Chin. Ann. of Math.
7B (4) 1986

INVARIANT TESTS OF EXISTENCE OF THE LINEAR RELATIONSHIP WITH LEFT $O(n)$-INVARIANT ERRORS

BIAN GUORUI (卞国瑞)* ZHANG YAOTING (张尧庭)**

Abstract

This paper gives the invariant tests of the existence of a linear relationship among row vectors of the mean matrix of the multivariate linear models with the left $O(n)$-invariant errors. Some asymptotic properties of these testing methods are also discussed.

§ 1. Introduction

In the multivariate linear models, to test whether there is a linear relationship among row vectors of the mean matrix is an important problem. In the papers [1] and [2], we discussed this problem by supposing the error's distribution is the normal distribution $N(O, I_n, \Sigma)$, and reduced the models and problems to the following two canonical forms:

Model I.

$$\underset{n\times k}{Y}=\underset{n\times k}{M}+\underset{n\times k}{\epsilon},\quad M=\begin{pmatrix}\theta\\ \eta\\ O\end{pmatrix}\begin{matrix}p\\ q\\ n-p-q\end{matrix} \tag{1.1}$$

and the hypothesis tested is H_0: there is a matrix $\underset{q\times p}{C}$ such that $\eta=C\theta$.

Model II.

$$\underset{n\times k}{Y}=\underset{n\times k}{M}+\underset{n\times k}{\epsilon},\quad M=\begin{pmatrix}\theta\\ 0\end{pmatrix}\begin{matrix}p\\ n-p\end{matrix} \tag{1.2}$$

and the hypothesis tested is H_0: there is a matrix $\underset{r\times p}{\Gamma}$ $(r<p)$, $\Gamma'\Gamma=I$ such that $\Gamma\theta=0$.

We also discussed the maximum likelihood estimators and the likelihood ratio tests for the previous two models with the left $O(n)$-invariant errors in the paper [3]. In this paper, we discuss mainly the Model I, and derive several statistics for testing H_0 in the case that the error's distribution is left $O(n)$-invariant. We also

Manuscript received July 24, 1984.
* Department of Mathematics, Fudan University, Shanghai, China.
** Department of Mathematics, Wuhan University, Wuhan China.

discuss some asymptotic properties of these testing methods in the case that the error's distribution is normal.

Let $O(n)$, $\mathscr{S}(k)$ and GL (k) denote the set of $n\times n$ orthogonal matrices, the set of $k\times k$ positive definite matrices, and the set of $k\times k$ nonsingular matrices, respectively. Let $\mathscr{L}(X)$ be the distribution of $n\times k$ random matrix X. X is called left $O(n)$-invariant if $\mathscr{L}(X)=\mathscr{L}(\Gamma X)$ for all $\Gamma\in O(n)$.

If the distribution of X is left $O(n)$-invariant and its density function exists, then the density function will be able to fie expressed as $f(X'X)$.

In this paper, the model I is expressed as follows:

$$\underset{n\times k}{Y}=\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}\begin{matrix} p \\ q \\ n-p-q \end{matrix}=M+\epsilon,\quad M=\begin{pmatrix} \theta \\ \eta \\ 0 \end{pmatrix}\begin{matrix} p \\ q \\ n-p-q \end{matrix}, \tag{1.3}$$

$n\geqslant p+q+k$, and we assume that the density function of Y is the family as follows:

$$\mathscr{F}_1=\{|\Sigma|^{-\frac{n}{2}}f(\Sigma^{-\frac{1}{2}}[(Y_1-\theta)'(Y_1-\theta)+(Y_2-\eta)'(Y_2-\eta)+Y_3'Y_3]\Sigma^{-1/2'})\mid$$
$$\theta\in R^{pk},\ \eta\in R^{qk},\ \Sigma\in\varphi(k)\}. \tag{1.4}$$

We are interested in testing the following hypothesis:

$$H_0:\ \exists \underset{q\times p}{C}\ \text{such that}\ \eta=C\theta. \tag{1.5}$$

§ 2. Several Lemmas

Lemma 1. *For any matrix* $\underset{n\times m}{C}$,

$$(I_n+CC')^{-1}=I_n-C(I_m+C'C)^{-1}C'$$

and

$$C(I_m+C'C)^{-1}=(I_n+CC')^{-1}C.$$

Lemma 2. *Suppose A is a $(p+q)\times(q+p)$ non-negative definite matrix, $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_{p+q}$ are the eigenvalues of A, $t_1, t_2,\cdots, t_{p+q}$ are the eigenvectors of A corresponding to $\lambda_1, \lambda_2, \cdots, \lambda_{p+q}$, respectively,*

$$T=(t_1, t_2, \cdots, t_{p+q})=\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{matrix} p \\ q \end{matrix}$$
$$\qquad\qquad\qquad\qquad p \qquad q$$

and $\det T_{22}\neq 0$.

Denote

$$M(B)=(I_q+BB')^{-\frac{1}{2}}(-B, I_q)A\begin{pmatrix} -B \\ I_q \end{pmatrix}(I_q+BB')^{-\frac{1}{2}}.$$

Then

(i) $\underset{B\in\mathscr{B}}{\text{Min}}$ trace $M(B)=\sum\limits_{j=p+1}^{p+q}\lambda_j$;

(ii) $\underset{B\in\mathscr{B}}{\text{Min}}\det M(B)=\prod\limits_{j=p+1}^{p+q}\lambda_j$;

(iii) $\underset{B\in\mathscr{B}}{\operatorname{Min}} \det (I_q+M(B))=\prod_{j=p+1}^{p+q}(1+\lambda_j)$;

(iv) $\underset{B\in\mathscr{B}}{\operatorname{Min}} \lambda_1(MB))=\lambda_{p+1}$,

where $\lambda_1(M)$ denots the largest eigenvalue of M;

(v) $\underset{B\in\mathscr{B}}{\operatorname{Max}} \operatorname{trace} (I+M(B))^{-1}=\sum_{j=p+1}^{p+q}(1+\lambda_j)^{-1}$

and all previous extremums are attained when $B=B_0=-(T'_{22})^{-1}T'_{12}$, where $\mathscr{B}$ is the set of $q\times p$ matrices.

Lemma 3. *Suppose A is a $(p+q)\times(p+q)$ nonnegative definite matrix, the meanings of λ_i, T and T_{ij} are the same as those of Lemma 2, and $\det T_{11}\neq 0$. Denote*

$$N(D)=(I_p+DD')^{-\frac{1}{2}}(I_p,\ -D)A\begin{pmatrix} I_p \\ -D'\end{pmatrix}(I_p+DD')^{-\frac{1}{2}}.$$

Then

(i) $\underset{D\in\mathscr{D}}{\operatorname{Max}} \operatorname{trace} N(D)=\sum_{j=1}^{p}\lambda_j$;

(ii) $\underset{D\in\mathscr{D}}{\operatorname{Max}} \det N(D)=\prod_{j=1}^{p}\lambda_j$;

(iii) $\underset{D\in\mathscr{D}}{\operatorname{Max}} \det(I_p+N(D))=\prod_{j=1}^{p}(1+\lambda_j)$;

(iv) $\underset{D\in\mathscr{D}}{\operatorname{Max}} \lambda_p(N(D))=\lambda_p$,

where $\lambda_p(N)$ is the smallest eiginvalue of N;

(v) $\underset{D\in\mathscr{D}}{\operatorname{Min}} \operatorname{trace} (I_p+N(D))^{-1}=\sum_{j=1}^{p}(1+\lambda_j)^{-1}$

and all previous extremums are attained when $D=D_0=-(T'_{11})^{-1}T'_{21}$, where $\mathscr{D}$ is the set of $p\times q$ matrices.

It is easy to verify the previous lemmas, so these proofs are omitted.

§ 3. Invariant Tests

The hypothesis (1.5) can be expressed as follows:

$$H_0=\bigcup_C H_0(C), \tag{3.1}$$

where the hypothesis $H_0(C)$ is

$$H_0(C):\ \eta=C\theta. \tag{3.2}$$

Therefore, for every fixed matrix C, $H_0(C)$ is an ordinarily linear hypothesis; it has been discussed by many authors and several testing statistics have been obtained by using invariant principle and various criterions. Now we use the Intersection-Union principle on the basis of the testing region of $H_0(C)$ to derive the testing region of H_0.

Suppose $T(C)$ is a testing statistic of $H_0(C)$ and $T(C)>d_\alpha$ is the corresponding rejection region. Because the hypothesis H_0 is the union of all hypotheses $H_0(C)$, if one wants to reject H_0 then he will have to reject $H_0(C)$ for every C. Consequently, by Intersection-Union principle we see that the testing statistic of H_0 is

$$T = \operatorname*{Min}_{C} T(C)$$

and the corresponding rejection region is

$$T > d,$$

where n is chosen such that the level of significance equals to d, i.e.

$$P_{H_0}(T > d) \leqslant d.$$

In the following, firstly various statistics $T(C)$ for $H_0(C)$ are derived, where C is a known matrix, by invariant principle, and then we derive the corresponding T from $T(C)$.

Denote

$$D = \begin{pmatrix} I_p \\ C \end{pmatrix}.$$

Then

$$DD^+ = \begin{pmatrix} I_p \\ C \end{pmatrix} (I_p + C'C)^{-1} (I_p, C')$$

and DD^+ is a projective matrix. So there exists an orthogonal matrix U, $U'U = I_{p\times q}$ such that

$$DD^+ = U \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} U'.$$

Let

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \begin{matrix} p \\ q \end{matrix} = U' \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}.$$

Then

$$U'(DD^+) \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} W_1 \\ 0 \end{pmatrix},$$

$$U'(I_{p+q} - DD^+) \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ W_2 \end{pmatrix}$$

$$EW_1 = U_1' \begin{pmatrix} \theta \\ \eta \end{pmatrix},$$

$$EW_2 = U_2' \begin{pmatrix} \theta \\ \eta \end{pmatrix},$$

where $U = (\underset{p}{U_1}, \underset{q}{U_2})$.

Because $\mathscr{M}(D) = \mathscr{M}(U_1) \perp \mathscr{M}(U_2)$, where $\mathscr{M}(A)$ expresses the linear subspace generated by the column vectors of A, it is obvious that $\eta = C\theta$ is equivalent to

$EW_2=0$. Hence, the previous testing problem may be reduced to the following testing problem: the joint density function family of W_1, W_2 and Y_3 is

$$\mathscr{F}_1=\{|\Sigma|^{-\frac{n}{2}}f(\Sigma^{-\frac{1}{2}}[(W_1-M_1)'(W_1-M_1)+(W_2-M_2)'(W_2-M_2)+Y_3'Y_3]\Sigma^{-\frac{1}{2}'})\mid M_1\in R^{pk},\ M_2\in R^{qk},\ \Sigma\in\mathscr{S}(k)\} \tag{3.3}$$

and the tested hypothesis $H_0^*(C)$

$$H_0^*(C):\ M_2=0. \tag{3.4}$$

Let G be a transformation group acting on (W_1, W_2, Y_3),

$$G=O(p)\times O(q)\times O(n-p-q)\times GL(k)\times R^{pk}.$$

For every tranformation $g\in G$,

$$g(W_1, W_2, Y_3)=(\Gamma_1W_1A'+\alpha,\ \Gamma_2W_2A',\ \Gamma_3Y_3A'),$$

where $\Gamma_1\in O(p)$, $\Gamma_2\in O(q)$, $\Gamma_3\in O(n-p-q)$, $A\in Gl(k)$ and $\alpha\in R^{pk}$, and the corresponding induced transformation acting on the parameter space (M_1, M_2, Σ) is $\bar{g}$:

$$\bar{g}(M_1, M_2, \Sigma)=(\Gamma_1M_1A'+\alpha,\ \Gamma_2M_2A',\ A\Sigma A').$$

It is easy to verify that the testing problem, described by (3.3) and (3.4), remains invariant under G and that the maximal invariant statistics are the all eigenvalues of the following equation (see [1] and [4])

$$\det(W_2'W_2-\lambda Y_3'Y_3)=0. \tag{3.5}$$

Because

$$\begin{aligned}W_2'W_2&=(Y_1', Y_2')(I_{p+q}-DD^+)\begin{pmatrix}Y_1\\Y_2\end{pmatrix}\\&=(Y_1', Y_2')\left(I_{p+q}-\begin{pmatrix}I_p\\C\end{pmatrix}(I_p+C'C)^{-1}(I_p, C)\right)\begin{pmatrix}Y_1\\Y_2\end{pmatrix}\\&=(Y_2-CY_1)'(I_q+CC')^{-1}(Y_2-CY_1),\end{aligned} \tag{3.6}$$

the equation (3.5) may be rewritten as

$$\det\left((Y_2-CY_1)'(I_q+CC')^{-1}(Y_2-CY_1)-\lambda_3Y_3'Y_3\right)=0. \tag{3.7}$$

Hereafter we assume that $n\geqslant p+q+k$ and $p\leqslant k\leqslant p+q$. Then $P(\det(Y_3'Y_3)=0)=0$ holds.

Let $\lambda_1^*\geqslant\lambda_2^*\geqslant\cdots\geqslant\lambda_{p+q}^*\geqslant0$ be the eigenvalues of the matrix A,

$$A=\begin{pmatrix}Y_1\\Y_2\end{pmatrix}(Y_3'Y_3)^{-1}(Y_1', Y_2'), \tag{3.8}$$

and $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_k\geqslant0$ be the eigenvalues of the matrix $(Y_1'Y_1+Y_2'Y_2)(Y_3'Y_3)^{-1}$. Because the non-zero eigenvalues of A are the same as those of $(Y_1'Y_1+Y_2'Y_2)\times(Y_3'Y_3)^{-1}$, the following results hold:

$$\lambda_i^*=\lambda_i,\quad \text{if}\quad 1\leqslant i\leqslant k,$$
$$\lambda_j^*=0,\quad \text{if}\quad k<j\leqslant p+q.$$

Below, we derive the corresponding testing statistics by various criterions.

(1) Lawley-Hotelling criterion.

The rejection region of $H_0(C)$ based on Lawley-Hotelling criterion is $T_1(C)>d_\alpha$, where

$$T_1(C)=\text{trace}((Y_2-CY_1)'(I_q+CC')^{-1}(Y_2-CY_1)(Y_3'Y_3)^{-1})$$
$$=\text{trace}\left((I_q+CC')^{-\frac{1}{2}}(-C,\ I_q)A\begin{pmatrix}-C'\\ I_q\end{pmatrix}(I_q+CC')^{-\frac{1}{2}}\right).$$

Using Iemma 2 (i) and $P(\det T_{22}=0)=0$, we see that the testing statistic of H_0 associated with $T_1(C)$ is

$$T_1=\underset{C\in\mathscr{B}}{\text{Min}}\ T_1(C)=\sum_{j=p+1}^{p+q}\lambda_j^*=\sum_{j=p+1}^{k}\lambda_j, \tag{3.9}$$

where $\mathscr{B}$ is the set of all $q\times p$ matrices.

(2) Wilks-P. L. Hsu criterion.

The rejection region of $H_0(C)$ based on Wilks, P. L. Hsu criterion is $T_2(C)>d_\alpha$, where

$$T_2(C)=\det(I_k+(Y_2-CY_1)'(I_q+CC')^{-1}(Y_2-CY_1)(Y_3'Y_3)^{-1})$$
$$=\det\left((I_q+(I_q+CC')^{-\frac{1}{2}}(-C,\ I_q)A\begin{pmatrix}-C'\\ I_q\end{pmatrix}(I_q+CC')^{-\frac{1}{2}}\right).$$

Using Lemma 2(iii) and $P(\det T_{22}=0)=0$, we see that the testing statistic of H_0 associated with $T_2(C)$ is

$$T_2=\underset{C\in\mathscr{B}}{\text{Min}}\ T_2(C)=\prod_{j=p+1}^{p+q}(1+\lambda_j^*)=\prod_{j=p+1}^{*}(1+\lambda_j), \tag{3.10}$$

where $\mathscr{B}$ is the same as that of (3.9). This testing statistic T_2 is the same as the likelihood ratio test statistic (see [3]).

(3) Generalized correlation coefficient criterion.

The rejection region of $H_0(C)$ based on the generalized correlation coefficient criterion is $T_3(C)>d_\alpha$, where

$$T_3(C)=\text{trace}([Y_2-CY_1)'(I_q+CC')^{-1}(Y_2-CY_1)+Y_3'Y_3]^{-1}$$
$$\times(Y_2-CY_1)'(I_q+CC')^{-1}(Y_2-CY_1))$$
$$=\sum_{j=p+1}^{p+q}\frac{\lambda_j((Y_2-CY_1)'(I_q+CC')^{-1}(Y_2-CY_1)(Y_3'Y_3)^{-1})}{1+\lambda_j((Y_2-CY_1)'(I_q+CC')^{-1}(Y_2-CY_1)(Y_3'Y_3)^{-1})},$$

where $\lambda_1(x)\geqslant\lambda_2(x)\geqslant\cdots\geqslant\lambda_m(x)$ express the order eigenvalues of the $m\times m$ matrix X.

Because $x/(1+x)$ is a monotone increasing function with regard to x when $x\geqslant0$, using Lemma 2 and $P(\det T_{22}=0)=0$ we see that the testing statistic of H_0 associated with $T_3(C)$ is

$$T_3=\underset{C\in\mathscr{B}}{\text{Min}}\ T_3(C)=\sum_{j=p+1}^{p+q}\frac{\lambda_j^*}{1+\lambda_j^*}=\sum_{j=p+1}^{*}\frac{\lambda_j}{1+\lambda_j}. \tag{3.11}$$

(4) Roy's extremum criterion.

The rejection region of $H_0(C)$ based on Roy's extremum criterion is $T_4(C)>d_\alpha$, where

$$T_4(C)=\lambda_1((Y_2-CY_1)'(I_q+CC')^{-1}(Y_2-CY_1)(Y_3'Y_3)^{-1})$$
$$=\lambda_1\left((I_q+CC')^{-\frac{1}{2}}(-C,\ I_q)A\begin{pmatrix}-C'\\ I_q\end{pmatrix}(I_q+CC')^{-\frac{1}{2}}\right).$$

Using Lemma 2(iv) and $P(\det T_{22}=0)=0$, we see that the testing statistic of H_0 associated with $T_4(C)$ is

$$T_4=\operatorname*{Min}_{C\in\mathscr{B}} T_4(C)=\lambda^*_{p+1}=\lambda_{p+1}. \tag{3.12}$$

(5) Pillai criterion.

The rejection region of $H_0(C)$ based on Pillai criterion is $T_5(C)>d_\alpha$, where

$$T_5(C)=\operatorname{trace}((Y_3'Y_3)[(Y_2-CY_1)'(I_q+CC')^{-1}(Y_2-CY_1)+Y_3'Y_3]^{-1})$$
$$=\operatorname{trace}(\{Y_3'Y_3)^{-1}(Y_2-CY_1)'(I_q+CC')^{-1}(Y_2-CY_1)+I)^{-1}$$
$$=\operatorname{trace}(I_q+(I_q+CC')^{-1/2}(-C,\ I_q)A\begin{pmatrix}-C\\ I_q\end{pmatrix}(I_q+CC')^{-1/2'})^{-1}+(k-q)$$
$$=T_5(C)+(k-q).$$

Here $T(C)$ is

$$T_5(C)=\operatorname{trace}(I_q+CC')^{-1/2}(-C,\ I_q)A\begin{pmatrix}-C'\\ I_q\end{pmatrix}(I_q+CC')^{-1/2'})^{-1}.$$

Using Lemma 3(v) for $T_5(C)$ and $P(\det T_{11}=0)=0$, we see that the testing statistic of H_0 associated with $T_5'(C)$ is

$$T_5=\operatorname*{Min}_{C\in\mathscr{B}} T_5(C)=\sum_{j=1}^{q}(1+\lambda_j^*)^{-1}=\sum_{j=1}^{q}(1+\lambda_j)^{-1}. \tag{3.13}$$

§ 4. Some Asymptotic Properties

In this section, we assume that $\underset{n\times k}{Y}$ is a normal random matrix, i.e. the distribution of Y is the normal distribution $N(M,\ I_n\ \Sigma)$, where $M=\begin{pmatrix}\theta\\ \eta\\ 0\end{pmatrix}$, and $n\geqslant p+q+k$, $p\leqslant k\leqslant p+q$.

Theorem 4.1. *Suppose*

$$\underset{n\times k}{Y}=\underset{k}{\begin{pmatrix}Y_1\\ Y_2\\ Y_3\end{pmatrix}}\begin{matrix}p\\ q\\ n-p-q\end{matrix}\sim N(M_n,\ I_n,\ \Sigma),$$

$$M_n=\begin{pmatrix}\theta\\ \eta\\ 0\end{pmatrix}\begin{matrix}p\\ q\\ n-p-q\end{matrix}. \tag{4.1}$$

Let $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_k$ be all eigenvalues of the following equation (4.2)

$$\det(Y_1'Y_1+Y_2'Y_2-Y_3'Y_3)=0 \tag{4.2}$$

and $\tau_1^{(n)} \geqslant \tau_2^{(n)} \geqslant \cdots \geqslant \tau_k^{(n)} \geqslant 0$ *be the eigenvalues of the following equation*

$$\det\left(\frac{1}{n} M_n' M_n - \tau\Sigma\right) = 0.$$

If $\lim \tau_i^{(n)} = \tau_i$, $i = 1, \cdots, k$, and $\tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_s > 0$, $\tau_{s+1} = \cdots = \tau_k = 0$, then

(i) $\lambda_1, \cdots, \lambda_s$ and $\lambda_{s+1}, \cdots, \lambda_k$ are asymptotically independent;

(ii) the joint distribution density of $n\lambda_{s+1}, \cdots, n\lambda_k$ is asymptotically given by

$$C_1 \prod_{i=s+1}^{k} x_i^{\frac{p+q-k-1}{2}} e^{-\sum_{i=s+1}^{k} x_i/2} \prod_{i<j} (x_i - x_j),$$

$$\text{if } x_{s+1} > x_{s+2} > \cdots > x_k > 0 \tag{4.3}$$

(iii) further, if $\tau_1 = \tau_2 = \cdots = \tau_{k_1} > \tau_{k_1+1} = \cdots = \tau_{k_1+k_2} > \cdots > \tau_{k_1} + \cdots + k_{m-1} + 1 = \cdots = \tau_{k_1} + \cdots + k_m$, $k_1 + k_2 + \cdots + k_m = s$, then $(\lambda_1, \cdots, \lambda_{k_1})$, $(\lambda_{k_1+1}, \cdots, \lambda_{k_1+k_2})$, $\cdots$, $(\lambda_{k_1+\cdots+k_{m-1}+1}, \cdots, \lambda_{k_1+\cdots+k_m})$ are asymptotically independent and the asymptotic distribution density of $\sqrt{n}\,(\lambda_{k_1+\cdots+_j+1} = \tau_{k_1+\cdots k_j+1}, \cdots, \lambda_{k_1+\cdots k_j+1} - \tau_{k_1+\cdots+k_j+1}) / \sqrt{2\tau^2_{k_1+\cdots+k_{j+1}} + 4\tau_{k_1+\cdots+k_{j+1}}}$ is

$$C_2 e^{-\sum_j y_j^2} \prod_{i>j} (y_i - y_j), \quad if\ y_{k_1+\cdots+k_j+1} > \cdots > y_{k_1+\cdots+k_{j+1}} > 0, \tag{4.4}$$

where C_1 and C_2 are the regularizing constants.

(The proof of this theorem can be found in [5]).

Theorem 4.2. *Suppose* $Y \sim N(M, I_p, \Sigma)$, $M = \begin{pmatrix} \theta \\ \eta \\ 0 \end{pmatrix}$, $n > p+q+k$, $p \leqslant k \leqslant p+q$, *and* $rk\ \theta = p$. *Then when the hypothesis* (3.2) *is true, the asymptotic distributions of* T_1, $T_2 - 1$ *and* T_3 *defined by* (3.9), (3.10) *and* (3.11) *are the same,* nT_1 *is asymptotic chi-square distribution with* $(k-p)q$ *degree of freedom, and therefore the testing methods* T_1, T_2 *and* T_3 *are asymptotically equivalent.*

Proof Because $rk\theta = p$, $\theta'\theta + \eta'\eta = \theta'(I_p + C'C)\theta$ holds when H_0 is true. Hence

$$H_0 \text{ is true} \Leftrightarrow rk(\theta'\theta + \eta'\eta) = p.$$

From this, we get $\tau_1 > \cdots > \tau_p > 0$ and $\tau_{p+1} = \cdots = \tau_k = 0$ when H_0 is true. Using Theorem 4.1 we prove that $\lambda_1 \cdots \lambda_p$ and $\lambda_{p+1}, \cdots, \lambda_k$ are asymptotically independent and that the joint density function of $n\lambda_{p+1}, \cdots, n\lambda_k$ is asymptotically given by

$$c_1 \prod_{j=p+1}^{k} x_j^{(p+q-k-1)/2} e^{-\sum_{j=p+1}^{k} x_i/2} \prod_{i<j} (x_i - x_j), \quad \text{if } x_{p+1} > x_{p+2} > \cdots > x_k > 0. \tag{4.5}$$

From (4.2), it is easy to verify that $nT_1 = n \sum_{j=p+1}^{k} \lambda_j$ is asymptotically $\chi^2((k-p)q)$ distribution.

Since

$$T_2 = \prod_{j=p+1}^{k} (1+\lambda_j) = 1 + T_1 + \sum_{i \neq j} \lambda_i \lambda_j + \cdots + \lambda_{p+1} \cdots \lambda_k$$

$$n\left(\sum_{i \neq j} \lambda_i \lambda_j + \cdots + \lambda_{p+1} \cdots \lambda_k\right) \xrightarrow[n \to \infty]{\Pr} 0$$

and

$$T_3=\sum_{j=p+1}^{k}\frac{\lambda_j}{1+\lambda_j}=T_1-\sum_{j=p+1}^{k}\frac{\lambda_j}{1+\lambda_j}$$

$$n\sum_{j=p+1}^{k}\frac{\lambda_j^2}{1+\lambda_j}\xrightarrow[n\to\infty]{\Pr}0,$$

we see that the asymptotic distributions of T_1, T_2-1 and T_3 are the same and that T_1, T_2 and T_3 are asymptotically equivalent.

Theorem 4.3. *Suppose the conditions of Theorem 4.2 hold. When H_0 is true, let $F(x)$ be the asymptotic distribution of nT_4, then*

(i) *if $k-p=2m$,*

$$F(x)=\frac{c}{m!}\int_0^x\cdots\int_0^x \det\begin{pmatrix} F_1(t_2) & g_1(t_2) & F_1(t_4) & \cdots & g_1(t_{2m}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{2m}(t_2) & g_{2m}(t_2) & F_{2m}(t_4) & \cdots & g_{2m}(t_{2m}) \end{pmatrix}\prod_{j=1}^{m}dt_{2j}, \tag{4.6}$$

(ii) *if $k-p=2m+1$,*

$$F(x)=\frac{c}{m!}\int_0^x\cdots\int_0^x \det\begin{pmatrix} F_1(t_2) & g_1(t_2) & F_1(t_4) & \cdots & g_1(t_{2m}) & F_1(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{2m+1}(t_2) & g_{2m+1}(t_2) & F_{2m+1}(t_4) & \cdots & g_{2m+1}(t_{2m}) & F_{2m+1}(x) \end{pmatrix}\prod_{j=1}^{m}dt_{2j}$$

$$=\frac{c}{m!}\sum_{i=1}^{2m+1}(-1)^{i+1}F_i(x)\int_0^x\cdots\int_0^x C_i\prod_{j=1}^{m}dt_{2j}, \tag{4.7}$$

where D_i is the subdeterminant corresponding to $F_i(x)$, c is given by

$$c=\pi^{(k-p)2/2}\prod_{j=1}^{kp}\frac{\Gamma\left(\frac{p+q-k+2)-1}{2}\right)}{\Gamma\left(\frac{k-p-j+1}{2}\right)\Gamma\left(\frac{q-j+1}{2}\right)}, \tag{4.8}$$

$g_j(x)$ is the density function of $\chi^2(p+q-k+2j-1)$ and $F_j(x)$ is the distribution function of $\chi^2(p+q-k+2j-1)$, $j=1, \cdots, k-p$.

Proof When H_0 is true, the joint density function of $\sqrt{n}\,\lambda_{p+1}, \cdots, \sqrt{n}\,\lambda_k$ is asymptotically given by (4.5), where

$$C_1=\frac{\pi^{(k-p)2/2}}{2^{(k-p)q/2}}\prod_{j=1}^{k-p}\frac{1}{\Gamma\left(\frac{k-p-j+1}{2}\right)\Gamma\left(\frac{q-j+1}{2}\right)}.$$

Then, the results of this theorem are obtained soon by using the results of [6].

Theorem 4.4. *If the conditions of Theorem 4.2 hold, then testing methods T_1, T_2, T_3 and T_4 are all consistent tests.*

Proof Here we give the proof only for T_4, the proofs of T_1, T_2 and T_3 are all similar to that of T_4. For the testing statistic T_4, the corresponding rejection region is

$$T_4=\lambda_{p+1}>C_{\alpha n},$$

where $C_{\alpha n}$ is chosen such that $P_{H^0}(\lambda_{p+1}>C_{\alpha n})=\alpha$.

When H_0 is true, the asymptotical distribution density of $n\,\lambda_{p+1}$ is determined exactly by (4.3) (here $s=p$). Then we get $nC_{an}\underset{n\to\infty}{\longrightarrow} C_a$, $0\leqslant C_a<\infty$. However, when H_0 is not true, from (4.4), it is clear that $\sqrt{n}\,(\lambda_{p+1}-\tau_{p+1})$ has non-degenerate asymptotic distribution denoted by $F_1(x)$, where $\tau_{p+1}>0$. Hence, when H_0 is not true,

$$P(\lambda_{p+1}>C_{an})=P(\sqrt{n}\,(\lambda_{p+1}-\tau_{p+1})>\sqrt{n}\,(C_{an}-\tau_{p+1})).$$

Since $\sqrt{n}\,(C_{an}-\tau_{p+1})\underset{n\to\infty}{\longrightarrow}-\infty$, we obtain

$$P(\lambda_{p+1}>C_{an})\underset{v\to\infty}{\longrightarrow}1,$$

i.e., the testing method based on T_4 is a consistent test.

For the Model II, we can also obtain results similar to those of the Model I.

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