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A NOTE ON A PROBLEM OF BOAS R. P.*

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Abstract

To answer the rest part of the problem of Boas R. P. on derivative of polynomial, it is shown that if $p(z)$ is a polynomial of degree n such that $\max\limits_{|z|\leqslant 1}|p(z)|\leqslant 1$ and $p(z)\neq 0$ in $|z|\leqslant k$, $0<k\leqslant 1$, then $|p'(z)|\leqslant n/(1+k^n)$ for $|z|\leqslant 1$. The above estimate is sharp and the equation holds for $p(z)=(z^n+k^n)/(1+k^n)$.

§ 1. Introduction

If $p(z)$ is a complex polynomial of degree n, then we have the following famous result known as Bernstein's inequality

$$\operatorname*{Max}_{|z|=1}|p'(z)|\leqslant n\operatorname*{Max}_{|z|=1}|p(z)|. \tag{1}$$

It was conjectured by P. Erdös and proved later by P. D. Lax[1] that if $p(z)\neq 0$ in $|z|<1$, then

$$\operatorname*{Max}_{|z|=1}|p'(z)|\leqslant (n/2)\operatorname*{Max}_{|z|=1}|p(z)|. \tag{2}$$

In this direction, R. P. Boas asked (see [2]) how large can

$$\operatorname*{Max}_{|z|\leqslant k}|p'(z)|/\operatorname*{Max}_{|z|\leqslant k}|p(z)|$$

be if $p(z)$ is an arbitrary polynomial of degree n not vanishing in $|z|<k$, where k is a given positive number.

A partial answer was given by M. A. Malik[3], who proved the following

Theorem A. *If $p(z)$ polynomial of degree n such that $|p(z)|\leqslant 1$ for $|z|=1$ and $p(z)\neq 0$ in $|z|<k$, then*

$$|p'(z)|\leqslant n/(1+k) \text{ for } |z|\leqslant 1, \tag{3}$$

provided $k\geqslant 1$.

Q. I. Rahman[2] pointed out that in the case $k<1$, the precise answer to the above mentioned question of R.P. Boas is not known.

§ 2. The Main Result

The purpose of this note is to answer the rest of the above mentioned question.

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We shall prove the following

Theorem 1. *If $p(z)$ is a polynomial of degree n such that* $\max_{|z|<1}|p(z)|=1$ *and* $p(z)\neq 0$ *in* $|z|<k$ $(0<k\leqslant 1)$, *then*

$$|p'(z)|\leqslant n/(1+k^n) \text{ for } |z|\leqslant 1. \tag{4}$$

The estimate (4) is sharp and the equality holds for

$$p(z)=(z^n+k^n)/(1+k^n).$$

To prove Theorem 1 we need some lemmas.

Lemma 1 (P. D. Lax[1]). *If $p(z)$ has no roots inside the unit circle, we have* $|p'(z)|\leqslant|q'(z)|$ *as* $|z|=1$, *where* $q(z)=z^n\overline{p(1/\bar{z})}$.

Lemma 2. *If $p(z)$ is a polynomial of degree n, then on* $|z|=1$,

$$|p'(z)|+|q'(z)|\leqslant n\max_{|z|=1}|p(z)|, \tag{5}$$

where $q(z)=z^n\overline{p(1/\bar{z})}$.

This lemma is a special case of a result due to N. K. Govil and Q. I. Rahman (see [4, Lemma 10]).

Proof of Theorem 1 First we prove that on $|z|=1$,

$$k^n|p'(z)|\leqslant|q'(z)|. \tag{6}$$

In fact, $p_1(z)\equiv p(kz)$ has no roots inside the unit circle. Using Lemma 1, we have

$$|p_1'(z)|\leqslant|q_1'(z)| \text{ on } |z|=1,$$

where

$$q_1(z)=z^n\overline{p_1(1/\bar{z})}=z^n\overline{p(k/\bar{z})}=k^nq(z/k).$$

Hence

$$|p'(kz)|\leqslant k^{n-2}|q'(z/k)|. \tag{7}$$

For a polynomial $F(z)$ of degree n, it is known that[5]

$$R^{-n}\max_{|z|=R}|F(z)|\leqslant r^{-n}\max_{|z|=r}|F(z)|,\ (0<r<R). \tag{8}$$

Using (8), we obtain

$$|q'(z/k)|\leqslant \max|q(z)|/k^{n-1} \tag{9}$$

and

$$\max_{|z|=1}|p'(z)|\leqslant\max_{|z|=1}|p'(kz)|/k^{n-1}. \tag{10}$$

A substitution of the above estimates (9) and (10) into (7) gives (6).

Now (5) and (6) imply that on $|z|=1$,

$$(1+k^n)|p'(z)|\leqslant n\max_{|z|=1}|p(z)|.$$

The proof is completed.

The following result is an L_2 analogue of Theorem 1.

Theorem 2. *If $p(z)$ is a polynomial of degree n such that $p(z)\neq 0$ in* $|z|<k$ $(0<k\leqslant 1)$, *then*

$$\left\{(2\pi)^{-1}\int_{-\pi}^{\pi}|p'(e^{i\theta})|^2d\theta\right\}^{\frac{1}{2}}\leqslant(1+k^n)^{-\frac{1}{2}}\left\{(2\pi)^{-1}\int_{-\pi}^{\pi}|p(e^{i\theta})|^2d\theta\right\}^{\frac{1}{2}}.$$

The above estimate is sharp and the equality holds for $p(z)=(z^n+k^n)/(1+k^n)$.

The proof follows from (6) and Parseval's formula. We omit the detail.

References

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