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REMARKS ON HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN A HIGHER DIMENSIONAL PSEUDO-SPHERE*

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Abstract

A necessary condition to be satisfied by the metric of an n-manifold minimally immersed in an $(n+1)$-pseudo-sphere is obtained, and a sufficient condition for a complete hypersurface in a pseudo-sphere with constant mean curvature to be totally umbilical is given.

§ 1. Introduction

Let H^{n+1} be an $(n+1)$-dimensional unit pseudo-sphere, i.e., a complete simply connected space with constant sectional curvature -1. In this note, we give a necessary condition to be satisfied by the metric of an n-dimensional Riemannian manifold minimally immersed in H^{n+1}, and give a sufficient condition for a complete hypersurface in H^{n+1} with constant mean curvature to be totally umbilical. In [1] Barbosa and Do Carmo proved that if g is the induced metric of a minimal surface in H^3 and K is the Gauss curvature of g, then the Gauss curvature $\hat{K}$ of $\hat{g}=-Kg$ satisfies $\hat{K}\leqslant 1$ (cf. [1], Proposition 2.2). We extend it to higher dimension as follows:

Theorem 1. *Let g be the induced metric of a minimal hypersurface in H^{n+1} and let R denote the scalar curvatures of g. Then the scalar curvature $\hat{R}$ of the conformal metric $\hat{g}=\sigma g$, where either $\sigma=-R$ or $\sigma=-R-2n(n-1)/3$, satisfies*

$$\hat{R}\geqslant 2n-3. \tag{1}$$

On the other hand, as a generalization of the Hilbert-Liebmann Theorem, Yau proved that if M is a compact hypersurface in H^{n+1} which has constant scalar curvature and positive sectional curvature, then M is totally umbilical ([2], Theorem 11). Now, by employing Omori-Yau's maximum principle, we prove the

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following theorem.

Theorem 2. *Let M be a connected complete hypersurface in $H^{n+1}(n\geqslant 3)$ with constant mean curvature k. If the scalar curvature R of M satisfies*

$$R\geqslant\frac{n-2}{n-1}n^2k^2-(n-2)(n+1), \tag{2}$$

then M is totally umbilical.

Moreover, it is possible to generalize Theorem 1 and Theorem 2 to higher codimension, and we shall discuss it in another paper.

§ 2. Fundamental Formulas

Throughout this paper, we follow closely the notations and the exposition in [2], unless otherwise stated. Let M be a hypersurface in H^{n+1} and let $e_1, \cdots, e_{n+1}$ be a local field of orthonormal frames in H^{n+1} such that, restricted to M, the vector e_{n+1} is normal to M. Then, the second fundamental form B and the mean curvature k for M can be written as (cf. [2])*

$$B=\Sigma h_{ij}\omega_i\omega_j e_{n+1}, \quad k=\frac{1}{n}\Sigma h_{ii}.$$

The Gauss–Codazzi equations for M are

$$R_{ijkl}=\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl}+h_{ik}h_{jl}-h_{il}h_{jk}, \tag{3}$$

$$h_{ijk}=h_{ikj}. \tag{4}$$

It follows from (3) that the scalar curvature R of M is

$$R=-n(n-1)+n^2k^2-\|B\|^2, \tag{5}$$

where $\|B\|^2=\Sigma(h_{ij})^2$. We denote by Δ the Laplacian relative to the induced metric on M. If $k=$constant, then (cf. [8])

$$\frac{1}{2}\Delta(\|B\|^2)=\|\nabla B\|^2-\|B\|^4-n(\|B\|^2+nk^2)+nkW, \tag{6}$$

where

$$\|\nabla B\|^2=\Sigma(h_{ijk})^2, \quad W=\Sigma h_{ij}h_{jk}h_{ki}. \tag{7}$$

Setting

$$l_{ij}=h_{ij}-k\delta_{ij}, \quad L=(l_{ij}), \quad f^2=\operatorname{tr} L^2(f>0), \tag{8}$$

we have $\operatorname{tr} L=0$ and $f^2=\|B\|^2-nk^2$, so that M is totally umbilical iff $f^2=0$ identically. Repeating the same calculation as Okumura has done in [3], one can get

$$\begin{aligned}\frac{1}{2}\Delta f^2 &\geqslant \|\nabla B\|^2+f^2\left\{nk^2-n-\frac{n-2}{\sqrt{n(n-1)}}n|k|f-f^2\right\}\\ &\geqslant -f^2\left\{f^2+\frac{n-2}{\sqrt{n(n-1)}}n|k|f-n(k^2-1)\right\}.\end{aligned} \tag{9}$$

Here, as shown in (8), f is a nonnegative function on M.

* We shall agree the range of Latin indices with $\{1, 2, \cdots, n\}$.

§ 3. The Proof of Theorem 1

First of all, for a minimal hypersurface M in H^{n+1}, it follows from (5) that $-R>0$ and that $-R-2n(n-1)/3>0$. Thus, we can define a conformal metric $\hat{g}=\sigma g$ on M, where $\sigma=-R$ or $\sigma=-R-2n(n-1)/3$. As well known, the scalar curvature $\hat{R}$ of $\hat{g}$ satisfies[4]

$$\sigma\hat{R}=R-(n-1)\Delta\log\sigma-\frac{1}{4}(n-1)(n-2)|\nabla\log\sigma|^2. \tag{10}$$

We now, for preciseness, consider the case that $\sigma=-R$. From (5) and (10) we have

$$-\sigma\hat{R}=\sigma+(n-1)\frac{\Delta\sigma}{\sigma}+\frac{(n-1)(n-6)|\nabla\sigma|^2}{4\sigma^2}, \tag{11}$$

where, in view of (5),

$$|\nabla\sigma|^2=|\nabla(\|B\|^2)|^2=4\sum_k(\sum_{i,j}h_{ij}h_{ijk})^2. \tag{12}$$

At any point of M, let $h_{ij}=\lambda_i\delta_{ij}$. From (5) and (12) it turns out that

$$\frac{1}{4}|\nabla\sigma|^2=\sum_k(\sum_i\lambda_ih_{iik})^2\leqslant\|B\|^2(\sum_{i,k}h^2_{iik})\leqslant\sigma(\sum_{i,k}h^2_{iik}) \tag{13}$$

at that point.

On the other hand, from (4) and (7) we have

$$\|\nabla B\|^2\geqslant3\sum_{i\neq k}h^2_{iik}+\sum_k h^2_{kkk}=2\sum_{i\neq k}h^2_{iik}+\sum_{i,k}h^2_{iik}. \tag{14}$$

For a fixed index k, using the condition that M is minimal, one can easily get

$$\sum_i h^2_{iik}=\sum_{i\neq k}h^2_{iik}+(\sum_{i\neq k}h_{iik})^2\leqslant\sum_{i\neq k}h^2_{iik}+(n-1)\sum_{i\neq k}h^2_{iik}=n\sum_{i\neq k}h^2_{iik}. \tag{15}$$

Summing for k in (15), we have

$$\sum_{i\neq k}h^2_{iik}\geqslant\frac{1}{n}\sum_{i,k}h^2_{iik},$$

which together with (13) and (14) yields

$$\|\nabla B\|^2\geqslant(n+2)|\nabla\sigma|^2/4n\sigma. \tag{16}$$

Tnus, it follows from (5), (6), (16) and $k=0$ that

$$\begin{aligned}\frac{1}{2}\Delta\sigma=\frac{1}{2}\Delta(\|B\|^2)&=\|\nabla B\|^2-\|B\|^4-n\|B\|^2\\&=\|\nabla B\|^2-\sigma^2+n(n-1)\sigma+n(n-2)\|B\|^2\\&\geqslant(n+2)|\nabla\sigma|^2/4n\sigma-\sigma^2,\end{aligned}$$

i.e.,

$$\Delta\sigma/\sigma\geqslant(n+2)|\nabla\sigma|^2/2n\sigma^2-2\sigma. \tag{17}$$

Substituting (17) into (11), wet get

$$-\sigma\hat{R}\geqslant\sigma-2(n-1)\sigma+(n-1)(n-2)^2|\nabla\sigma|^2/4n\sigma^2\geqslant\sigma-2(n-1)\sigma,$$

which implies (1). This proves Theorem 1 for the case that $\sigma=-R$. For the case that $\sigma=-R-2n(n-1)/3$, we see from (5) that $\sigma>0$. In this case, (11) will be replaced by the following

$$-\sigma\hat{R}=\sigma+\frac{2}{3}n(n-1)+(n-1)\frac{\Delta\sigma}{\sigma}+\frac{(n-1)(n-6)|\nabla\sigma|^2}{4\sigma^2}$$
$$\geqslant\sigma+(n-1)\frac{\Delta\sigma}{\sigma}+\frac{(n-1)(n-6)|\nabla\sigma|^2}{4\sigma^2}.$$

The remainder of the proof is just the same as the above, and we omit it here. Hence, Theorem 1 is proved completely.

§ 4. The Proof of Theorem 2

The proof of Tneorem 2 is based on the following

Generalized Maximum Principle (Omori-Yau)[5,6]. *Let M be a complete Riemannian manifold with Ricci curvature bounded below and f be a C^2-function bounded above on M. Then, there exists a sequence $\{x_t\}$ $(t=1, 2, \cdots)$ on M such that*

$$\lim_{t\to\infty} f(x_t)=\sup f,\quad \lim_{t\to\infty}|\nabla f|(x_t)=0,\quad \lim_{t\to\infty}\Delta f(x_t)\leqslant 0.$$

We now prove Theorem 2. By virtue of (5) the condition (2) is equivalent to

$$f^2\leqslant\frac{n}{n-1}k^2-2, \tag{18}$$

where $f^2=\|B\|^2-nk^2\geqslant 0$ and $f\geqslant 0$ as in (8).

On putting

$$b^2=n^2k^2-4(n-1), \tag{19}$$

we see from (18) that $b^2>n^2k^2-2n(n-1)\geqslant n(n-1)f^2$ for $n\geqslant 3$, so we can assume that $b>0$. Then from (9), by a direct calculation, one can easily get

$$\frac{1}{2}\Delta f^2\geqslant\|\nabla B\|^2-f^2\left\{f+\frac{n}{2\sqrt{n(n-1)}}[(n-2)|k|+b]\right\}$$
$$\times\left\{f+\frac{n}{2\sqrt{n(n-1)}}[(n-2)|k|-b]\right\}$$
$$\geqslant f^2\left\{\frac{n}{2\sqrt{n(n-1)}}[(n-2)|k|+b]+f\right\}$$
$$\times\left\{\frac{n}{2\sqrt{n(n-1)}}[b-(n-2)|k|]-f\right\}. \tag{20}$$

Using the fact that $n|k|b<n^2k^2-2(n-1)$ for $n>1$, we have

$$\frac{n}{2(n-1)}(n-2)|k|b<\frac{n}{2(n-1)}k^2(n^2-2n+2)-\frac{n}{n-1}k^2-n+2,$$

i.e.,

$$\frac{n}{n-1}k^2-2<\frac{1}{4}\frac{n}{n-1}[b-(n-2)|k|]^2$$

for $n\geqslant 3$.

Now, since k is constant, there exists a positive number ε, such that

$$\left(\frac{n}{n-1}k^2-2\right)^{1/2}\leqslant\frac{1}{2}\frac{n}{\sqrt{n(n-1)}}[b-(n-2)|k|]-\varepsilon,$$

which together with (18) yields

$$\sup f \leqslant \frac{1}{2} \frac{n}{\sqrt{n(n-1)}} [b-(n-2)|k|] - \varepsilon. \tag{21}$$

On the other hand, by using the result of [7], we see that the inequality (2) implies that the sectional curvatures of M are nonnegative. Furthermore, the inequality (18) shows that f^2 is bounded above. Thus, we can apply Omori-Yau's generalized maximum principle to f^2, i.e., there exists a sequence of points $\{x_t\}$ $(t=1, 2, \cdots)$ on M such that

$$\lim_{t\to\infty} f^2(x_t) = \sup f^2, \quad \lim_{t\to\infty} |\nabla f^2|(x_t) = 0, \quad \lim_{t\to\infty} \Delta f^2(x_t) \leqslant 0. \tag{22}$$

Since f is a nonnegative function on M, we have $\sup f^2 = (\sup f)^2$. Thus, from (22) and (20) we conclude

$$0 \geqslant (\sup f^2) \left\{ \frac{n}{2\sqrt{n(n-1)}} [b+(n-2)|k|] + \sup f \right\} \times \left\{ \frac{n}{2\sqrt{n(n-1)}} [b-(n-2)|k|] - \sup f \right\},$$

which together with (21) implies that $\sup f^2 = 0$, so that $f^2 = 0$ on M everywhere. Theaefore, M is totally umbilical and Theorem 2 is proved.

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