1} - t_i) x_i] = 0, \quad i = 1, \cdots, N-1;$$
$$Q_N x_N + W_N^{-1}[x_N - T(t_N - t_{N-1}) x_{N-1}] = 0. \tag{3.14}$$

From (3.14) we find the optimal solution of (3.10) recursively determined by (3.11) with the given initial state x_0.

Furthermore, the second-order Fréchet derivatives of the quadratic functional (3.6) form a non-negative self-adjoint operator in $\mathscr{L}(X^N)$ as follows:

$$\Psi=[D^2_{x_ix_j}\hat{J}]_{i,j=1,\cdots,N}$$

$$=\left[\begin{array}{c|c|c|c|c|c}
\begin{array}{c}Q_1+W_1^{-1}+T^*(t_2-t_1)W_2^{-1}T(t_2-t_1)\end{array} & -T^*(t_2-t_1)W_2^{-1} & 0 & \cdots & 0 & 0\\
-W_2^{-1}T(t_2-t_1) & Q_2+W_2^{-1}+T^*(t_3-t_2)W_3^{-1}T(t_3-t_2) & -T^*(t_3-t_2)W_3^{-1} & \cdots & 0 & 0\\
0 & -W_3^{-1}T(t_3-t_2) & Q_3+W_3^{-1}+T^*(t_4-t_3)W_4^{-1}T(t_4-t_3) & \cdots & 0 & 0\\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & \cdots & -W_N^{-1}T(t_N-t_{N-1}) & Q_N+W_N^{-1}
\end{array}\right]$$

$$\geqslant 0. \tag{3.15}$$

Therefore, the set $\{x_1, \cdots, x_N\}$ determined by (3.11) is exactly the minimum solution of (3.10).

Thus we obtain the closed-loop result of Problem (II) described below.

Theorem 4. *Under the hypothesis (3.1), $u(\cdot)$ is the optimal control of Problem (II) if and only if it is the state feedback given by*

$$u(t)=-R^{-1}B^*T^*(t_i-t)H_iT(t_i-t_{i-1})x(t_{i-1}), \quad t\in[t_{i-1}, t_i),\ i=1, \cdots, N, \tag{3.16}$$

where $x(\cdot)$ is the corresponding state trajectory, operators H_i are given by

$$H_N=Q_N(W_NQ_N+I)^{-1},$$

$$H_i=[Q_i+T^*(t_{i+1}-t_i)H_{i+1}T(t_{i+1}-t_i)]$$
$$\cdot[W_iQ_i+W_iT^*(t_{i+1}-t_i)H_{i+1}T(t_{i+1}-t_i)+I]^{-1}, \quad i=1, \cdots, N-1. \tag{3.17}$$

Proof The results of Lemma 1 to Lemma 3 imply that the optimal control $u(\cdot)$ of Problem (II) (its existence and uniqueness have been proved by Theorem 1) is given by (3.16) and vice versa. By (3.12) and (3.13), we have

$$H_N=W_N^{-1}-(W_NQ_NW_N+W_N)^{-1}=Q_N(W_NQ_N+I)^{-1},$$

$$H_i=W_i^{-1}-[W_iQ_iW_i+W_i+W_iT^*(t_{i+1}-t_i)H_{i+1}T(t_{i+1}-t_i)W_i]^{-1}$$
$$=[Q_i+T^*(t_{i+1}-t_i)H_{i+1}T(t_{i+1}-t_i)]$$
$$\cdot[W_iQ_i+W_iT^*(t_{i+1}-t_i)H_{i+1}T(t_{i+1}-t_i)+I]^{-1}, \quad 1\leqslant i\leqslant N-1.$$

Remark. In case the hypothesis (3.1) does not hold, results in § 4 are available.

§ 4. Closed-Loop Solution of Problem (III)

The feature of closed-loop solutions of Problems (I) and (II), (2.5)—(2.6) and (3.16)—(3.17), respectively, is the explicit expressions of their feedback operators $P_i(t)$ and H_i. However, the feedback operator of Problem (III) will be characterized by the solution of an operator linear integral equation inherently because of the appearance of state integral term in the criterion.

Theorem 5. *For each given $x_0 \in X$, there exists a unique optimal control of Problem (III). $u(\cdot)$ is the optimal control if and only if*

$$u(t) = -R^{-1}B^*\Big[\sum_{k=i}^{N} T^*(t_k - t)Q_k x(t_k) + \int_t^{t_f} T^*(\sigma - t)Qx(\sigma)d\sigma\Big],$$
$$t \in [t_{i-1}, t_i),\ i = 1, \cdots, N, \tag{4.1}$$

where $x(\cdot)$ is the corresponding state trajectory.

Proof Analogous to § 1 Theorem 1 and [3] Theorem 2.

Theorem 6. *For each given $x_0 \in X$, $u(\cdot)$ is the optimal control of Problem (III) if and only if it is the state feedback given by*

$$u(t) = -R^{-1}B^*P_i(t)x(t), \quad t \in [t_{i-1}, t_i),\ i = 1, \cdots, N, \tag{4.2}$$

where $\{P_i(\cdot)\}_{i=1}^N$ is the unique strongly continuous and self-adjoint solution of the follawing operator Riccati integral equations

$$P_N(t) = T^*(t_f - t)Q_N T(t_f - t) + \int_t^{t_f} T^*(\sigma - t)[Q - P_N(\sigma)BR^{-1}B^*P_N(\sigma)]T(\sigma - t)d\sigma, \quad t \in [t_{N-1}, t_f];$$
$$P_i(t) = T^*(t_i - t)[Q_i + P_{i+1}(t_i)]T(t_i - t) + \int_t^{t_i} T^*(\sigma - t)[Q - P_i(\sigma)BR^{-1}B^*P_i(\sigma)]T(\sigma - t)d\sigma,$$
$$t \in [t_{i-1}, t_i],\ i = 1, \cdots, N-1. \tag{4.3}$$

Let $P(t) = P_i(t)$, $t = [t_{i-1}, t_i)$, $i = 1, \cdots, N$. The optimal state strajectory is given by

$$x(t) = G(t, s)x(s) = G(t, 0)\,x_0, \quad 0 \leqslant s \leqslant t \leqslant t_f, \tag{4.4}$$

where $G(t, s)$ is the mild evolution operator generated by $A\text{-}BR^{-1}B^*P(t)$.

Proof Here $P(t)$, $t \in [0, t_f]$, given by (4.3) is piecewise strongly continuous and its norm is uniformly bounded. These results can be proved similarly to § 1, Theorems 2 and 3.

We must find out the discrete point state feedback required by this Problem (III).

Lemma 4. *$u(\cdot)$ is the optimal control of Problem (III) if and only if*

$$u(t) = -R^{-1}B^*F_i(t)x(t_{i-1}), \quad t \in [t_{i-1}, t_i),\ i = 1, \cdots, N, \tag{4.5}$$

where $x(t_i)$ is the corresponding state value at t_i, and

$$F_i(t) = \sum_{k=i}^{N} T^*(t_k - t)Q_k G(t_k, t_{i-1}) + \int_t^{t_f} T^*(\sigma - t)QG(\sigma, t_{i-1})d\sigma, \quad t \in [t_{i-1}, t_i],\ i = 1, \cdots, N. \tag{4.6}$$

Proof By Substituting (4.4) into (4.1), the optimal control $u(\cdot)$ of Problem (III) must satisfy (4.5). On the other hand, if we begin with x_0, the control function determined by (4.5) and (4.6) is unique.

Lemma 5. *The sequenc e of operator functions $\{F_i(\cdot)\}_{i=1}^N$ defined by (4.6) is strongly continuous solution of the following system of operator Fredholm linear integral* *equations,*

$$F_i(t)+\int_{t_{i-1}}^{t_i} K_i(t,\sigma)BR^{-1}B^*F_i(\sigma)d\sigma=K_i(t,t_{i-1}),\quad t\in[t_{i-1},t_i],\ i=1,\cdots,N,\tag{4.7}$$

where

$$K_N(t,\sigma)=T^*(t_f-t)Q_NT(t_f-\sigma)+\int_t^{t_f}T^*(\eta-t)QT(\eta-\sigma)d\eta,\ (t,\sigma)\in[t_{N-1},t_f]^2;\tag{4.8$_N$}$$

$$K_i(t,\sigma)=T^*(t_i-t)[Q_i+F_{i+1}(t_i)]T(t_i-\sigma)$$
$$+\int_t^{t_i}T^*(\eta-t)QT(\eta-\sigma)d\eta,\ (t,\sigma)\in[t_{i-1},t_i]^2,\quad i=1,\cdots,N-1.\tag{4.8$_i$}$$

Proof Obviously, $F_i(\cdot)$ defined by $(4.6)_i$ on $[t_{i-1},t_i]$ is strongly continuous, $i=1,\cdots,N$. As $G(t,s)$ satisfies (2.15) similar to the case in § 1, from (4.4), (4.5) and (4.6), we can establish the following relation of $G(\cdot,\cdot)$:

$$G(t,t_{N-1})=T(t-t_{N-1})-\int_{t_{N-1}}^{t}T(t-s)BR^{-1}B^*T^*(t_f-s)Q_NG(t_f,t_{N-1})ds$$
$$-\int_{t_{N-1}}^{t_f}\Big(\int_{t_{N-1}}^{\min(t,\eta)}T(t-s)BR^{-1}B^*T^*(\eta-s)ds\Big)QG(\eta,t_{N-1})d\eta,$$
$$t\in[t_{N-1},t_N].\tag{4.9}$$

Now we verify directly that $F_N(\cdot)$ defind by $(4.6)_N$ satisfies the integral equation $(4.7)_N$. In fact

$$\int_{t_{N-1}}^{t_f}K_N(t,\sigma)BR^{-1}B^*F_N(\sigma)d\sigma$$
$$=T^*(t_f-t)Q_N\int_{t_{N-1}}^{t_f}T(t_f-\sigma)BR^{-1}B^*T^*(t_f-\sigma)Q_NG(t_f,t_{N-1})d\sigma$$
$$+T^*(t_f-t)Q_N\int_{t_{N-1}}^{t_f}T(t_f-\sigma)BR^{-1}B^*\int_\sigma^{t_f}T^*(\eta-\sigma)QG(\eta,t_{N-1})d\eta\, d\sigma$$
$$+\int_{t_{N-1}}^{t_f}\int_t^{t_f}T^*(\eta-t)QT(\eta-\sigma)d\eta BR^{-1}B^*T^*(t_f-\sigma)Q_NG(t_f,t_{N-1})d\sigma$$
$$+\int_{t_{N-1}}^{t_f}\int_t^{t_f}T^*(\eta-t)QT(\eta-\sigma)d\eta BR^{-1}B^*\int_\sigma^{t_f}T^*(\xi-\sigma)QG(\xi,t_{N-1})d\xi\, d\sigma$$
$$=T^*(t_f-t)Q_N\Big\{\int_{t_{N-1}}^{t_f}T(t_f-\sigma)BR^{-1}B^*T^*(t_f-\sigma)Q_NG(t_f,t_{N-1})d\sigma$$
$$+\int_{t_{N-1}}^{t_f}\Big(\int_{t_{N-1}}^{\eta}T(t_f-\sigma)BR^{-1}B^*T^*(\eta-\sigma)d\sigma\Big)QG(\eta,t_{N-1})d\eta\Big\}$$
$$+\int_{t_{N-1}}^{t_f}T^*(\eta-t)Q\Big\{\int_{t_{N-1}}^{\eta}T(\eta-\sigma)BR^{-1}B^*T^*(t_f-\sigma)Q_NG(t_f,t_{N-1})d\sigma$$
$$+\int_{t_{N-1}}^{t_f}\Big(\int_{t_{N-1}}^{\min(\eta,\xi)}T(\eta-\sigma)BR^{-1}B^*T^*(\xi-\sigma)d\sigma\Big)QG(\xi,t_{N-1})d\xi\Big\}d\eta$$
$$=T^*(t_f-t)Q_N\{T(t_f-t_{N-1})-G(t_f,t_{N-1})\}$$
$$+\int_{t_{1-N}}^{t_f}T^*(\eta-t)Q\{T(\eta-t_{N-1})-G(\eta,t_{N-1})\}d\eta$$
$$=K_N(t,t_{N-1})-F_N(t),\quad t\in[t_{N-1},t_N].\tag{4.10}$$

In this verification, we have used the interchange of the order of integration and the relation (4.9). Hence, (4.10) indicates that $F_N(\cdot)$ satisfies the equation $(4.7)_N$.

When $i=1, \cdots, N-1$, according to (4.6), $F_i(\cdot)$ is given by

$$
\begin{aligned}
F_i(t) &= T^*(t_i-t)\Big[Q_i+\sum_{k=i+1}^{N} T^*(t_k-t_i)Q_kG(t_k, t_i)\Big]G(t_i, t_{i-1}) \\
&\quad +\int_t^{t_i} T^*(\sigma-t)QG(\sigma, t_{i-1})d\sigma+T^*(t_i-t)\int_{t_i}^{t_f} T^*(\sigma-t_i)QG(\sigma, t_i)d\sigma G(t_i, t_{i-1}) \\
&= T^*(t_i-t)\,[Q_i+F_{i+1}(t_i)]\,G(t_i, t_{i-1}) \\
&\quad +\int_t^{t_i} T^*(\sigma-t)QG(\sigma, t_{i-1})d\sigma, \quad t\in[t_{i-1}, t_i].
\end{aligned} \tag{4.11}
$$

By $(4.8)_i$ and (4.11), replacing t_{N-1}, t_N and Q_N respectively by t_{i-1}, t_i and $Q_i+F_{i+1}(t_i)$ in (4.9) and (4.10), through an analogous verification, we can conclude that $F_i(\cdot)$ satisfies the equation $(4.7)_i$, $i=1, \cdots, N-1$.

Lemma 6. *For $i=1, \cdots, N$, the strongly continuous solution of the operator integral equation $(4.7)_i$ is unique.*

Proof It is enough to show that the strongly continuous solution $\widetilde{F}_i(\cdot)$ of the following homogeneous equation is identically zero, $i=N, N-1, \cdots, 1$,

$$\widetilde{F}_i(t)+\int_{t_{i-1}}^{t_i} K_i(t, \sigma)BR^{-1}B^*\widetilde{F}_i(\sigma)d\sigma=0, \quad t\in[t_{i-1}, t_i]. \tag{4.12}$$

By (4.12), the function $\psi_i(t)=R^{-1}B^*\widetilde{F}_i(t)$ satisfies the following equation

$$R\psi_i(t)+\int_{t_{i-1}}^{t_i} B^*K_i(t, \sigma)B\psi_i(\sigma)d\sigma=0, \quad t\in[t_{i-1}, t_i]. \tag{4.13}$$

We define a sequence of self-adjoint linear bounded operators on $\mathscr{U}_i$ as follows:

$$
\begin{aligned}
V_i &= RI_{\mathscr{U}_i}+M_i^*QM_i+M_{1i}^*[Q_i+F_{i+1}(t_i)]M_{0i}, \quad i=1, \cdots, N-1, \\
V_N &= RI_{\mathscr{U}_N}+M_N^*QM_N+M_{1N}^*Q_NM_{1N},
\end{aligned} \tag{4.14}
$$

where $M_i\in\mathscr{L}(\mathscr{U}_i; L^2(t_{i-1}, t_i\ X))$ and $M_{1i}\in\mathscr{L}(\mathscr{U}_i; X)$ are defined as follows:

$$[M_i\varphi](t)=\int_{ti-1}^{t} T(t-\sigma)B\varphi(\sigma)d\sigma, \ t\in[t_{i-1}, t_i]; \qquad M_{1i}\varphi=\int_{t_{i-1}}^{t_i} T(t_i-\sigma)B\varphi(\sigma)d\sigma.$$

Then, considering the concrete formof the operator V_i, we know that the equation (4.13) is just

$$V_i\psi_i(\cdot)=0. \tag{4.15}$$

However, it is easy to see that V_i is coercively positive, so it is invertible, thus $\psi_i(\cdot)=0$ in $\mathscr{U}_i$ and in $C([t_{i-1}, t_i]; U)$ as it is strongly continuous. Then substitute this into (4.12), we obtain $\widetilde{F}_i(t)\equiv0$, $t\in[t_{i-1}, t_i]$. Q. E. D.

Based on these lemmas, we obtain finally the closed-loop result of Problem (III).

Theorem 7. *For each given $x_0\in X$, $u(\cdot)$ is the optimal control of Problem (III) if and only if it is the state feedback given by*

$$u(t)=-R^{-1}B^*F_i(t)x(t_{i-1}), \quad t\in[t_{i-1}, t_i), \ i=1, \cdots, N, \tag{4.16}$$

where $\{F_i(\cdot)\}_{i=1}^N$ is the unique strongly continuous solution of the system of operator *integral equations* (4.7).

Proof This is the synthesis of the results obtained in Lemma 4 to Lemma 6.

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