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# ON THE EXPECTED SAMPLE SIZES OF SOME POWER ONE TESTS FOR NORMAL MEAN WITH UNKNOWN VARIANCE

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**Abstract**

Suppose that $x_1, x_2, \cdots$ are i. i. d. random variables on a probability space $(\Omega, \mathscr{F}, P_{\theta\sigma})$ and $x_1$ is normally distributed with mean $\theta$ and variance $\sigma^2$, both of which are unknown. Given $\theta_0$ and $0<\alpha<1$, we propose a concrete stopping rule $T$ w. r. t. the $\{x_n, n\geqslant 1\}$ such that

$$P_{\theta\sigma}(T<\infty)\leqslant\alpha \text{ for all } \theta\leqslant\theta_0,\ \sigma>0,$$
$$P_{\theta\sigma}(T<\infty)=1 \text{ for all } \theta>\theta_0,\ \sigma>0,$$
$$\lim_{\theta\downarrow\theta_0}(\theta-\theta_0)^2\left(\ln_2\frac{1}{\theta-\theta_0}\right)^{-1}E_{\theta\sigma}T=2\sigma^2P_{\theta_0\sigma}(T=\infty),$$

where $\ln_2 x=\ln(\ln x)$.

## § 1. Introduction and Summary

We suppose that $x_1, x_2, \cdots$ are i. i. d. random variables on a probability space $(\Omega, \mathscr{F}, P_{\theta\sigma})$ and $x_1$ is normally distributed with mean $\theta$ and variance $\sigma^2$, both of which are unknown. Given $\theta_0$ and $0<\alpha<1$, by a size $\alpha$ test of power one of the hypothesis $H_0$: $\theta\leqslant\theta_0$ against $H_1$: $\theta>\theta_0$ we mean a stopping rule $T$ w. r. t. the sequence $x_1, x_2, \cdots$ such that

$$\begin{aligned} &P_{\theta\sigma}(T<\infty)\leqslant\alpha \quad \text{for all} \quad \theta\leqslant\theta_0,\ \sigma>0,\\ &P_{\theta\sigma}(T<\infty)=1 \quad \text{for all} \quad \theta>\theta_0,\ \sigma>0. \end{aligned} \tag{1.1}$$

We stop sampling at stage $T$ and reject $H_0$ (We do not reject $H_0$ as long as we continue sampling).

Among such rules we wish to find one which in some sense minimizes $E_{\theta\sigma}T$ as $\theta\downarrow\theta_0$.

As shown in [1] and [2], for any fixed $\sigma$, the following inequality

$$\varlimsup_{\theta\downarrow\theta_0}(\theta-\theta_0)^2\left(\ln_2\frac{1}{\theta-\theta_0}\right)^{-1}E_{\theta\sigma}T\geqslant 2\sigma^2P_{\theta_0\sigma}(T=\infty) \tag{1.2}$$

always holds for any stopping rule $T$ w. r. t. the $x_1, x_2, \cdots$. Here $\ln_2 x=\ln(\ln x)$. [2]

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also suggested some concrete rule $T_0$ (depending on $\sigma$) which achieves "the lower bound":

$$\lim_{\theta\downarrow\theta_0}(\theta-\theta_0)^2\left(\ln_2\frac{1}{\theta-\theta_0}\right)^{-1}E_{\theta\sigma}T_0=2\sigma^2P_{\theta_0\sigma}(T_0=\infty).$$

It is natural to ask whether similar results hold or not when $\sigma$ is unknown, [3] proved the answer of this problem is affirmative. The results obtained may be described as follows. Without loss of generality, we suppose $\theta_0=0$.

1) Let $b_n=\sqrt{2n(\ln_2 n+c\ln_3 n+o(\ln_3 n)}\left(c>\frac{3}{2}\right)$,

$$T_1=\inf\{n:n\geqslant m,\ S_n/v_n\geqslant b_n\},$$

where $m\geqslant 2$,

$$S_n=\sum_1^n x_i,\ \bar{x}_n=\frac{1}{n}S_n,\ v_n=\sqrt{\frac{1}{n-1}\sum_{i=1}^n(x_i-\bar{x}_n)^2}.$$

For $\alpha\in(0,1)$, if $m$ is so selected that $P_{0\sigma}(T_1=\infty)\leqslant\alpha$ for all $\sigma>0$, then

$$\lim_{\theta\downarrow 0}\theta^2\left(\ln_2\frac{1}{\theta}\right)^{-1}E_{\theta\sigma}T_1=2\sigma^2P_{0\sigma}(T_1=\infty)$$

for all $\sigma>0$.

2) Let $g(r)$ be a density function on $(0,\infty)$ satifying

$$g(r)=\frac{\delta}{r\left(\ln\frac{1}{r}\right)\left(\ln_2\frac{1}{r}\right)^{1+\delta}}\quad\text{for all}\quad r\in(0,r_0),$$

where $r_0\in(0,e^{-e})$, $\delta>0$, and being bounded on $(r_0,\infty)$. Set

$$Z_n=\int_0^\infty R_n(r)g(r)dr,$$

where $R_n(r)=U_n(r)/U_n(0)$, $U_n(r)=\int_0^\infty\frac{1}{u}e^{nf(u,T_n,r)}du$, $T_n=\bar{x}_n\Big/\left(n^{-1}\sum_1^n x_i^2\right)^{\frac{1}{2}}$ and $f(u,y,r)=-\frac{1}{2}u^2+ryu+\ln u-\frac{1}{2}r^2$. Let

$$T_\alpha^*=\inf\left\{n:n\geqslant m,\ Z_n\geqslant\frac{1}{\alpha}\right\}\quad(0<\alpha<1).$$

Then $P_{\theta\sigma}(T_\alpha^*<\infty)\leqslant P_{0\sigma}(T_\alpha^*<\infty)\leqslant\alpha$ for all $\theta\leqslant 0$, $\sigma>0$; $P_{\theta\sigma}(T_\alpha^*<\infty)=1$ for all $\theta>0$, $\sigma>0$; $\lim_{\theta\downarrow 0}\theta^2\left(\ln_2\frac{1}{\theta}\right)^{-1}E_{\theta\sigma}T_\alpha^*=2\sigma^2P_{0\sigma}(T_\alpha^*=\infty)$ for all $\sigma>0$ and

$$\lim_{\alpha\to 0}\frac{1}{|\ln\alpha|}E_{\theta\sigma}T_\alpha^*=2/\ln\left(1+\frac{\theta^2}{\sigma^2}\right)$$

for all $\theta>0$, $\sigma>0$.

Although these two stopping rules $T_1$ and $T_\alpha^*$ have theoretical optimality as $\theta\downarrow 0$ (comparing with (1.2)), we meet great difficulty in using them in practice. The difficulty of using $T_1$ lies in how to find out $m$ satisfying $P_{0\sigma}(T_1<\infty)\leqslant\alpha$ for pregiven $\alpha\in(0,1)$. Since the expression of $Z_n$ is very complex, it is also hard to use $T_\alpha^*$.

To avoid these trouble, in the present paper we suggest the following power one test, which is a natural generalization of the result in [2].

Let $m$ be an integer number $(m\geq 2)$. $\theta_1=\theta_2=\cdots=\theta_m=\theta_0$ ($\theta_0$ being given) and

$$\theta_{n+1}=\theta_0+\left(\frac{S_n}{n}-\theta_0\right)I\{S_n-n\theta_0\geq\sqrt{n}\,a_nv_n\}\quad(n\geq m),\tag{1.3}$$

where $S_n=\sum_1^n x_i$, $v_n=\sqrt{\frac{1}{n-1}\sum_1^n(x_i-\bar{x}_n)^2}$, $\bar{x}_n=\frac{1}{n}S_n$, $a_n=\sqrt{2\ln_2^+ n+3\ln_3^+ n}$, $\ln_2 n=\ln(\ln n)$, $\ln_3 n=\ln(\ln_2 n)$, $x^+=\max(0, x)$ and $I(A)$ is indicator function of $A$. Set

$$T_2=\inf\left\{n:n\geq m+1,\ \sum_{i=m+1}^n(\theta_i-\theta_0)x_i-\frac{1}{2}\sum_{i=m+1}^n(\theta_i^2-\theta_0^2)\geq cv_m^2\right\},\tag{1.4}$$

where $c>0$. Our main result is

**Theorem A.**

$$P_{\theta\sigma}(T_2<\infty)\leq\left(\frac{m-1}{m-1+2c}\right)^{\frac{m-1}{2}}\quad \textit{for all}\quad \theta\leq\theta_0,\ \sigma>0;\tag{1.5}$$

$$P_{\theta\sigma}(T_2<\infty)=1\quad \textit{for all}\quad \theta>\theta_0,\ \sigma>0;\tag{1.6}$$

$$\lim_{\theta\downarrow\theta_0}(\theta-\theta_0)^2\left(\ln_2\frac{1}{\theta-\theta_0}\right)^{-1}E_{\theta\sigma}T_2=2\sigma^2P_{\theta_0\sigma}(T_2=\infty)\tag{1.7}$$

*for all* $\sigma>0$.

Thus for given $\alpha\in(0, 1)$, we easily find $m$ and $c>0$ such that

$$\left(\frac{m-1}{m-1+2c}\right)^{\frac{m-1}{2}}\leq\alpha.$$

(1.7) shows that $T_2$ is asymptotically optimal as $\theta\downarrow\theta_0$.

What we do below is to prove Theorem A.

At first, it is simple to prove (1.5). Set

$$W_n=\exp\left\{-\frac{1}{2\sigma^2}\sum_1^n(x_i-\theta_i)^2\right\}\Big/\exp\left\{-\frac{1}{2\sigma^2}\sum_1^n(x_i-\theta_0)^2\right\}.$$

By [2], under $P_{\theta\sigma}(\theta\leq\theta_0)$, $(W_n, n\geq 1)$ is a supermartingale. Noting $T_2=\inf\{n:n\geq m+1, W_n\geq\exp\{cv_m^2/\sigma^2\}$, we have, for $\theta\leq\theta_0$, $P_{\theta\sigma}(T_2<\infty)=E_{\theta\sigma}\{P_{\theta\sigma}(T_2<\infty\,|\,v_m^2)\}\leq E_{\theta\sigma}(\exp\{-cv_m^2/\sigma^2\})=\int_0^\infty e^{-\frac{c}{m-1}u}u^{\frac{m-1}{2}-1}e^{-\frac{u}{2}}\left(2^{\frac{m-1}{2}}\Gamma\left(\frac{m-1}{2}\right)\right)^{-1}du=\left(\frac{m-1}{m-1+2c}\right)^{\frac{m-1}{2}}$. This proves (1.5).

As for (1.6) and (1.7), we will prove them in § 2 and § 3.

We suppose $\theta_0=0$ (if $\theta_0\neq 0$, replace $x_i$ and $\theta$ by $x_i-\theta_0$ and $\theta-\theta_0$ respectively).

## § 2. Some Lemmas

Suppose $\theta_0=0$. Then $T_2$ defined by (1.4) becomes

$$T_2=\inf\left\{n:n\geq m,\ \sum_1^n\left(\theta_ix_i-\frac{1}{2}\theta_i^2\right)\geq cv_m^2\right\}.\tag{2.1}$$

Using the method in [2], but more complicated computation being needed, we can obtain the expression

$$E_{\theta\sigma}T_2=\theta^{-2}E_{\theta\sigma}\Big(\sum_{n=1}^{T_2}(\theta_{n+1}-\theta)^2\Big)+\theta^{-2}\cdot 2E_{\theta\sigma}\Big(\sum_{n=1}^{T_2}\Big[\theta_n x_n-\frac{1}{2}\theta_n^2\Big]^2\Big) \tag{2.2}$$

and estimate the order of magnitude of two terms on the right side of (2.2).

We always use the notations $\varphi(x)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right)$ and $\Phi(x)=\int_{-\infty}^{x}\varphi(t)dt$.

**Lemma 1.** *For all $n\geqslant m$ and $\theta>0$,*

$$E_{\theta\sigma}(\theta_{n+1}-\theta)^2=\frac{\sigma^2}{n}E_{\theta\sigma}\left\{1-\Phi\left(\frac{a_nv_n-\sqrt{n}\,\theta}{\sigma}\right)+\left(\frac{a_nv_n-\sqrt{n}\,\theta}{\sigma}\right)\varphi\left(\frac{a_nv_n-\sqrt{n}\,\theta}{\sigma}\right)\right\}+\theta^2E_{\theta\sigma}\Phi\left(\frac{a_nv_n-\sqrt{n}\,\theta}{\sigma}\right), \tag{2.3}$$

$$E_{\theta\sigma}(\theta_{n+1}-\theta)^4\leqslant\frac{19\sigma^4+48a_n^4\sigma^4}{n^2}, \tag{2.4}$$

*where $\theta_{n+1}$ is defined by* (1.3).

*Proof* (2.3) can be proved by calculating. In order to prove (2.4), we need to use inequality

$$\theta^4\Phi\left(\frac{a_nv-\sqrt{n}\,\theta}{\sigma}\right)\leqslant\frac{16\sigma^4+16a_n^4v^4}{n^2}\quad(\theta>0),$$

which is not hard to prove. So

$$\begin{aligned}E_{\theta\sigma}(\theta_{n+1}-\theta)^4&=E_{\theta\sigma}\left\{\left(\frac{S_n-n\theta}{n}\right)^4I(S_n\geqslant\sqrt{n}\,a_nv_n)+\theta^4I(S_n<\sqrt{n}\,a_nv_n)\right\}\\&\leqslant E_{\theta\sigma}\left[\left(\frac{S_n-n\theta}{n}\right)^4+\theta^4\Phi\left(\frac{a_nv_n-\sqrt{n}\,\theta}{\sigma}\right)\right]\\&\leqslant\frac{3\sigma^4}{n^2}+\frac{16\sigma^4+48a_n^4\sigma^4}{n^2}=\frac{19\sigma^4+48a_n^4\sigma^4}{n^2}.\end{aligned}$$

**Corollary 1.** $\sum_1^{\infty}E_{\theta\sigma}(\theta_{n+1}-\theta)^4$ *is bounded in $\theta>0$.*

**Lemma 2.** *Let $Y_n=\sum_1^n\left(\theta_kx_k-\frac{1}{2}\theta_k^2\right)$ $(n\geqslant1)$, and $h(t)=t+\varphi(t)/\Phi(t)$. Then*

$$\int_{\{T_2<\infty\}}(Y_{T_2}-cv_m^2)dP_{\theta\sigma}\leqslant\sigma h\left(\frac{\theta}{\sigma}\right)\left\{1+\sum_{n=1}^{\infty}E_{\theta\sigma}(\theta_{n+1}-\theta)^4\right\}, \tag{2.5}$$

$$\int_{\{T_2<\infty\}}Y_{T_2}dP_{\theta\sigma}=O(1)\quad(\theta\downarrow0). \tag{2.6}$$

*Proof* We directly obtain (2.6) from (2.5) and Corollary 1. *So it suffices to prove* (2.5). *It is obvious that*

$$\{T_2=n\}=\left\{T_2>n-1,\ \theta_nx_n-\frac{1}{2}\theta_n^2+Y_{n-1}\geqslant cv_m^2\right\}.$$

Hence

$$\int_{\{T_2=n\}}(Y_{T_2}-cv_m^2)\,dP_{\theta\sigma}$$

$$=\int_{\substack{\{T_2>n-1,\ \theta_n>0\\ x_n\geqslant\frac{1}{2}\theta_n+\frac{1}{\theta_n}(cv_m^2-Y_{n-1})\}}}\theta_n\Big[x_n-\frac{\theta_n}{2}-\frac{1}{\theta_n}(cv_n^2-Y_{n-1})\,dP_{\theta\sigma}$$

$$=\int_{\{T_2>n-1\}}I(\theta_n>0)\cdot\theta_n E_{\theta\sigma}[I(x_n\geqslant\lambda)\cdot(x_n-\lambda)]\Big|_{\lambda=\frac{1}{2}\theta_n+\frac{1}{\theta_n}(cv_m^2-Y_{n-1})}dP_{\theta\sigma}$$

$$=\int_{\{T_2>n-1\}}I(\theta_n>0)\cdot\theta_n\cdot\sigma\left[P_{\theta\sigma}(x_n\geqslant\lambda)h\Big(\frac{\theta-\lambda}{\sigma}\Big)\right]\Big|_{\lambda=\frac{1}{2}\theta_n+\frac{1}{\theta_n}(cv_m^2-Y_{n-1})}.\,dP_{\theta\sigma}$$

Since $h(t)$ is increasing, we have

$$\int_{\{T_2=n\}}(Y_{T_2}-cv_m^2)\,dP_{\theta\sigma}$$

$$\leqslant\sigma h\Big(\frac{\theta}{\sigma}\Big)\int_{\{T_2>n-1\}}I(\theta_n>0)\cdot\theta_n I\Big(x_n\geqslant\frac{1}{2}\theta_n+\frac{1}{\theta_n}(cv_m^2-Y_{n-1})\Big)dP_{\theta\sigma}$$

$=\sigma h\Big(\frac{\theta}{\sigma}\Big)\int_{\{T_2=n\}}\theta_n dP_{\theta\sigma}$. Therefore $\int_{\{T_2<\infty\}}(Y_{T_2}-cv_m^2)\,dP_{\theta\sigma}$

$$\leqslant\sigma h\Big(\frac{\theta}{\sigma}\Big)\int_{\{T_2<\infty\}}\theta_{T_2}dP_{\theta\sigma}\leqslant\sigma h\Big(\frac{\theta}{\sigma}\Big)\int_{\{T_2<\infty\}}\Big[1+\sum_{n=1}^{\infty}(\theta_{n+1}-\theta)^4\Big]dP_{\theta\sigma}.$$

Thus (2.5) is proved.

**Lemma 3.** *For all $\theta>0$ and $\sigma>0$, $E_{\theta\sigma}T_2<\infty$ and (2.2) holds.*

*Proof* Set $\tau_l=\min(T_2, l)$. Since $\Big(\sum_1^n\theta_k(x_k-\theta),\ \mathscr{F}_n,\ \geqslant m,\ P_{\theta\sigma}\Big)$ is a martingale (where $\mathscr{F}_n=\sigma\{x_1, \cdots, x_n\}$), we have $E_{\theta\sigma}\Big(\sum_1^{\tau_l}\theta_k(x_k-\theta)\Big)=E_{\theta\sigma}[\theta_m\cdot(x_m-\theta)]=0$. On the other hand

$$2\sum_1^{\tau_l}\Big(\theta_k x_k-\frac{1}{2}\theta_k^2\Big)+\sum_1^{\tau_l}(\theta_k-\theta)^2=2\Big(\sum_1^{\tau_l}(\theta_k x_k-\theta_k\theta)\Big)+\tau_l\theta^2.$$

Hence

$$E_{\theta\sigma}\tau_l=\theta^{-2}E_{\theta\sigma}\Big(\sum_1^{\tau_l}(\theta_k-\theta)^2\Big)+2\theta^{-2}E_{\theta\sigma}\Big(\sum_1^{\tau_l}\Big(\theta_k x_k-\frac{1}{2}\theta_k^2\Big)\Big). \tag{2.7}$$

It is obvious that

$$E_{\theta\sigma}\Big(\sum_1^{\tau_l}\Big(\theta_k x_k-\frac{1}{2}\theta_k^2\Big)\Big)\leqslant\int_{\{T_2<\infty\}}\sum_1^{T_2}\Big(\theta_k x_k-\frac{1}{2}\theta_k^2\Big)dP_{\theta\sigma}+E_{\theta\sigma}(cv_m^2). \tag{2.8}$$

On the other hand

$$E_{\theta\sigma}\Big(\sum_1^{\tau_l}(\theta_k-\theta)^2\Big)=\sum_{n=1}^{l}\Big[\int_{\{\tau_l>n-1\}}\sum_1^n(\theta_k-\theta)^2dP_{\theta\sigma}-\int_{\{\tau_l>n\}}\sum_1^n(\theta_k-\theta)^2dP_{\theta\sigma}\Big]$$

$$=\sum_{n=1}^{l-1}\int_{\{\tau_l>n\}}(\theta_{n+1}-\theta)^2dP_{\theta\sigma}$$

$$\leqslant\sum_{n=1}^{l-1}\Big[\int_{\{|\theta_{n+1}-\theta|>\eta\}}(\theta_{n+1}-\theta)^2dP_{\theta\sigma}+\eta^2P_{\theta\sigma}(\tau_l>n)\Big]$$

$$\leqslant\eta^{-2}\sum_{n=1}^{\infty}E_{\theta\sigma}(\theta_{n+1}-\theta)^4+\eta^2E_{\theta\sigma}\tau_l\quad(\eta>0).$$

Thus it follows from (2.7) and (2.8) that

$$E_{\theta\sigma}\tau_l\leqslant(1-\eta^2/\theta^2)^{-1}\left\{\frac{1}{\theta^2\eta^2}\sum_{n=1}^{\infty}E_{\theta\sigma}(\theta_{n+1}-\theta)^4\right.$$
$$\left.+\frac{2}{\theta^2}\int_{\{T_2<\infty\}}\sum_{1}^{T_2}\left(\theta_k x_k-\frac{1}{2}\theta_k^2\right)dP_{\theta\sigma}+2\theta^{-2}c\sigma^2\right\}.$$

Letting $l\to\infty$, we obtain $E_{\theta\sigma}T_2=\lim_l E_{\theta\sigma}\tau_l<\infty$ by lemma 1 and lemma 2. Letting $l\to\infty$ in (2.7) and noting

$$\lim_l\int_{\{T_2>l\}}\sum_{1}^{l}\left(\theta_k x_k-\frac{1}{2}\theta_k^2\right)dP_{\theta\sigma}=0,$$

we obtain (2.2).

**Lemma 4.** *Let* $n_1=\left[\frac{1}{\theta^2}\left(\ln_2\frac{1}{\theta}\right)^{-1}\right]$ $(\theta>0)$. *Then*

$$\sum_{n=1}^{n_1}E_{\theta\sigma}(\theta_{n+1}-\theta)^2=o\left(\ln_2\frac{1}{\theta}\right)\quad(\theta\downarrow 0).\tag{2.9}$$

*Proof* We obtain from (2.3)

$$E_{\theta\sigma}(\theta_{n+1}-\theta)^2\leqslant\frac{\sigma^2}{n}\int_0^\infty\left[1-\Phi\left(\frac{a_n\sqrt{u}}{\sqrt{n-1}}-\frac{\sqrt{n}}{\sigma}\theta\right)\right]dG_{n-1}(u)$$
$$+\frac{\sigma^2}{n}\int_0^\infty\frac{a_n\sqrt{u}}{\sqrt{n-1}}\varphi\left(\frac{a_n\sqrt{u}}{\sqrt{n-1}}-\frac{\sqrt{n}\,\theta}{\sigma}\right)dG_{n-1}(u)+\theta^2,$$

where $G_n(u)$ is $\chi^2$-distribution function with degree of freedom $n$.

At first, we proceed to estimate

$$\int_0^\infty\left[1-\Phi\left(\frac{a_n\sqrt{u}}{\sqrt{n-1}}-\frac{\sqrt{n}\,\theta}{\sigma}\right)\right]dG_{n-1}(u).\tag{2.10}$$

It is easy to see that there is $d>0$ satisfying $a_n\sqrt{n}\,\theta/\sigma\leqslant d$ for all $n\leqslant n_1$ and small enough $\theta$. Set $b_n=4d^2(n-1)/(\ln_3^2 n)$, then $u\geqslant 4a_n^2n(n-1)\theta^2/(\ln_3^2 n)\sigma^2$ if $u\geqslant b_n$, and $a_n\sqrt{b_n}/\sqrt{n-1}-\sqrt{n}\,\theta/\sigma>a_nd/\ln_3 n>0$. Hence

$$\int_{b_n}^\infty\left[1-\Phi\left(\frac{a_n\sqrt{u}}{\sqrt{n-1}}-\frac{\sqrt{n}\,\theta}{\sigma}\right)\right]dG_{n-1}(u)$$
$$\leqslant\int_{b_n}^\infty\frac{\ln_3 n}{a_n d}\varphi\left(\frac{a_n\sqrt{u}}{\sqrt{n-1}}-\frac{\sqrt{n}}{\sigma}\theta\right)dG_{n-1}(u)$$
$$=\frac{\ln_3 n}{a_n d}\int_{b_n}^\infty\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left[\frac{a_n^2u}{n-1}+\frac{n}{\sigma}\theta^2\right.\right.$$
$$\left.\left.-2\frac{a_n\sqrt{n}\,\theta}{\sqrt{n-1}\,\sigma}\sqrt{u}\right]\right\}u^{\frac{n-1}{2}-1}e^{-\frac{1}{2}u}\left[2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)\right]^{-1}du$$
$$\leqslant\frac{\ln_3 n}{a_nd\sqrt{2\pi}}\left[2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)\right]^{-1}$$
$$\cdot\int_{b_n}^\infty\exp\left\{-\frac{1}{2}\left[1+\frac{1}{n-1}(2\ln_2 n+2\ln_3 n)\right]u\right\}u^{\frac{n-1}{2}-1}du$$
$$\leqslant\frac{\ln_3 n}{a_nd\sqrt{2\pi}}\left(1+\frac{2\ln_2 n+2\ln_3 n}{n-1}\right)^{-\frac{n-1}{2}}=o\left(\frac{1}{(\ln n)\ln_2 n}\right).$$

On the other hand

$$\int_0^{b_n}\left[1-\Phi\left(\frac{a_n\sqrt{u}}{\sqrt{n-1}}-\frac{\sqrt{n}\,\theta}{\sigma}\right)\right]dG_{n-1}(u)$$
$$\leqslant\int_0^{b_n}\left[2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)\right]^{-1}u^{\frac{n-1}{2}-1}e^{-\frac{u}{2}}du$$
$$=\frac{b_n^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)}=o\left(\frac{1}{\sqrt{n}}\right).$$

Therefore the in tegral in (2.10) is $o\left(\frac{1}{(\ln n)\ln_2 n}\right)$. We now proceed to estimate the integral

$$\int_0^\infty\frac{a_n}{\sqrt{n-1}}\sqrt{u}\,\exp\left\{-\frac{1}{2}\left(\frac{a_n\sqrt{u}}{\sqrt{n-1}}-\frac{\sqrt{n}\,\theta}{\sigma}\right)^2\right\}dG_{n-1}(u). \tag{2.11}$$

It is obvious that

$$\int_{b_n}^\infty\frac{a_n}{\sqrt{n-1}}\sqrt{u}\,\exp\left\{-\frac{1}{2}\left(\frac{a_n\sqrt{u}}{\sqrt{n-1}}-\frac{\sqrt{n}\,\theta}{\sigma}\right)^2\right\}dG_{n-1}(u)$$
$$\leqslant\frac{a_n}{\sqrt{n-1}}\left[2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)\right]^{-1}\int_{b_n}^\infty u^{\frac{n}{2}-1}\exp\left\{-\frac{1}{2}\left(1+\frac{2\ln_2 n+2\ln_3 n}{n-1}\right)u\right\}du$$
$$\leqslant\frac{a_n}{\sqrt{n-1}}\left[2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)\right]^{-1}2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\cdot\left(1+\frac{2\ln_2 n+2\ln_3 n}{n-1}\right)^{-\frac{n}{2}}$$
$$=O\left(\frac{1}{(\ln n)\sqrt{\ln_2 n}}\right)\quad(n\to\infty).$$

When $n$ is large enough, we have $b_n<n/e$. Hence

$$\int_0^\infty\frac{a_n}{\sqrt{n-1}}\sqrt{u}\,\exp\left\{-\frac{1}{2}\left(\frac{a_n\sqrt{u}}{\sqrt{n-1}}-\frac{\sqrt{n}\,\theta}{\sigma}\right)^2\right\}dG_{n-1}(u)$$
$$\leqslant\int_0^{n/e}\frac{a_n\sqrt{u}}{\sqrt{n-1}}dG_{n-1}(u)=\frac{a_n}{\sqrt{n-1}}\left(2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)\right)^{-1}\left(\frac{n}{e}\right)^{\frac{n}{2}}\cdot\frac{2}{n}$$
$$=o\left(\frac{1}{(\ln n)\sqrt{\ln_2 n}}\right).$$

In virtue of the discussion above, there is $M>0$ such that

$$E_{\theta\sigma}(\theta_{n+1}-\theta)^2\leqslant\frac{M}{n\,(\ln n)\sqrt{\ln_2 n}}+\theta^2$$

for all small enough $\theta$ and $e<n\leqslant n_1$.

Therefore

$$\sum_{n=3}^{n_1}E_{\theta\sigma}(\theta_{n+1}-\theta)^2\leqslant M\sum_{n=3}^{n_1}\frac{1}{n\,(\ln n)\sqrt{\ln_2 n}}+n_1\theta^2=o\left(\ln_2\frac{1}{\theta}\right)\quad(\theta\downarrow 0).$$

So, (2.9) is true.

**Lemma 5.** *Let* $\theta>0$, $\delta>0$, $\lambda>1$ *and* $n_2=\left[2\lambda\sigma^2\theta^{-2}\ln_2\frac{1}{\theta}\right)$. *Then*

$$\sum_{n_2+1}^{\infty}\int_{\{|\theta_{n+1}-\theta|>\delta\theta\}}(\theta_{n+1}-\theta)^2dP_{\theta\sigma}=o(1)\quad(\theta\downarrow 0). \tag{2.12}$$

*Proof*

$$\sum_{n_2+1}^{\infty}\int_{\{+|\theta_{n+1}-\theta|>\delta\theta\}}(\theta_{n+1}-\theta)^2\,dP_{\theta\sigma}$$

$$\leqslant\frac{1}{\delta^2\theta^2}\sum_{n_2+1}^{\infty}E_{\theta\sigma}(\theta_{n+1}-\theta)^4$$

$$\leqslant\frac{1}{\delta^2\theta^2}\sum_{n_2+1}^{\infty}E_{\theta\sigma}\Big(\frac{S_n-n\theta}{n}\Big)^4+\frac{\theta^2}{\delta^2}\sum_{n_2+1}^{\infty}E_{\theta\sigma}\Big[\Phi\Big(\frac{a_nv_n-\sqrt{n}\,\theta}{\sigma}\Big)\Big]$$

$$\leqslant\frac{1}{\delta^2\theta^2}\sum_{n_2+1}^{\infty}\frac{3\sigma^4}{n^2}+\frac{\theta^2}{\delta^2}\sum_{n_2+1}^{\infty}\int_{\{v_n\leqslant\sqrt{1+\eta}\,\sigma\}}\Phi\Big(\frac{a_nv_n-\sqrt{n}\,\theta}{\sigma}\Big)dP_{\theta\sigma}$$

$$+\frac{\theta^2}{\delta^2}\sum_{n_2+1}^{\infty}P_{\theta\sigma}\,(v_n>\sqrt{1+\eta}\,\sigma)$$

$$\triangleq\mathrm{I}+\mathrm{II}+\mathrm{III}\quad\Big(\text{Here }\eta\in\Big(0,\frac{\lambda-1}{\lambda+1}\Big)\Big).$$

We now proceed to prove that I, II and III are $o(1)$ as $\theta\downarrow 0$.

At first

$$I=\frac{3\sigma^4}{\delta^2\theta^2}\sum_{n_2+1}^{\infty}\frac{1}{n^2}\leqslant\frac{3\sigma^4}{\delta^2\theta^2}\cdot\frac{1}{n_2}=o(1).$$

Secondly, set

$$Y_i=\frac{x_1+\cdots+x_i-ix_{i+1}}{\sqrt{i(i+1)}},\ z_i=Y_i^2-1\quad(i=1,2,\cdots).\tag{2.13}$$

It is well-known that $z_1, z_2, \cdots$ is independent sequence with common distribution and $E_{01}z_1=0$, $E_{01}z_1^2<\infty$. Since $v_n^2=\frac{1}{n-1}\sum_1^{n-1}Y_i^2$, we obtain

$$\mathrm{III}=\frac{\theta^2}{\delta^2}\sum_{n_2+1}^{\infty}P_{\theta\sigma}(v_n^2>(1+\eta)\sigma^2)=\frac{\theta^2}{\delta^2}\sum_{n_2+1}^{\infty}P_{01}\Big(\frac{1}{n-1}\sum_1^{n-1}z_i>\eta\Big).$$

Since $\frac{1}{n}\sum_1^n z_i$ is completely convergent to zero, we have $\sum_{n=2}^{\infty}P_{01}\Big(\frac{1}{n-1}\sum_1^{n-1}z_i>\eta\Big)<\infty$, which implies $\mathrm{III}=o(1)$ as $\theta\downarrow 0$.

At last we consider II. We first point out that for any fixed

$$\varepsilon\in\Big(0,1-\sqrt{\frac{\lambda+1}{2\lambda}(1+\eta)}\Big),$$

$$a_n\sqrt{1+\eta}-\frac{\sqrt{n}\,\theta}{\sigma}<-\frac{\varepsilon\theta\sqrt{n}}{\sigma}\tag{2.14}$$

for all small enough $\theta$ and $n\geqslant n_2$.

In fact, since $\frac{t}{\ln t}$ is increasing when $t$ is large enough and

$$\lim_{\theta\downarrow 0}\frac{\theta}{\sigma}\sqrt{n_2}/[(1+\eta)(\lambda+1)\ln_2 n_2]^{\frac{1}{2}}=\sqrt{\frac{2\lambda}{(1+\eta)(\lambda+1)}},$$

we have

$$\frac{\theta}{\sigma}\sqrt{n}/[(1+\eta)(\lambda+1)\ln_2 n]^{\frac{1}{2}}>1/(1-\varepsilon)\tag{2.15}$$

for all small enough $\theta$ and $n\geqslant n_2$.

Since $(\lambda-1)\ln_2 n>3\ln_3 n$ for large $n$, it follows from (2.15) that

$$\sqrt{(1+\eta)(2\ln_2 n+3\ln_3 n)}-\sqrt{n}\,\theta/\sigma<-\varepsilon\sqrt{n}\,\theta/\sigma.$$

Thus (2.14) is proved.

On the other hand, setting $f(x)=\sqrt{1+\eta}(2\ln_2 x+3\ln_3 x)^{1/2}-\sqrt{x}\cdot\theta/\sigma$, it is easy to see $f(n_2)\to-\infty$ as $\theta\downarrow 0$. Hence $f(n_2)<-1$ when $\theta$ is small enough. It can be shown that $f'(x)<0$ when $\theta$ is small enough and $x\geqslant n_2$. Therefore

$$\sqrt{1+\eta}\,a_n-\sqrt{n}\,\theta/\sigma<-1 \tag{2.16}$$

for all small $\theta$ and $n\geqslant n_2$.

By (2.14) and (2.16), $\Phi(\sqrt{1+\eta}\,a_n-\sqrt{n}\,\theta/\sigma)\leqslant\varphi\left(\frac{-\varepsilon\theta\sqrt{n}}{\sigma}\right)$ and

$$\begin{aligned}\text{II}&\equiv\frac{\theta^2}{\delta^2}\sum_{n_2+1}^{\infty}\int_{\{v_n<\sqrt{1+\eta}\,\sigma\}}\Phi\left(\frac{a_n v_n-\sqrt{n}\,\theta}{\sigma}\right)dP_{\theta\sigma}\\
&\leqslant\frac{\theta^2}{\delta^2}\sum_{n_2+1}^{\infty}\Phi(a_n\sqrt{1+\eta}-\sqrt{n}\,\theta/\sigma)\leqslant\frac{\theta^2}{\delta^2}\sum_{n_2+1}^{\infty}\varphi\left(\frac{-\varepsilon\theta\sqrt{n}}{\sigma}\right)\\
&\leqslant\frac{\theta^2}{\delta^2}\sum_{n_2+1}^{\infty}\exp\left\{-\frac{1}{2\sigma^2}\varepsilon^2\theta^2 n\right\}\\
&\leqslant\frac{\theta^2}{\delta^2}\exp\left\{-\frac{\varepsilon^2\theta^2}{2\sigma^2}n_2\right\}\cdot(1-e^{-\frac{1}{2}\varepsilon^2\theta^2/\sigma^2})^{-1}=o(1)\quad(\theta\to 0).\end{aligned}$$

This finishes the proof of Lemma 5.

**Lemma 6.** *Let $x_1, x_2, \cdots$ be independent random variables with common distribution $N(\theta,\sigma^2)$. Then for any integer $i>0$ and $\varepsilon\in\left(0,\frac{1}{4}\right)$,*

$$P_{\theta\sigma}\{\text{there exists } n \text{ such that } i\leqslant n\leqslant 2i\ln_2 i$$

*and*

$$|s_n-n\theta|\geqslant\sqrt{2(1-\varepsilon)n(\ln_2 n)}\cdot v_n\}=o((\ln i)^{4\varepsilon-1}). \tag{2.17}$$

*Proof* $P_{\theta\sigma}$ {there exists $n$ such that $i\leqslant n\leqslant 2i\ln_2 i$ and $|s_n-n\theta|\geqslant\sqrt{2(1-\varepsilon)n\ln_2 n}\,v_n\}\leqslant P_{01}$ {there exists $n$ such that $i\leqslant n\leqslant 2i\ln_2 i$ and $v_n^2<1-\varepsilon\}+P_{01}\{v_n^2\geqslant 1-\varepsilon)$ for all $n\in[i, 2i\ln_2 i]$ and there exists some $n$ such that $i\leqslant n\leqslant 2i\ln_2 i$, $|s_n|\geqslant\sqrt{2(1-\varepsilon)n\ln_2 n}\,v_n\}\leqslant P_{01}\{\max_{i\leqslant n\leqslant 2i\ln_2 i}|v_n^2-1|\geqslant\varepsilon\}+P_{01}$ {there exists some $n$ such that $i\leqslant n\leqslant 2i\ln_2 i$ and $|s_n|\geqslant\sqrt{2(1-\varepsilon)n\ln_2 n}\,v_n\}$. It is known that the last term in the expression above is $o((\ln i)^{3\varepsilon-1})$ (cf. [2], p. 423).

On the other hand, using (2.13) and Kolmogoroff inequality, we obtain

$$\begin{aligned}P_{01}(\max_{i\leqslant n\leqslant 2i\ln_2 i}|v_n^2-1|\geqslant\varepsilon)&\leqslant P_{01}\left(\max_{i\leqslant n\leqslant 2i\ln_2 i}\left|\sum_1^{n-1}z_i\right|\geqslant(i-1)\varepsilon\right)\\
&\leqslant(i-1)^{-2}\varepsilon^{-2}2i(\ln_2 i)E_{01}z^2\\
&=o((\ln i)^{4\varepsilon-1}).\end{aligned}$$

Thus the Lemma 6 is proved.

## § 3. Proof of Theorem A

Since the conclusions (1.5) and (1.6) of Theorem A have been proved above, it suffices to prove (1.7), i. e. $\lim\limits_{\theta\downarrow 0}\frac{1}{\theta^2}\left(\ln_2\frac{1}{\theta}\right)^{-1}E_{\theta\sigma}T_2=2\sigma^2P_{0\sigma}(T_2=\infty)$ for all $\sigma>0$.

At first, we analyse $E_{\theta\sigma}\left(\sum\limits_1^{T_2}(\theta_{n+1}-\theta)^2\right)$. Let $\delta>0$, $n_1=\left[\theta^{-2}\left(\ln_2\frac{1}{\theta}\right)^{-1}\right]$, $\lambda>1$ and $n_2=\left[2\lambda\sigma^2\theta^{-2}\ln_2\frac{1}{\theta}\right]$. Then

$$E_{\theta\sigma}\left(\sum_1^{T_2}(\theta_{n+1}-\theta)^2\right)\leqslant\sum_{n=0}^{\infty}\int_{\{T_2>n\}}(\theta_{n+1}-\theta)^2\,dP_{\theta\sigma}\leqslant\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4,$$

where

$$\Sigma_1=\sum_{n=0}^{n_1}E_{\theta\sigma}(\theta_{n+1}-\theta)^2,$$

$$\Sigma_2=\sum_{n=n_1+1}^{n_2}\int_{\{T_2>n_1\}}\theta_{n+1}-\theta)^2\,dP_{\theta\sigma},$$

$$\Sigma_3=\sum_{n=n_2+1}^{\infty}\int_{\{|\theta_{n+1}-\theta|>\delta\theta\}}(\theta_{n+1}-\theta)^2\,dP_{\theta\sigma},$$

$$\Sigma_4=\sum_{n=n_2+1}^{\infty}\int_{\{T_2>n,\,|\theta_{n+1}-\theta|\leqslant\delta\theta\}}(\theta_{n+1}-\theta)^2\,dP_{\theta\sigma}.$$

By Lemmas 4 and 5, $\Sigma_1=o\left(\ln_2\frac{1}{\theta}\right)$ and $\Sigma_3=o\left(\ln_2\frac{1}{\theta}\right)$ as $\theta\downarrow 0$. It is easy to see that

$$\Sigma_4\leqslant\delta^2\theta^2\sum_{n=1}^{\infty}P_{\theta\sigma}(T_2>n)\leqslant\delta^2\theta^2E_{\theta\sigma}T_2.$$

We now proceed to estimate $\Sigma_2$.

$$\Sigma_2\leqslant\sum_{n=n_1+1}^{n_2}\left[\int_{\{T_2>n_1\}}\left(\frac{s_n-n\theta}{n}\right)^2dP_{\theta\sigma}+\int_{\{T_2>n_1\}}\theta^2\,dP_{\theta\sigma}\right]$$

$$\leqslant\sum_{n=n_1+1}^{n_2}\frac{\sigma^2}{n}+\theta^2n_2P_{\theta\sigma}(T_2>n_1)$$

$$\leqslant\sigma^2\ln\frac{n_2}{n_1}+2\lambda\sigma^2\left(\ln_2\frac{1}{\theta}\right)P_{\theta\sigma}(T_2>n_1).$$

Since

$$\ln\frac{n_2}{n_1}=o\left(\ln_2\frac{1}{\theta}\right)$$

and

$$\overline{\lim_{\theta\to 0}}\,P_{\theta\sigma}(T_2>n_1)\leqslant P_{0\sigma}(T_2=\infty),$$

we obtain

$$\Sigma_2\leqslant 2\lambda\sigma^2P_{0\sigma}(T_2=\infty)\cdot\ln_2\frac{1}{\theta}+o\left(\ln_2\frac{1}{\theta}\right).$$

Therefore

$$E_{\theta\sigma}\left(\sum_1^{T_2}(\theta_{n+1}-\theta)^2\right)\leqslant 2\lambda\sigma^2P_{0\sigma}(T_2=\infty)\ln_2\frac{1}{\theta}+\delta^2\theta^2E_{\theta\sigma}(T_2)+o\left(\ln_2\frac{1}{\theta}\right)\quad(\theta\downarrow 0).$$

Combining this with Lemma 2 and Lemma 3 yields

$$\theta^2(1-\delta^2)E_{\theta\sigma}T_2\leqslant 2\lambda\sigma^2 P_{0\sigma}(T_2=\infty)\cdot \ln_2\frac{1}{\theta}+o\left(\ln_2\frac{1}{\theta}\right)+O(1).$$

So,

$$\overline{\lim_{\theta\downarrow 0}}\,\theta^2\left(\ln_2\frac{1}{\theta}\right)^{-1}E_{\theta\sigma}T_2\leqslant\frac{2\lambda\sigma^2}{1-\delta^2}P_{0\sigma}(T_2=\infty).$$

Letting $\delta\downarrow 0$ and $\lambda\downarrow 1$, we obtain

$$\overline{\lim_{\theta\downarrow 0}}\,\theta^2\left(\ln_2\frac{1}{\theta}\right)^{-1}E_{\theta\sigma}T_2\leqslant 2\sigma^2 P_{0\sigma}(T_2=\infty). \tag{3.1}$$

To prove the reverse inequality, we suppose $n_0<m_1=\left[\frac{\lambda\sigma^2}{\theta^2}\right]$ and $m_2=2m_1\ln_2 m_1$, where $\lambda\in(0,1)$. It is not hard to see that $\{m_1<T_2\leqslant m_2\}\subset\{$There exists $n$ such that $m_1<n\leqslant m_2$ and $\theta_n\neq 0\}\subset\{$There exists $n$ such that $m_1\leqslant n\leqslant m_2$ and $s_n\geqslant\sqrt{n}\,a_n v_n\}$. Let $\varepsilon\in\left(0,\frac{1-\lambda}{4}\right)$. By Lemma 6, $P_{0\sigma}(m_1<T_2\leqslant m_2)=o((\ln m_1)^{4\varepsilon-1})$ as $\theta\downarrow 0$. Given $a>0$, it can be shown that

$$\begin{aligned}P_{\theta\sigma}(m_1<T_2\leqslant m_2)&\leqslant P_{\theta\sigma}\left(\frac{s_{m_2}-m_2\theta}{\sqrt{m_2}\,\sigma}>a\right)+P_{\theta\sigma}\left(\frac{s_{m_2}-m_2\theta}{\sqrt{m_2}\,\sigma}\leqslant a,\ m_1<T_2\leqslant m_2\right)\\&\leqslant 1-\Phi(a)+\exp\left\{\frac{\theta^2 m_2}{2\sigma^2}+\frac{a\theta\sqrt{m_1}}{\sigma}\right\}P_{0\sigma}(m_1<T_2\leqslant m_2)\\&=1-\Phi(a)+o(1)\ (\theta\downarrow 0).\end{aligned}$$

Since $a>0$ is arbitrary, we have

$$\lim_{\theta\downarrow 0}P_{\theta\sigma}(m_1<T_2\leqslant m_2)=0.$$

On the other hand

$$\begin{aligned}P_{\theta\sigma}(n_0\leqslant T_2\leqslant m_1)&\leqslant 1-\Phi(a)+\exp\left\{\frac{\theta^2 m_1}{2\sigma^2}+\frac{a\theta\sqrt{m_1}}{\sigma}\right\}\cdot P_{0\sigma}(n_0<T_2\leqslant m_1)\\&\leqslant 1-\Phi(a)+\exp\left\{\frac{\lambda}{2}+a\sqrt{\lambda}\right\}\cdot P_{0\sigma}(n_0<T_2<\infty).\end{aligned}$$

Noting (4.2) and $E_{\theta\sigma}T_2\geqslant m_2P_{\theta\sigma}(T_2>m_2)=m_2\{P_{\theta\sigma}(T_2>n_0)-P_{\theta\sigma}(n_0<T_2\leqslant m_1)-P_{\theta\sigma}(m_1<T_2\leqslant m_2)\}$, we obtain

$$\lim_{\theta\downarrow 0}\frac{1}{m_2}E_{\theta\sigma}T_2\geqslant P_{0\sigma}(T_2=\infty)-\left\{1-\Phi(a)+\exp\left\{\frac{\lambda}{2}+a\sqrt{\lambda}\right\}P_{0\sigma}(n_0<T_2<\infty)\right\}.$$

Letting $n_0\to\infty$, we have

$$\lim_{\theta\downarrow 0}\frac{1}{m_2}E_{\theta\sigma}T_2\geqslant P_{0\sigma}(T_2=\infty)-\{1-\Phi(a)\}.$$

Since $a$ is arbitrary and $m_2\sim 2\lambda\sigma^2\cdot\frac{1}{\theta^2}\ln_2\frac{1}{\theta}\ (\theta\downarrow 0)$, we have

$$\lim_{\theta\downarrow 0}\theta^2\left(\ln_2\frac{1}{\theta}\right)^{-1}E_{\theta\sigma}T_2\geqslant 2\lambda\sigma^2P_{0\sigma}(T_2=\infty). \tag{3.3}$$

Letting $\lambda\uparrow 1$ in (4.3), we have

$$\lim_{\theta\downarrow 0}\theta^2\left(\ln_2\frac{1}{\theta}\right)^{-1}E_{\theta\sigma}T_2\geqslant 2\sigma^2P_{0\sigma}(T_2=\infty). \tag{3.4}$$

Combining (4.4) with (4.1) yields (1.7). Thus we conclude the proof of Theorem A.

**Remark 1°** From Lemam 3 and $\sum_{k=1}^{T_2}\left(\theta_k x_k-\frac{1}{2}\theta_k^2\right)\geqslant cv_m^2$, it follows that

$$E_{\theta\sigma}T_2\geqslant 2\theta^{-2}C\sigma^2 \quad \text{for all} \quad \theta>0,\ \sigma>0. \tag{3.5}$$

We point out that this inequality is asymptotically sharp as $c\to\infty$. In fact, observing the proof of Lemma 3, we obtain

$$E_{\theta\sigma}T_2\leqslant(1-\eta^2\theta^{-2})^{-1}\left\{\theta^{-2}\eta^{-2}\sum_{n=0}^{\infty}E_{\theta\sigma}(\theta_{n+1}-\theta)^4\right.$$
$$\left.+2\theta^{-2}\int_{\{T_2<\infty\}}\sum_{1}^{T_2}\left(\theta_k x_k-\frac{1}{2}\theta_k^2\right)dP_{\theta\sigma}+2\theta^{-2}c\sigma^2\right\}.$$

So, from Lemma 2 it follows that

$$E_{\theta\sigma}T_2\leqslant(1-\eta^2\theta^{-2})^{-1}\theta^{-2}(M+2c\sigma^2),$$

where $M$ is independent of $c$. Therefore

$$\varlimsup_{C\to\infty}\theta^{+2}c^{-1}\sigma^{-2}E_{\theta\sigma}T_2\leqslant(1-\eta^2\theta^{-2})^{-1}.$$

Letting $\eta\downarrow 0$, we have

$$\varlimsup_{C\to\infty}2^{-1}c^{-1}\sigma^{-2}\theta^2E_{\theta\sigma}T_2\leqslant 1,$$

which and (4.5) imply

$$E_{\theta\sigma}T_2\sim\frac{2c\sigma^2}{\theta^2} \quad (c\to\infty).$$

**Remark 2°** (1.7) shows that $T_2$ is asymptotically optimal as $\theta\downarrow\theta_0$. It is natural to ask whether $T_2$ has asymptotically optimality as $\alpha=P_{\theta\sigma}(T_2=\infty)\downarrow 0$ or equavalently $c\to\infty$. It remains to be solved.

## References

[1] Farrell, R., Asymptotic behavior of the expected sample size in certain one-sided tests, *Ann. Math. Statist.*, **35** (1964), 36—72.

[2] Robbins, H. and Siegmund, D., The expected sample size of some tests of power one, *Ann. of statistics*, **2**: 3 (1974), 415—436.

[3] Chen Jiading, Asymptotic optimality of some power-one tests for normal mean with unknown variance, *Scientia sinica, series A*, **28**: 9 (1985), 938—949.