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A REMARK ON KOLMOGOROV'S COMPARISON THEOREM

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Abstract

In approximation theory the theorem of Kolmogorov concerning the comparison of derivatives of differentiable functions defined on the real line is well-known. It plays an important rôle in establishing sharp inequalities between the norms of derivatives of a function. In this note we establish a comparison theorem of Kolmogorov type on a class of functions which are defined on the real line and can be continuated analytically in a stripped region containing the real line. As a consequence we have derived an inequality of Landau-Kolmogorov type on this function class, and moreover, we have applied it to get the exact estimation for the Kolmogorov's N-widths of the analytic function class.

§ 1. Preliminaries

In approximation theory, Kolmogorov's theorem concerning the comparison of derivatives of differentiable functions is well known[1]. It serves as a basic tool for establishing some sharp inequalities between norms of derivatives. In this remark we will give some comparison theorems of Kolmogorov type for a class of analytic functions and will apply them to the computation of Kolmogorov's width numbers. As in our previous paper[2], we consider the set $\tilde{H}_\delta(L_p)$ which is defined as follows. For $1<p\leqslant+\infty$, $f(x)\in\tilde{H}_\delta(L_p)\Leftrightarrow$

$$f(x)=\frac{1}{2\pi}\int_0^{2\pi}H(x-t)h(t)dt, \tag{1}$$

$\|h\|_p=\left\{\int_0^{2\pi}|h(t)|^p dt\right\}^{\frac{1}{p}}\leqslant 1$ ($1\leqslant p<+\infty$), $\|h\|_\infty=\text{ess sup}\ |h(t)|$. For $p=1$, $f(x)\in\tilde{H}_\delta(L_1)\Leftrightarrow$

$$f(x)=\frac{1}{2\pi}\int_0^{2\pi}H(x-t)d\lambda(t), \tag{2}$$

where $\lambda(t)\in V[0, 2\pi]$, $\int_0^{2\pi}|d\lambda|\leqslant 1$. The kernel is

$$H(x)=1+4\sum_{k=1}^{\infty}\frac{\cos kx}{\operatorname{ch} k\delta},\ \delta>0. \tag{3}$$

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Denote $E_n(f)_p=\min_{t_n\in T_n}\|f-t_n\|_p$, $T_n=\text{span}\left\{1, \begin{matrix}\sin t\\ \cos t\end{matrix}, \cdots, \begin{matrix}\sin(n-1)t\\ \cos(n-1)t\end{matrix}\right\}$, Н. И. Ахиезер[3] proved

$$E_n(H_\delta(L_\infty))_\infty=\sup_{f\in H_\delta(L_\infty)}E_n(f)_\infty=(2\pi)^{-1}E_n(H)_1=\|f_{n\delta}\|_\infty, \tag{4}$$

where

$$f_{n\delta}(x)=\frac{1}{2\pi}\int_0^{2\pi}H(x-t)\ \text{sgn}\cos nt\,dt=\frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{\cos(2\nu+1)nx}{(2\nu+1)\text{ch}(2\nu+1)n\delta}. \tag{5}$$

We call $f_{n\delta}$ standard in $H_\delta(L_\infty)$. It has the following properties:

(1) $f_{n\delta}\left(x+\frac{\pi}{n}\right)=-f_n(x)$.

(2) $\xi_k=\frac{\pi}{2n}+\frac{k\pi}{n}$ $(k=0, \cdots, 2n-1)$ are zeros of $f_{n\delta}$ in $[0, 2\pi)$. $x_k=\frac{k\pi}{n}$ $(k=0, \cdots, 2n-1)$ are extremal points of $f_{n\delta}$ in $[0, 2\pi)$.

(3) $f_{n\delta}(x_k)=(-1)^k\|f_{n\delta}\|_\infty$.

(4) $f_{n\delta}$ is strictly monotonic in $\Delta_k=\left(\frac{k\pi}{n}, \frac{k+1}{n}\pi\right)$, $k=0, \pm1, \pm2, \cdots$.

§ 2. Comparison Theorems of Kolmogorov Type on $H_\delta L_\infty$

Theorem 1. *Let $f(x)\in H_\delta(L_\infty)$ be such that $\|f\|_\infty\leqslant\|f_{n\delta}\|_\infty$ for some positive integer n and $f(a)=f_{n\delta}(\alpha)$ for $a, \alpha\in R$. Then*

$$|f'(a)|\leqslant|f'_{n\delta}(\alpha)|. \tag{6}$$

Proof Without loss of generality we may assume $a=\alpha$. Suppose that Theorem 1 is not true. Then for some $f(x)\in H_\delta(L_\infty)$, positive integer n and $\alpha\in R$, we have $\|f\|_\infty\leqslant\|f_{n\delta}\|_\infty$, $f(\alpha)=f_{n\delta}(\alpha)$ and $|f'(\alpha)|>|f'_{n\delta}(\alpha)|$. The continuity of f' and $f'_{n\delta}$ ensures the existence of $\rho>1$, $\beta\in R$ such that

$$\frac{1}{\rho}f(\beta)=f_{n\delta}(\beta),\ \frac{1}{\rho}|f'(\beta)|>|f'_{n\delta}(\beta)|.$$

Denote $\bar{f}(x)=\frac{1}{\rho}f(x)$. We have $\bar{f}\in H_\delta(L_\infty)$ and $\|\bar{f}\|_\infty<\|f_{n\delta}\|_\infty$. It is enough to consider one possible case $\bar{f}(\beta)=f_{n\delta}(\beta)\geqslant0$, $\bar{f}'(\beta)>f'_{n\delta}(\beta)>0$. Let Δ_k be the interval which contains β. By simple geometrical consideration we see that on Δ_k the graphs of $\bar{f}$ and $f_{n\delta}$ intersect at least three times, while on each of the other intervals $\Delta_j(j\in\{0, 1, \cdots, 2n-1\}\setminus\{k\})$ these graphs intersect at least once. So, for $g(x)=f_{n\delta}(x)-\bar{f}(x)$ we have $S_c^-(g)\geqslant2n+2$. On the other hand, it is known[4] that the kernel $H(x-t)$ is totally positive. Hence it possesses the cyclic variation-diminishing property (CVD). Thus, on account of $S_c^-\left(\text{sgn}\cos nt-\frac{1}{\rho}h(t)\right)\leqslant2n$ and

$$g(x)=\frac{1}{2\pi}\int_0^{2\pi}H(x-t)\left[\text{sgn}\cos nt-\frac{1}{\rho}h(t)\right]dt$$

we have $S_c^-(g) \leqslant 2n$. This is a contradiction. Theorem 1 is proved.

Corollary 1. *Let $f \in H_\delta(L_\infty)$ be such that $\|f\|_\infty \leqslant \|f_{n\delta}\|_\infty$ for some n, and $f(\xi_0) = f_{n\delta}(\eta_0)$, $f(\xi_1) = f_n(\eta_1)$ for $\xi_0, \xi_1, \eta_0, \eta_1 \in R$, where η_0, η_1 are contained in one interval of monotonity of $f_{n\delta}$. Then*

$$|\xi_0 - \xi_1| \geqslant |\eta_0 - \eta_1|.$$

Corollary 2. *Let $f \in H_\delta(L_\infty)$ be such that $\|f\|_\infty \leqslant \|f_{n\delta}\|_\infty$ for some n and $f(\xi_0) = f_{n\delta}(\eta_0)$ for $\xi_0, \eta_0 \in R$ and $\eta_0 \in \Delta_k$. Then*

(1) *for the case $f_{n\delta}\downarrow$ in Δ_k we have*

$$f(\xi_0+u) \leqslant f_{n\delta}(\eta_0+u), \quad 0 \leqslant u \leqslant \frac{k+1}{n}\pi - \eta_0,$$

$$f(\xi_0-u) \geqslant f_{n\delta}(\eta_0-u), \quad 0 \leqslant u \leqslant \eta_0 - \frac{k}{n}\pi,$$

(2) *for the case $f_{n\delta}\downarrow$ in Δ_k we have*

$$f(\xi_0+u) \geqslant f_{n\delta}(\eta_0+u), \quad 0 \leqslant u \leqslant \frac{k+1}{n}\pi - \eta_0,$$

$$f(\xi_0-u) \leqslant f_{n\delta}(\eta_0-u), \quad 0 \leqslant u \leqslant \eta_0 - \frac{k}{n}\pi.$$

Corollary 3. *Let $f \in H_\delta(L_\infty)$ be such that $\|f\|_\infty \leqslant \|f_{n\delta}\|_\infty$ for some n. Then*

$$\|f'\|_\infty \leqslant \|f'_{n\delta}\|_\infty. \tag{7}$$

Let us put

$$F_{n\delta}(x) = \frac{4}{\pi} \sum_{\nu=0}^{\infty} \frac{\sin(2\nu+1)nx}{n(2\nu+1)^2 \operatorname{ch}(2\nu+1)n\delta}. \tag{8}$$

It is obvious that $F'_{n\delta}(x) = f_{n\delta}(x)$.

Theorem 2. *Let $f(x) \in H_\delta(L_\infty)$, and $F(x)$ be a periodic integral of f such that $\|F\|_\infty \leqslant \|F_{n\delta}\|_\infty$ for some n, $F(a) = F_{n\delta}(\alpha)$ for $a, \alpha \in R$. Then*

$$|f(a)| \leqslant |f_{n\delta}(\alpha)|. \tag{9}$$

Proof The argument is quite similar to that of Theorem 1. Suppose $a=\alpha$, $|F'(\alpha)| > |F'_{n\delta}(\alpha)|$. Then there exist $\rho > 1$ and β such that

$$\frac{1}{\rho} F(\beta) = F_{n\delta}(\beta), \quad \frac{1}{\rho}|F'(\beta)| > |F'_{n\delta}(\beta)|.$$

For $G(x) = F_{n\delta}(x) - \frac{1}{\rho} F(x)$ we have $S_c^-(G) \geqslant 2n+2$. Hence by Rolle's theorem $S_c^-(G') \geqslant 2n+2$. But for

$$G'(x) = F'_{n\delta}(x) - \frac{1}{\rho} F'(x) = f_{n\delta}(x) - \frac{1}{\rho} f(x) = \frac{1}{2\pi}\int_0^{2\pi} H(x-t)\left[\operatorname{sgn} \cos nt - \frac{1}{\rho} h(t)\right] dt$$

we have just proved $S_c^-(G') \leqslant 2n$. This is absurd. Thus Theorem 2 is proved.

§ 3. Some Sharp Inequalities Derived from the Comparison Theorem

For a 2π-periodic summable function f denote by $p(f, t)$ the non-increasing rearrangement of $|f|$ (cf. [1]). We have

Theorem 3. *Let $f\in H_\delta(L_\infty)$ and F be a periodic integral of f such that $\|F\|_\infty\leqslant\|F_{n\delta}\|_\infty$ for some n. Then*

$$\int_0^x p(f,t)dt\leqslant\int_0^x p(f_{n\delta},t)dt,\ 0\leqslant x\leqslant 2\pi. \tag{10}$$

The proof of (10) is essentially similar to that given by Корнейчук[1] for the class $\widetilde{W}^r_\infty$. We omit the details.

By (10) and one lemma of Chong Kong-ming[5] we have

Theorem 4. *Let $f\in H_\delta(L_\infty)$ and F be a periodic integral of f such that $\|F\|_\infty\leqslant\|F_{n\delta}\|_\infty$ for some n. Then for every p, $1\leqslant p\leqslant+\infty$, we have*

$$\|f\|_p\leqslant\|f_{n\delta}\|_p. \tag{11}$$

In what follows we give some applications of (11). Consider one subset of $H_\delta(L_\infty)$ defined by

$$H_\delta(L_\infty)\cap T_n^\perp\stackrel{df}{=\!=}\left\{f\in H_\delta(L_\infty):\int_0^{2\pi}f(t)\begin{matrix}\sin kt\\ \cos kt\end{matrix}dt=0,\ k=0,\cdots,n-1\right\}.$$

Let F be the periodic integral of $f\in H_\delta(L_\infty)\cap T_n^\perp$ such that $\int_0^{2\pi}F(t)dt=0$. Then we have $\int_0^{2\pi}F(t)\begin{matrix}\sin kt\\ \cos kt\end{matrix}dt=0$ for $k=0,\cdots,n-1$. It is easily seen that $F(x)$ may be presented by 2π-periodic convolution, in which the kernel is the composition of $H(x-u)$ and $D_1(u-t)$, where $D_1(u)=\sum_{\nu=1}^\infty\frac{\sin\nu u}{\nu}$ is the Bernoulli polynomial of degree 1. The composition $(H*D_1)(x-t)$ satisfies the Markov-Nicolsky condition A_n (cf[1]). Therefore we have

$$\|F\|_\infty\leqslant\|F_{n\delta}\|_\infty.$$

Now by Theorem 4 we obtain

Theorem 5. *For $n=1,2,3,\cdots$ and p, $1\leqslant p\leqslant+\infty$,*

$$\sup_{f\in H_\delta(L_\infty)\cap T_n^\perp}\|f\|_p=\|f_{n\delta}\|_p. \tag{12}$$

By applying the duality theorem of the best approximation of convolution class[1] we have

Theorem 6. *For $n=1,2,3,\cdots$ and p, $1\leqslant p\leqslant+\infty$,*

$$E_n(H_\delta(L_p))_1\stackrel{df}{=\!=}\sup_{f\in H_\delta(L_p)}E_n(f)_1=\sup_{f\in H_\delta(L_\infty)\cap T_n^\perp}\|f\|_{p'}=\|f_{n\delta}\|_{p'}\left(\frac{1}{p}+\frac{1}{p'}=1\right). \tag{13}$$

A. Pinkus[6] obtained some exact relations for the Kolmogorov $2n$-width of 2π-periodic convolution class with a periodic TP kernel. For the $(2n-1)$-width the exact formulas are not obtained, because the inequalities (11) (knows as the Taikov inequality) can not be established in general. For $H(x-t)$ we have

Theorem 7. *For $n=1,2,3,\cdots$, any p, $1\leqslant p\leqslant+\infty$,*

(1) $d_{2n-1}[H_\delta(L_p);L_1]=d_{2n}[H_\delta(L_p);L_1]=\|f_{n\delta}\|_{p'}$ (14)

(2) $d^{2n-1}[H_\delta(L_\infty);L_p]=d^{2n}[H_\delta(L_\infty);L_p]=\|f_{n\delta}\|_p$. (15)

Proof By A. Pinkus[6] we get

$$\|f_{n\delta}\|_{p'}=d_{2n}[H_\delta L_p);\ L_1]\leqslant d_{2n-1}[H_\delta(L_p);\ L_1].$$

Comparing with (13) we have

$$d_{2n-1}[H_\delta(L_p);\ L_1]\leqslant E_n(H_\delta(L_p))_1=\|f_{n\delta}\|_{p'}.$$

(14) is proved. T_n yields an extremal subspace. (15) can be derived readily by duality theorems of the Kolmogorov and Gelfand widths numbers.

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