Chin. Ann. of Math.
7B (4) 1986

GLOBAL EXISTENCE OF THE SOLUTIONS OF NONLINEAR PARABOLIC EQUATIONS IN EXTERIOR DOMAINS

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Abstract

This paper deals with the following IBV problem of nonlinear parabolic equation:

$$\begin{cases} u_t=\Delta u+F(u,\ D_x u,\ D_x^2 u),\ (t,\ x)\in R^+\times\Omega, \\ u(0,\ x)=\varphi(x),\ x\in\Omega, \\ u|_{\partial\Omega}=0, \end{cases}$$

where Ω is the exterior domain of a compact set in R^n with smooth boundary and F satisfies $|F(\lambda)|=o(|\lambda|^2)$, near $\lambda=0$. It is proved that when $n\geqslant 3$, under the suitable smoothness and compatibility conditions, the above problem has a unique global smooth solution for small initial data. Moreover, It is also proved that the solution has the decay property $\|u(t)\|_{L^\infty(\Omega)}=o(t^{-\frac{n}{2}})$, as $t\to+\infty$.

§ 1. Introduction

In the recent years, a great deal of attention has been paid to the research on the global existence of the solution for nonlinear evolution equations. As far as we are aware the initial boundary value problem in interior domains has been studied in greater details, while that in exterior domains has not received the same amount of attention. Recently, Y. Tsutsumi[4] discussed the following semilinear Schrödinger equation

$$\begin{cases} u_t+i\Delta u=i\lambda|u|^p u,\ (t,\ x)\in Q, \\ u(0,\ x)=\varphi(x),\ x\in\Omega, \\ u|_{\Sigma}=0, \end{cases} \tag{1.1}$$

where the domain Ω is the exterior domain of a compact set in R^n with smooth boundary Γ, $Q=(0,\ \infty)\times\Omega$, $\Sigma=(0,\ \infty)\times\partial\Omega$, λ is a real constant. He proved that when $n\geqslant 3$ and p is an even integer with $p\geqslant 2$, problem (1.1) has a unique global solution for small initial data under a certain assumption on the shape of the domain Ω.

Manuscript received April 4, 1984. Revised October 6, 1984.
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In this paper we consider the following initial boundary value problem for the nonlinear parabolic equation in an exterior domain

$$\begin{cases} u_t-\Delta u=F(u,\ D_x u,\ D_x^2 u), & (t,\ x)\in Q, \\ u(0,\ x)=\varphi(x), & x\in\Omega, \\ u|_{\Sigma}=0, \end{cases} \tag{1.2}$$

where Ω is the same as above. Our main result is the following

Theorem 1.1. *Let $n\geqslant 3$ and N be an integer with $N\geqslant 3n+3\left[\frac{n}{2}\right]+23$. We assume that $F\in C^{2N}(R\times R^n\times R^{n^2})$ satisfying*

$$|F(\lambda)|=O(|\lambda|^2) \quad \text{near } \lambda=0 \tag{1.3}$$

and $\varphi(x)\in H^{2N}(\Omega)\cap W^{2N,1}(\Omega)$. Then there exists a positive constant ε such that if $\varphi(x)$ satisfies

$$\|\varphi\|_{H^{2N}(\Omega)}+\|\varphi\|_{W^{2N,1}(\Omega)}<\varepsilon \tag{1.4}$$

and the compatibility condition, problem (1.2) has a unique global solution $u(t,\ x)$ satisfying

$$\begin{cases} \dfrac{\partial^k u}{\partial t^k}\in C([0,\ +\infty);\ H^{2(N-k)}(\Omega)\cap H_0^1(\Omega))\cap L^2(0,\ +\infty;\ H^{2(N-k)+1}(\Omega)), \quad 0\leqslant k\leqslant N-1, \\ \dfrac{\partial^N u}{\partial t^N}\in C([0,\ +\infty);\ L^2(\Omega))\cap L^2(0,\ +\infty;\ H_0^1(\Omega)). \end{cases} \tag{1.5}$$

Moreover, we have the following decay estimate

$$\|u(t)\|_{L^\infty(\Omega)}=O(t^{-\frac{n}{2}}) \quad \text{as } t\to+\infty. \tag{1.6}$$

Remark 1.1. The compatibility condition means

$$\left.\frac{\partial^j u}{\partial t^j}\right|_{t=0}\in H_0^1(\Omega) \quad (0\leqslant j\leqslant N-1). \tag{1.7}$$

It has been proved that problem (1.2) has a unique global solution when the space dimension $n\geqslant 1$ if Ω is a bounded open set in R^n and when $n\geqslant 3$ if $\Omega=R^n$ (see [7, 8]).

Our plan in the present paper is as follows. In section 2 we prove the local existence and uniqueness theorem. In section 3 we give the energy estimate. In section 4 we give the local energy decay for the homogeneous linear problem. In section 5 we give the decay estimates for the linear problem. Finally, in section 6 we establish a priori estimate by using the results of sections 3 and 5. By the technique of Matsumura and Nishida[3], combining the a priori estimate with the local existence theorem, we obtain Theorem 1.1.

We give several notations. Let X be a Banach space on G. We denote by $C([0,\ T];\ X)$ the space of continuous functions on $[0,\ T]$ valued in X. We denote $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x_j}$ by u_t and u_{x_j} respectively. $D_x^m u$ represents the vector $\left\{\left(\frac{\partial}{\partial x}\right)^\alpha u,\right.$

$|\alpha|=m\}$ where $\alpha=\{\alpha_1, \cdots, \alpha_n\}$ and $|\alpha|=\sum_{i=1}^{n}\alpha_i$. Throughout this paper we denote by C various universal constants.

§ 2. Local Existence and Uniqueness

In this section we shall prove the existence and uniqueness of the local solution for problem (1.2).

From (1.3) we know that there exists a positive constant γ_0, such that if $|\lambda|\leqslant\gamma_0$, assumption (1.3) holds. By Sobolev imbedding theorem we can choose a positive constant $E_0<\frac{1}{2}$ such that if $f\in H^{[\frac{n}{2}]+1}(\Omega)$ and satisfies $\|f\|_{H^{[\frac{n}{2}]+1}(\Omega)}\leqslant E_0$, then we have $\|f\|_{L^\infty(\Omega)}\leqslant\gamma_0$.

Before we give the local existence theorem, we define the following space X_T by

$$
\begin{aligned}
X_T=\Big\{(V_0(t, x), \cdots, V_N(t, x))\Big|\\
&\frac{\partial^k V_j}{\partial t^k}\in C([0, T]; H^{2(N-k-j)}(\Omega)\cap H_0^1(\Omega))\cap L^2(0, T; H^{2(N-k-j)+1}(\Omega)),\\
&\qquad 0\leqslant k\leqslant N-j-1,\ 0\leqslant j\leqslant N-1,\\
&\frac{\partial^{N-j}V_j}{\partial t^{N-j}}\in C([0, T]; L^2(\Omega))\cap L^2(0, T; H_0^1(\Omega)),\quad 0\leqslant j\leqslant N\Big\},
\end{aligned} \tag{2.1}
$$

which is equipped with the norm

$$
\begin{aligned}
\|V\|_{x_T}^2=\max_{0\leqslant t\leqslant T}\sum_{0\leqslant j\leqslant N}\sum_{0\leqslant k\leqslant N-j}\left\|\frac{\partial^k V_j}{\partial t^k}(t)\right\|^2_{H^{2(N-k-j)}(\Omega)}\\
+\int_0^T\sum_{0\leqslant j\leqslant N}\sum_{0\leqslant k\leqslant N-j}\left\|\frac{\partial^k V_j}{\partial t^k}(t)\right\|^2_{H^{2(N-k-j)+1}(\Omega)}dt.
\end{aligned} \tag{2.2}
$$

We easily see that X_T is a anach space.

For any $V=(v_0, v_1, \cdots, v_N)\in X_T$, set

$$
\begin{cases}
f_0(t, x)=F(u, D_x u, D_x^2 u)\,|_{u=v_0},\\
f_1(t, x)=\left(\frac{\partial}{\partial t}F\right)(u, D_x u, D_x^2 u, u_t, D_x u_t, D_x^2 u_t)\,|_{u=v_0, u_t=v_1}\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
f_j(t, x)=\left(\frac{\partial^j}{\partial t^j}F\right)(u, D_x u, D_x^2 u, \cdots, \partial_t^j u, D_x\partial_t^j u, D_x^2\partial_t^j u)\,|_{u=v_0, \cdots, \partial_t^j u=v_j}\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
0\leqslant j\leqslant N,
\end{cases} \tag{2.3}
$$

and we denote $u_j(x)\quad(0\leqslant j\leqslant N)$ by

$$\begin{cases} u_0=\varphi, \\ u_1=\Delta u_0+F(u,\ D_x u,\ D_x^2 u)\,|_{u=u_0}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ u_j=\Delta u_{j-1}+\left(\dfrac{\partial^{j-1}}{\partial t^{j-1}}F\right)(u,\ D_x u.\ D_x^2 u,\ \cdots,\ \partial_t^{j-1}u,\ D_x\partial_t^{j-1}u,\ D_x^2\partial_t^{j-1}u)\,|_{u=u_0,\ \cdots,\ \partial_t^{j-1}u=u_{j-1}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\leqslant j\leqslant N. \end{cases} \tag{2.4}$$

It is easily realized that if u is a smooth solution of problem (1.2) we have

$$\frac{\partial^j u}{\partial t^j}(t,\ x)\,|_{t=0}=u_j(x),\quad 0\leqslant j\leqslant N. \tag{2.5}$$

For some positive constant $E\leqslant E_0$, we define space $X_{T,E}$ by

$$X_{T,E}=\{(v_0(t,\ x),\ \cdots,\ v_N(t,\ x))\,|\,v\in X_T,\ v_j(0,\ x)=u_j(x)\ (0\leqslant j\leqslant N),\ \|v\|_{X_T}\leqslant E\}. \tag{2.6}$$

Theorem 2.1. *In problem* (1.2) *we assume that* $F\in C^{2N}(R\times R^n\times R^n)^2$ *and satisfies* (1.3), $\varphi\in H^{2N}(\Omega)$, *where* N *is an integer with* $N\geqslant\left[\frac{n}{2}\right]+2$. *Then, there exist two positive constants* T *and* δ (<1) *such that for the suitably small constant* $E(\leqslant E_0)$, *if* $\varphi(x)$ *satisfies*

$$\|\varphi\|_{H^{2N}(\Omega)}<\delta E \tag{2.7}$$

and the compatibility condition, problem (1.2) *has a unique local solution* $u(t,\ x)$ *satisfying*

$$U=(u,\ u_t,\ \cdots,\ \partial_t^N u)\in X_{T,E}. \tag{2.8}$$

In order to prove Theorem 2.1 we first prove the following lemmas.

Lemma 2.1. *Let* Ω *be an exterior domain of a compact set in* R^n *and let* u *be the solution belonging to* $H^1(\Omega)$ *of the elliptic equation*

$$\begin{cases} \Delta u=f, & x\in\Omega, \\ u|_\Gamma=0. \end{cases} \tag{2.9}$$

Let L *be an arbitrary nonnegative integer. If* $f\in H^L(\Omega)$, *then we have*

$$\|u\|_{H^{L+2}(\Omega)}\leqslant C(\|u\|_{L^2(\Omega)}+\|f\|_{H^L(\Omega)}), \tag{2.10}$$

where C *is a positive constant depending only on* n, L *and* Ω.

The above lemma is well known (see F. E. Browder [1]).

Lemma 2.2. *In the problem*

$$\begin{cases} u_t-\Delta u=f(t,\ x), & (t,\ x)\in Q, \\ u(0,\ x)=\varphi(x), & x\in\Omega, \\ u|_\Sigma=0, \end{cases} \tag{2.11}$$

where Ω *is the same as in problem* (1.1), *we assume that* $f\in L^2(0,\ T;\ H^{-1}(\Omega))$ *and* $\varphi\in L^2(\Omega)$. *Then, problem* (2.11) *has a unique solution* $u(t,\ x)\in C([0,T];\ L^2(\Omega))\cap L^2(0,\ T;\ H_0^1(\Omega))$. *Moreover, we have the following estimate*

$$\max_{0\le t\le T}\|u(t)\|^2_{L^2(\Omega)}+\int_0^T\|u(t)\|^2_{H^1(\Omega)}dt\le Ce^T\left(\|\varphi\|^2_{L^2(\Omega)}+\int_0^T\|f(t)\|^2_{H^{-1}(\Omega)}dt\right). \quad (2.12)$$

The proof of this lemma can be obtained by Galerkin's method and energy integral.

Next, we prove Theorem 2.1.

Proof of Theorem 2.1. We fix two positive constant T and $E(\le E_0)$ to be determine d later. Let J be the mapping which maps $V\in X_{T,E}$ into the solution $W=(w_1, \cdots, w_N)$ of the following problem

$$\begin{cases}\dfrac{\partial W_j}{\partial t}-\Delta W_j=f_j(t, x), & (t, x)\in Q_T=(0, T)\times\Omega,\\ W_j(0, x)=u_j(x), & x\in\Omega,\\ W_j|_{\Sigma_T}=0, & 0\le j\le N,\end{cases} \quad (2.13)$$

where $f_j(t, x)$ and $u_j(t, x)$ are determined by (2.3) and (2.4), respectively. We shall show that if δ, T and E are sufficiently small, J is a contraction mapping of $X_{T,E}$ to itself.

Noticing $E\le E_0<\dfrac{1}{2}$ and $N\ge\left[\dfrac{n}{2}\right]+2$, using (1.3) and the estimate of the norms for the composite function and product of functions (see [8]), for any $V\in X_{T,E}$ we have

$$\begin{cases}\dfrac{\partial^k f_j}{\partial t^k}\in C([0, T]; H^{2(N-k-j-1)}(\Omega))\cap L^2(0, T; H^{2(N-k-j)-1}(\Omega)),\\ \qquad 0\le k\le N-j-1,\ 0\le j\le N-1,\\ \dfrac{\partial^{N-j}f_j}{\partial t^{N-j}}\in L^2(0, T; H^{-1}(\Omega)), \quad 0\le j\le N\end{cases} \quad (2.14)$$

and

$$\max_{0\le t\le T}\sum_{j=0}^{N-1}\sum_{k=0}^{N-j-1}\left\|\frac{\partial^k f_j}{\partial t^k}(t)\right\|^2_{H^{2(N-k-j-1)}(\Omega)}+\int_0^T\sum_{j=0}^{N}\sum_{k=0}^{N-j}\left\|\frac{\partial^k f_j}{\partial t^k}(t)\right\|^2_{H^{2(N-k-j)-1}(\Omega)}\le C_1E^4, \quad (2.15)$$

where C_1 is a positive constant independent of T. Similarly, from (2.4) we get

$$\begin{cases}u_j(x)\in H^{2(N-j)}(\Omega)\cap H^1_0(\Omega), & 0\le j\in N-1.\\ u_N(x)\in L^2(\Omega).\end{cases} \quad (2.16)$$

From (2.3), (2.4) and (2.13) we get

$$\begin{cases}\dfrac{\partial}{\partial t}\left(\dfrac{\partial^k}{\partial t^k}W_j\right)-\Delta\left(\dfrac{\partial^k}{\partial t^k}W_j\right)=\dfrac{\partial^k}{\partial t^k}f_j, & (t, x)\in Q_T,\\ \left(\dfrac{\partial^k}{\partial t^k}W_j\right)(0, x)=\tilde{u}_{j,k}, & x\in\Omega,\\ \dfrac{\partial^k}{\partial t^k}W_j|_{\Sigma_T}=0, & 0\le j\le N,\end{cases} \quad (2.17)$$

for each integer k with $0\le k\le N-j$, where $\tilde{u}_{j,k}=\Delta^k u_j+\sum_{l=0}^{k-1}\Delta^l\partial_t^{k-1-l}f_j|_{t=0}$. Using Lemma 2.2, we conclude from (2.14) and (2.16) that

$$\left(\frac{\partial^{N-j}}{\partial t^{N-j}} W_j\right)(t, x) \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \tag{2.18}$$

and

$$\max_{0\leq t\leq T}\left\| \frac{\partial^{N-j}}{\partial t^{N-j}} W_j(t) \right\|^2_{L^2(\Omega)} + \int_0^T \left\| \frac{\partial^{N-j}}{\partial t^{N-j}} W_j(t) \right\|^2_{H^1(\Omega)} dt$$

$$\leq Ce^T\left(\|\tilde{u}_{j,N-j}\|^2_{L^2(\Omega)} + \int_0^T \left\| \frac{\partial^{N-j}}{\partial t^{N-j}} f_j(t) \right\|^2_{H^{-1}(\Omega)} dt \right). \tag{2.19}$$

Next, we shall prove that for each integer k with $0\leq k\leq N-j-1$, we have

$$\frac{\partial^k}{\partial t^k} W_j(t, x) \in C([0, T]; H^{2(N-k-j)}(\Omega) \cap H_0^1(\Omega)) \cap L^2(0, T; H^{2(N-k-j)+1}(\Omega)),$$

$$0\leq j\leq N-1 \tag{2.20}$$

and

$$\max_{0\leq t\leq T}\left\| \frac{\partial^k}{\partial t^k} W_j(t) \right\|^2_{H^{2(N-k-j)}(\Omega)} + \int_0^T \left\| \frac{\partial^k}{\partial t^k} W_j(t) \right\|^2_{H^{2(N-k-j)+1}(\Omega)} dt$$

$$\leq C(e^T+1)\Bigg(\sum_{i=k}^{N-j} \|\tilde{u}_{j,i}\|^2_{L^2(\Omega)} + \max_{0\leq t\leq T} \sum_{i=k}^{N-j-1} \left\| \frac{\partial^i}{\partial t^i} f_j(t) \right\|^2_{H^{2(N-i-j-1)}(\Omega)}$$

$$+ \int_0^T \sum_{i=k}^{N-j} \left\| \frac{\partial_i}{\partial t^i} f_j(t) \right\|^2_{H^{2(N-i-j)-1}(\Omega)} dt \Bigg), \tag{2.21}$$

where C is a positive constant independent of T. Now we show that if (2.20) and (2.21) hold for an integer k with $1\leq k\leq N-j-1$, (2.20) and (2.21) also hold with k replaced by $k-1$. In fact, we easily see from (2.17) and Lemma 2.2 that

$$\frac{\partial^{k-1}}{\partial t^{k-1}} W_j \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \tag{2.22}$$

and

$$\max_{0\leq t\leq T}\left\| \frac{\partial^{k-1}}{\partial t^{k-1}} W_j(t) \right\|^2_{L^2(\Omega)} + \int_0^T \left\| \frac{\partial^{k-1}}{\partial t^{k-1}} W_j(t) \right\|^2_{H^1(\Omega)} dt$$

$$\leq Ce^T\left(\|\tilde{u}_{j,k-1}\|^2_{L^2(\Omega)} + \int_0^T \left\| \frac{\partial^{k-1}}{\partial t^{k-1}} f_j(t) \right\|^2_{H^{-1}(\Omega)} dt \right). \tag{2.23}$$

From (2.20) and (2.22) we get

$$\frac{\partial^{k-1}}{\partial t^{k-1}} W_j \in C([0, T]; H^1(\Omega)). \tag{2.24}$$

From (2.17) we see that for all $t\in[0, T]$, $\frac{\partial^{k-1}}{\partial t^{k-1}} W_j$ satisfies

$$\begin{cases} \Delta\left(\dfrac{\partial^{k-1}}{\partial t^{k-1}} W_j\right) = \dfrac{\partial^k}{\partial t^k} W_j - \dfrac{\partial^{k-1}}{\partial t^{k-1}} f_j, & x\in\Omega. \\ \dfrac{\partial^{k-1}}{\partial t^{k-1}} W_j \Big|_\Gamma = 0. \end{cases} \tag{2.25}$$

Applying Lemma 2.1, we conclude from (2.14), (2.20) and (2.24) that

$$\frac{\partial^{k-1}}{\partial t^{k-1}} W_j(t, x) \in C([0, T]; H^{2(N-k-j+1)}(\Omega) \cap H_0^1(\Omega)) \cap L^2(0, T; H^{2(N-k-j+1)+1}(\Omega)), \tag{2.26}$$

and we conclude from (2.21) and (2.23) that

$$\max_{0\le t\le T}\left\|\frac{\partial^{k-1}}{\partial t^{k-1}}W_j(t)\right\|^2_{H^{2(N-k-j+1)}(\Omega)}+\int_0^T\left\|\frac{\partial^{k-1}}{\partial t^{k-1}}W_j(t)\right\|^2_{H^{2(N-k-j+1)+1}(\Omega)}dt. \tag{2.27}$$

$$\le C(1+e^T)\Big(\sum_{i=k-1}^{N-j}\|\tilde{u}_{j,i}\|^2_{L^2(\Omega)}+\max_{0\le t\le T}\sum_{i=k-1}^{N-j-1}\left\|\frac{\partial^i}{\partial t^i}f_j(t)\right\|^2_{H^{2(N-i-j-1)}(\Omega)}$$

$$+\int_0^T\sum_{i=k-1}^{N-1}\left\|\frac{\partial^i}{\partial t^i}f_j(t)\right\|^2_{H^{2(N-i-j)-1}(\Omega)}dt\Big).$$

It shows that (2.20) and (2.21) hold with k replaced by $k-1$. In the same way we can prove that (2.20) and (2.21) hold for $k=N-j-1$ from (2.18) and (2.19). Thus, an induction argument shows that (2.20) and (2.21) hold for each integer k with $0\le k\le N-j-1$. Combining these with (2.18) and (2.19) we obtain

$$W\in X_T, \tag{2.28}$$

$$\|W\|^2_{X_T}\le C_2(1+e^T)\Big(\sum_{j=0}^{N}\|u_j\|^2_{H^{2(N-j)}(\Omega)}+\max_{0\le t\le T}\sum_{j=0}^{N-1}\sum_{k=0}^{N-j-1}$$

$$\left\|\frac{\partial^k}{\partial t^k}f_j(t)\right\|^2_{H^{2(N-k-j-1)}(\Omega)}+\int_0^T\sum_{j=0}^{N}\sum_{k=0}^{N-j}\left\|\frac{\partial^k}{\partial t^k}f_j(t)\right\|^2_{H^{2(N-k-j)-1}(\Omega)}dt\Big), \tag{2.29}$$

where C_2 is a positive constant independent of T.

Using the estimate of the norms for the composite function and product functions and the fact that $N\ge\left[\frac{n}{2}\right]+2$ and $E<\frac{1}{2}$, we see that there exists a positive constant $\delta<1$ such that if $\|\varphi\|_{H^{2N}(\Omega)}<\delta E$, the following inequality holds:

$$\|u_j\|_{H^{2(N-j)}(\Omega)}\le E/\sqrt{6C_2N},\quad 0\le j\le N. \tag{2.30}$$

Thus, if let T be sufficiently small such that $e^T<2$, and $E<\min\left(E_0, \frac{1}{\sqrt{12C_1C_2}}\right)$, we have

$$\|W\|^2_{X_T}\le\frac{1}{2}E^2+3C_1C_2E^4<E^2. \tag{2.31}$$

Hence, J maps $X_{T,E}$ to iteslf. We easily see that $X_{T,E}$ is nonempty. When $\|\varphi\|_{H^{2N}(\Omega)}$ is sufficiently small, $(u_1(x), \cdots, u_N(x))e^{-t}\in X_{T,E}$, where $u_j(x)\,(0\le j\le N)$ are defined by (2.4).

Next, we prove that if T and E are sufficiently small, J is a contraction mapping of $X_{T,E}$ to itself. For any $V_1, V_2\in X_{T,E}$ let $W_1=JV_1$, $W_2=JV_2$, and $W=W_1-W_2=(w_0, \cdots, w_N)$. We have

$$\begin{cases}\frac{\partial W_j}{\partial t}-\Delta W_j=f_j(V_1)-f_j(V_2), & (t, x)\in Q_T,\\ W_j(0, x)=0, & x\in\Omega,\\ W_j|_{\Sigma_T}=0, & 0\le j\le N.\end{cases} \tag{2.32}$$

From (2.29) we have

$$\|W\|^2_{X_T}\le C_2(1+e^T)\Big(\max_{0\le t\le T}\sum_{j=0}^{N-1}\sum_{k=0}^{N-j-1}\left\|\frac{\partial^k}{\partial t^k}(f_j(V_1)-f_j(V_2))\right\|^2_{H^{2(N-k-j-1)}(\Omega)}$$

$$+\int_0^T\sum_{j=0}^{N}\sum_{k=0}^{N-j}\left\|\frac{\partial^k}{\partial t^k}(f_j(V_1)-f_j(V_2))\right\|^2_{H^{2(N-k-j)-1}(\Omega)}dt\Big). \tag{2.33}$$

Noticing (1.3), we see that if $e^T<2$, the following estimate holds:

$$\|W\|_{X_T}^2 \leqslant C_3(1+e^T)E^2\|V_1-V_2\|_{X_T}^2 \leqslant 3C_3E^2\|V_1-V_2\|_{X_T}^2, \tag{2.34}$$

where C_3 is a positive constant independent of T. Thus, letting E be sufficiently small such that $E \leqslant \min(E_0,\ 1/\sqrt{12C_1C_2},\ 1/\sqrt{6C_3})$, we obtain

$$\|W\|_{X_T}^2 < \frac{1}{2}\|V_1-V_2\|_{X_T}^2. \tag{2.35}$$

This shows that there exist δ, T and E such that J is a contraction mapping of $X_{T,E}$ to itself. Then mapping J has a unique fixed point $U=(u_0, u_1, \cdots, u_N) \in X_{T,E}$. $u_0(t, x)$ is a local solution of problem (1.2). The uniqueness follows from an inequality similar to (2.35). The Proof of Theorem 2.1 is complete.

§ 3. The Energy Estimate

In this section we shall establisn the energy estimate for the local solution of problem (1.2).

Lemma 3.1. *Let N be an integer with $N \geqslant \left[\frac{n}{2}\right]+2$. We assume that $F \in C^{2N}(R \times R^n \times R^{n^2})$ and $\varphi(x) \in H^{2N}(\Omega)$ satisfy (1.3) and the compatibility, respectively. Then, there exists a positive constant $E(\leqslant E_0)$ such that if $u(t, x)$ is the solution belonging to $\prod_{k=0}^{N} C^k([0, T];\ H^{2(N-k)}(\Omega))$ of problem (1.2) and satisfies*

$$\sum_{|\alpha|+2j \leqslant 2N} \|D_x^\alpha \partial_t^j u(t)\|_{L^2(\Omega)}^2 \leqslant E^2, \quad 0 \leqslant t \leqslant T, \tag{3.1}$$

the following energy inequality holds:

$$\sum_{|\alpha|+2j \leqslant 2N} \|D_x^\alpha \partial_t^j u(t)\|_{L^2(\Omega)}^2 \leqslant C_N \|\varphi\|_{H^{2N}(\Omega)}\, e^{C_N \int_0^t \sum_{|\alpha|+2j \leqslant 2\left[\frac{N}{2}\right]+3} \|D_x^\alpha \partial_t^j u(\tau)\|_{L^\infty(\Omega)} d\tau}, \quad 0 \leqslant t \leqslant T, \tag{3.2}$$

where C_N is a positive constant independent of T.

Proof For each integer L with $1 \leqslant L \leqslant N$, $\partial_t^L u$ satisfies

$$\begin{cases} \partial_t(\partial_t^L u) = \Delta(\partial_t^L u) + \partial_t^L F, & (t, x) \in Q_T, \\ (\partial_t^L u)(0, x) = \Delta^L \varphi + \sum_{j=0}^{L-1} \Delta^{L-1-j} \partial_t^j F(0, x), & x \in \Omega, \\ \partial_t^L u|_{\Sigma_T} = 0. \end{cases} \tag{3.3}$$

Taking the inner product with $\partial_t^L u$ on the both sides of (3.3), we get

$$\frac{1}{2}\frac{d}{dt}\|\partial_t^L u(t)\|_{L^2(\Omega)}^2 + \|\nabla \partial_t^L u(t)\|_{L^2(\Omega)}^2 \leqslant \int_\Omega \partial_t^L u \cdot \partial_t^L F\, dx. \tag{3.4}$$

Next, we shall estimate the right side of (3.4). If we let $F = u_{ij}u_{kl}$, $u_{ij}u_k$, $u_{ij}u$, u_iu_j, uu_i and u^2 where i, j, k and l are integers with $1 \leqslant i, j, k, l \leqslant n$, using Hölder inequality we obtain the following estimate, (3.5)—(3.10), respectively.

$$\int_\Omega \partial_t^L u \partial_t^L u^2\, dx \leqslant C\|\partial_t^L u(t)\|_{L^2(\Omega)} \sum_{j \leqslant L} \|\partial_t^j u(t)\|_{L^2(\Omega)} \sum_{j \leqslant \left[\frac{L}{2}\right]} \|\partial_t^j u(t)\|_{L^\infty(\Omega)}, \tag{3.5}$$

$$\int_\Omega \partial_t^L u\cdot\partial_t^L(uu_j)\,dx\leqslant C\|\partial_t^L u(t)\|_{L^2(\Omega)}\{\|u(t)\|_{L^\infty(\Omega)}\|\partial_t^L u_j(t)\|_{L^2(\Omega)}$$
$$+\sum_{k<L}\|\partial_t^k u(t)\|_{L^2(\Omega)}\sum_{k\leqslant[\frac{L}{2}]}\|\partial_t^k u_j(t)\|_{L^\infty(\Omega)}+\sum_{k\leqslant L-1}\|\partial_t^k u_j(t)\|_{L^2(\Omega)}\cdot\sum_{k\leqslant[\frac{L}{2}]}\|\partial_t^k u(t)\|_{L^\infty(\Omega)}\},\tag{3.6}$$

$$\int_\Omega \partial_t^L u\cdot\partial_t^L(u_iu_j)\,dx\leqslant C\|\partial_t^L u(t)\|_{L^2(\Omega)}\{\|\partial_t^L u_j(t)\|_{L^2(\Omega)}\|u_i(t)\|_{L^\infty(\Omega)}$$
$$+\|\partial_t^L u_i(t)\|_{L^2(\Omega)}\cdot\|u_j(t)\|_{L^\infty(\Omega)}+\sum_{k\leqslant L-1}\|\partial_t^k u_i(t)\|_{L^2(\Omega)}\sum_{k\leqslant[\frac{L}{2}]}\|\partial_t^k u_j(t)\|_{L^\infty(\Omega)}$$
$$+\sum_{k\leqslant L-1}\|\partial_t^k u_j(t)\|_{L^2(\Omega)}\sum_{k\leqslant[\frac{L}{2}]}\|\partial_t^k u_i(t)\|_{L^\infty(\Omega)}\},\tag{3.7}$$

$$\int_\Omega \partial_t^L u\cdot\partial_t^L(uu_{ij})\,dx$$
$$=\int_\Omega \partial_t^L u\cdot u\cdot\partial_t^L u_{ij}\,dx+\int_\Omega \partial_t^L u[\partial_t^L(uu_{ij})-u\partial_t^L u_{ij}]\,dx$$
$$\leqslant C\{\|\partial_t^L u_i(t)\|_{L^2(\Omega)}\|\partial_t^L u_j(t)\|_{L^2(\Omega)}\|u(t)\|_{L^\infty(\Omega)}+\|\partial_t^L u(t)\|_{L^2(\Omega)}\|\partial_t^L u_j(t)\|_{L^2(\Omega)}\|u_i(t)\|_{L^\infty(\Omega)}$$
$$+\|\partial_t^L u(t)\|_{L^2(\Omega)}\sum_{k<L}\|\partial_t^k u(t)\|_{L^2(\Omega)}\sum_{k\leqslant[\frac{L}{2}]}\|\partial_t^k u_{ij}(t)\|_{L^\infty(\Omega)}$$
$$+\|\partial_t^L u(t)\|_{L^2(\Omega)}\sum_{k\leqslant L-1}\|\partial_t^k u_{ij}(t)\|_{L^2(\Omega)}\sum_{k\leqslant[\frac{L}{2}]}\|\partial_t^k u(t)\|_{L^\infty(\Omega)}\},\tag{3.8}$$

$$\int_\Omega \partial_t^L u\cdot\partial_t^L(u_ku_{ij})\,dx$$
$$\leqslant C\{\|\partial_t^L u_j(t)\|_{L^2(\Omega)}\|\partial_t^L u_i(t)\|_{L^2(\Omega)}\|u_k(t)\|_{L^\infty(\Omega)}$$
$$+\|\partial_t^L u(t)\|_{L^2(\Omega)}\|\partial_t^L u_j(t)\|_{L^2(\Omega)}\|u_{ki}(t)\|_{L^\infty(\Omega)}$$
$$+\|\partial_t^L u(t)\|_{L^2(\Omega)}\|\partial_t^L u_k(t)\|_{L^2(\Omega)}\|u_{ij}(t)\|_{L^\infty(\Omega)}$$
$$+\|\partial_t^L u(t)\|_{L^2(\Omega)}(\sum_{l\leqslant L-1}\|\partial_t^l u_k(t)\|_{L^2(\Omega)}\sum_{l\leqslant[\frac{L}{2}]}\|\partial_t^l u_{ij}(t)\|_{L^\infty(\Omega)}$$
$$+\sum_{l\leqslant L-1}\|\partial_t^l u_{ij}(t)\|_{L^2(\Omega)}+\sum_{l\leqslant[\frac{L}{2}]}\|\partial_t^l u_k(t)\|_{L^\infty(\Omega)})\},\tag{3.9}$$

$$\int_\Omega \partial_t^L u\cdot\partial_t^L(u_{ij}u_{kl})\,dx$$
$$\leqslant C\{\|\partial_t^L u(t)\|_{L^2(\Omega)}\|\partial_t^L u_l(t)\|_{L^2(\Omega)}\|u_{ijk}(t)\|_{L^\infty(\Omega)}$$
$$+\|\partial_t^L u_k(t)\|_{L^2(\Omega)}\|\partial_t^L u_l(t)\|_{L^2(\Omega)}\|u_{ij}(t)\|_{L_\infty(\Omega)}$$
$$+\|\partial_t^L u(t)\|_{L^2(\Omega)}\|\partial_t^L u_j(t)\|_{L^2(\Omega)}\|u_{kli}(t)\|_{L^\infty(\Omega)}$$
$$+\|\partial_t^L u_i(t)\|_{L^2(\Omega)}\|\partial_t^l u_j(t)\|_{L^2(\Omega)}\|u_{kl}(t)\|_{L^\infty(\Omega)}$$
$$+\|\partial_t^l u(t)\|_{L^2(\Omega)}\sum_{j\leqslant L-1}\|\partial_t^j u_{kl}(t)\|_{L^2(\Omega)}\sum_{k\leqslant[\frac{L}{2}]}\|\partial_t^k u_{ij}(t)\|_{L^\infty(\Omega)}$$
$$+\|\partial_t^L u(t)\|_{L^2(\Omega)}\sum_{k\leqslant L-1}\|\partial_t^k u_{ij}(t)\|_{L^2(\Omega)}\sum_{j\leqslant[\frac{L}{2}]}\|\partial_t^j u_{kl}(t)\|_{L^\infty(\Omega)}\}.\tag{3.10}$$

Using the estimate of the norms for the composite function and the product of functions and the fact $E<\frac{1}{2}$, $N\geqslant\left[\frac{n}{2}\right]+2$ and (3.1), we conclude that if F satisfies (1.3), we have

$$\int_\Omega \partial_t^L u\cdot\partial_t^L F\,dx$$
$$\leqslant C_4\{\sum_{|\alpha|+2j\leqslant 2L}\|D_x^\alpha\partial_t^j u(t)\|^2_{L^2(\Omega)}\sum_{|\alpha|+2j\leqslant 2\left[\frac{L}{2}\right]+3}\|D_x^\alpha\partial_t^j u(t)\|_{L^\infty(\Omega)}$$
$$+\|D_x\partial_t^L u(t)\|^2_{L^2(\Omega)}\|u(t)\|_{W^{3,\infty}(\Omega)}\},\tag{3.11}$$

where C_4 is a positive constant independent of T. By the imbedding theorem we obtain from (3.1) and $M\geqslant\left[\frac{n}{2}\right]+2$

$$\|u(t)\|_{W^{3,\infty}(\Omega)}\leqslant C_5\|u(t)\|_{H^{\left[\frac{n}{2}\right]+4}(\Omega)}\leqslant C_5E,\tag{3.12}$$

where C_5 is a positive constant independent of T. Thus, if let E be sufficiently small such that $C_4C_5E<\frac{1}{2}$, from (3.4) and (3.11) we obtain

$$\frac{d}{dt}\|\partial_t^L u(t)\|^2_{L^2(\Omega)}+\|D_x\partial_t^L u(t)\|^2_{L^2(\Omega)}$$
$$\leqslant C\sum_{|\alpha|+2j\leqslant 2L}\|D_x^\alpha\partial_t^j u(t)\|^2_{L^2(\Omega)}\sum_{|\alpha|+2j\leqslant 2\left[\frac{L}{2}\right]+3}\|D_x^\alpha\partial_t^j u(t)\|_{L^\infty(\Omega)}.\tag{3.13}$$

On the other hand, noticing (1.2), (1.3) and the fact that $\varphi(x)\in H^{2N}(\Omega)$ and $N\geqslant\left[\frac{n}{2}\right]+2$ we can obtain in proper order

$$\|\partial_t^L u(0)\|_{L^2(\Omega)}\leqslant C\|\varphi\|_{H^{2L}(\Omega)},\quad 1\leqslant L\leqslant N.\tag{3.14}$$

Thus, from (3.13) and (3.14) we get

$$\|\partial_t^L u(t)\|^2_{L^2(\Omega)}$$
$$\leqslant C\{\|\varphi\|^2_{H^{2L}(\Omega)}+\int_0^t\sum_{|\alpha|+2j\leqslant 2L}\|D_x^\alpha\partial_\tau^j u(\tau)\|^2_{L^2(\Omega)}\sum_{|\alpha|+2j\leqslant 2\left[\frac{L}{2}\right]+3}\|D_x^\alpha\partial_\tau^j u(\tau)\|_{L^\infty(\Omega)}d\tau,$$
$$0\leqslant t\leqslant T.\tag{3.15}$$

The above estimate was obtained for $1\leqslant L\leqslant N$. We easily obtain (3.15) for $L=0$ by the same way. Therefore (3.15) holds for each integer L with $0\leqslant L\leqslant N$. Next, we shall show that if for an integer J with $0\leqslant J\leqslant N-1$ we have

$$\sum_{|\alpha|\leqslant 2J}\|D_x^\alpha\partial_t^{N-J}u(t)\|^2_{L^2(\Omega)}$$
$$\leqslant C\{\|\varphi\|^2_{H^{2N}(\Omega)}+\sum_{|\alpha|+2j\leqslant 2N}\|D_x^\alpha\partial_\tau^j u(t)\|_{L^2(\Omega)}\sum_{|\alpha|+2j\leqslant N+1}\|D_x^\alpha\partial_t^j u(t)\|^2_{L^\infty(\Omega)}$$
$$+\int_0^t\sum_{|\alpha|+2j\leqslant 2N}\|D_x^\alpha\partial_\tau^j u(\tau)\|^2_{L^2(\Omega)}\sum_{|\alpha|+2j\leqslant 2\left[\frac{N}{2}\right]+3}\|D_x^\alpha\partial_\tau^j u(\tau)\|_{L^\infty(\Omega)}d\tau\},\quad 0\leqslant t\leqslant T,\tag{3.16}$$

where C is a positive constant independent of T, the inequality (3.16) also holds with J replaced by $J+1$. In fact, we easily see that for $0\leqslant t\leqslant T$ $\partial_t^{N-J-1}u$ satisfies

$$\begin{cases}\Delta(\partial_t^{N-J-1}u)=\partial_t^{N-J}u-\partial_t^{N-J-1}F, & x\in\Omega.\\ \partial_t^{N-J-1}u|_{\partial\Omega}=0.\end{cases}\tag{3.17}$$

By Lemma 2.1 we have

$$\sum_{|\alpha|\leqslant 2(J+1)} \|D_x^\alpha \partial_t^{N-J-1} u(t)\|^2_{L^2(\Omega)}$$
$$\leqslant C\{\|\partial_t^{N-J-1} u(t)\|^2_{L^2(\Omega)} + \sum_{|\alpha|\leqslant 2J} \|D_x^\alpha \partial_t^{N-J} u(t)\|^2_{L^2(\Omega)} + \sum_{|\alpha|\leqslant 2J} \|D_x^\alpha \partial_t^{N-J-1} F(t)\|^2_{L^2(\Omega)}\},$$
$$0\leqslant t\leqslant T. \tag{3.18}$$

From (1.3) we have

$$\sum_{|\alpha|\leqslant 2J} \|D_x^\alpha \partial_t^{N-J-1} F(t)\|^2_{L^2(\Omega)} \leqslant C \sum_{|\alpha|+2j\leqslant 2N} \|D_x^\alpha \partial_t^j u(t)\|^2_{L^2(\Omega)} \sum_{|\alpha|+2j\leqslant N+1} \|D_x^\alpha \partial_t^j u(t)\|^2_{L^\infty(\Omega)}. \tag{3.19}$$

Thereby, from (3.15), (3.16) and (3.18) we see that (3.16) holds with J replaced by $J+1$. From (3.15) we already know that (3.16) holds for $J=0$. An induction argument gives

$$\sum_{|\alpha|+2j\leqslant 2N} \|D_x^\alpha \partial_t^j u(t)\|^2_{L^2(\Omega)}$$
$$\leqslant C_6\{\|\varphi\|^2_{H^{2N}(\Omega)} + \sum_{|\alpha|+2j\leqslant 2N} \|D_x^\alpha \partial_t^j u(t)\|^2_{L^2(\Omega)} \sum_{|\alpha|+2j\leqslant N+1} \|D_x^\alpha \partial_t^j u(t)\|^2_{L^\infty(\Omega)}$$
$$+\int_0^t \sum_{|\alpha|+2j\leqslant 2N} \|D_x^\alpha \partial_\tau^j u(\tau)\|^2_{L^2(\Omega)} \sum_{|\alpha|+2j\leqslant 2\left[\frac{N}{2}\right]+3} \|D_x^\alpha \partial_\tau^j u(\tau)\|_{L^\infty(\Omega)} d\tau\}, \quad 0\leqslant t\leqslant T. \tag{3.20}$$

By (3.1) and the imbedding theorem we get

$$\sum_{|\alpha|+2j\leqslant N+1} \|D_x^\alpha \partial_t^j u(t)\|^2_{L^\infty(\Omega)} \leqslant C \sum_{|\alpha|+2j\leqslant 2N} \|D_x^\alpha \partial_t^j u(t)\|^2_{L^2(\Omega)} \leqslant C_7 E^2. \tag{3.21}$$

Thus, taking

$$E = \min(E_0,\ 1/2C_4C_5,\ 1/2\sqrt{C_6C_7}) \tag{3.22}$$

we have

$$\sum_{|\alpha|+2j\leqslant 2N} \|D_x^\alpha \partial_t^j u(t)\|^2_{L^2(\Omega)}$$
$$\leqslant C_N\{\|\varphi\|^2_{H^{2N}(\Omega)} + \int_0^t \sum_{|\alpha|+2j\leqslant 2N} \|D_x^\alpha \partial_\tau^j u(\tau)\|^2_{L^2(\Omega)} \sum_{|\alpha|+2j\leqslant 2\left[\frac{N}{2}\right]+3} \|D_x^\alpha \partial_\tau^j u(\tau)\|_{L^\infty(\Omega)} d\tau\}, \tag{3.23}$$

where C_N is a positive constant dependent on n, N and Ω, but not on T. By Gronwall's inequality we conclude (3.2). The Proof of Lemma 3.1 is complete.

§ 4. The Local Energy Decay Estimate

In this section we shall investigate the local energy decay for solutions of the following parabolic equation

$$\begin{cases} u_t - \Delta u = 0, & (t, x)\in Q, \\ u(0, x) = \varphi(x), & x\in Q, \\ u|_\Sigma = 0, \end{cases} \tag{4.1}$$

where the domain Ω is the same as in problem (1.1). We denote the solution of problem (4.1) by $U(t)\varphi$, where $U(t)$ is the evolution operator associated with problem (4.1), namely

$$U(t)=\frac{1}{2\pi i}\int_{-d+i\infty}^{-d-i\infty} e^{-\tau t}(\tau+\Delta)^{-1}\,d\tau,\quad d>0. \tag{4.2}$$

In this section we denote by $L^2_a(\Omega)$ and $H^2_a(\Omega)$ the closed subspaces of $L^2(\Omega)$ and $H^2(\Omega)$, respectively, consisting of functions that vanish for $|x|>a$. Let $H^2_e(\Omega)$ be the Banach space $\{u;\ e^{-|x|^2}u(x)\in H^2(\Omega)\}$ with the norm $\|u\|_{H^2_e(\Omega)}=\|e^{-|x|^2}u\|_{L^2(\Omega)}$. Let $L(X, Y)$ be the Banach space consisting of all bounded linear operators from Banach space X to Banach space Y. We denote its norm by $\|\cdot\|_{L(X,Y)}$. Let R be a positive constant such that $\partial\Omega\subset\{x\in R^n;\ |x|<R\}$. For $r>R$ we write $\Omega_r=\{x\in\Omega;\ |x|<r\}$.

Lemma 4.1. *Let $n\geqslant 3$ and Ω be the same as in problem (1.1). Then, for two positive constants a and b with $a, b>R$ there exists a positive constant C such that*

$$\|U(t)\|_{L(L^2_a(\Omega),L^2(\Omega_b))}\leqslant Ct^{-\frac{n}{2}},\quad t\geqslant 1, \tag{4.3}$$

where C is a positive constant dependent only on n, a, b and Ω.

We may assume that $b\geqslant a>R+1$.

Proof In order to estimate $U(t)$ we first investigate the resolvent $(\tau+\Delta)^{-1}$. Let D be the entire complex plane if n is odd and the Riemann surface on which the function $\ln k$ is single-valued if n is even. Let D^+ be the region $\{k;\ k\in D,\ 0<\arg k<\pi,\ k\neq 0\}$. The resolvent $(k^2+\Delta)^{-1}$ is an $L(L^2(\Omega);\ H^2(\Omega))$-valued analytic function with respect to $k\in D^+$. About $(k^2+\Delta)^{-1}$ we have the following two lemmas.

Lemma 4.2. *Let $n\geqslant 3$. Then the resolvent $(k^2+\Delta)^{-1}$ admits a meromorphic extension to D as an $L(L^2_a(\Omega);\ H^2_e(\Omega))$-valued function and the set of all poles of the meromorphic extension has no limit point in D.*

We also denote the extension by $(k^2+\Delta)^{-1}$.

The Lemma has proved by B. R. Vainberg[6].

Lemma 4.3. *Let $n\geqslant 3$. Then there exists a positive constant $\delta_1<1$ such that*

(1) *If n is odd,*

$$(k^2+\Delta)^{-1}=\sum_{j=0}^{\infty}B_{2j}k^{2j}+\sum_{j=\frac{n-3}{2}}^{\infty}B_{2j+1}k^{2j+1}. \tag{4.4}$$

in the region $W=\{k;\ k\in D,\ |k|<\delta_1\}$, where the operators $B_j(j=0, 1, \cdots)\in L(L^2_a(\Omega);\ H^2_e(\Omega))$ and the expansion (4.4) converges uniformly and absolutely in the operator norm;

(2) *If n is even,*

$$(k^2+\Delta)^{-1}=\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}B_{mj}(k^{n-2}\ln k)^m k^{2j} \tag{4.5}$$

in the region $W=\left\{k;\ k\in D,\ |k|<\delta_1,\ -\frac{\pi}{2}<\arg k<\frac{3\pi}{2}\right\}$, where the operator $B_{m,j}(m, j=0, 1, \cdots)\in L(L^2_a(\Omega);\ H^2_e(\Omega))$ and the expansion (4.5) converges uniformly and *absolutely in operator norm.*

This Lemma has proved by Y. Tsutsumi[5].

Remark 4.1. As a consquence of Lemma 4.3, the meromorphic extension $(k^2+\Delta)^{-1}$ has no pole and is bounded in a neighbourhood of $k=0$.

Next, we shall translate the results on $(k^2+\Delta)^{-1}$ into those on $(\tau+\Delta)^{-1}$, and complete the proof of Lemma 4.1.

Proof of Lemma 4.1. Taking $\delta_0<\frac{\delta_1}{2}$, we consider the region D_k on the k-plane, which is hatched in Figure 1. From Remark 4.1 we can choose δ_0 so small that $(k^2+\Delta)^{-1}$ has no pole in the region D_k. Under the mapping $\tau=k^2$, the region D_k is taken one to one onto the region D_τ on the τ-plane, which is hatched in Figure 2.

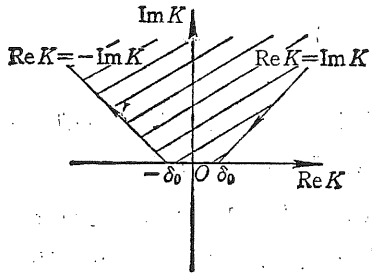

Fig. 1. k-plane

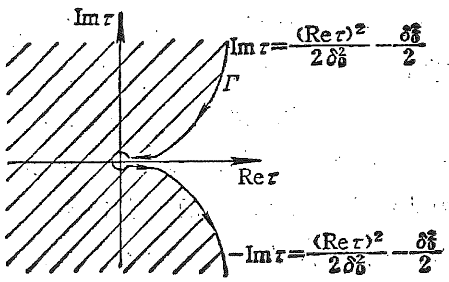

Fig. 2. τ-plane

It is well known that there exist three positive constants ξ, η and M such that

$$\|(\lambda-\Delta)^{-1}\|_{L(L^2(\Omega);L^2(\Omega))}\leqslant\frac{M}{|\lambda|}\tag{4.6}$$

for all $\lambda\in\left\{\lambda;\ |\arg(\lambda-\xi)|<\frac{\pi}{2}+\eta\right\}$. Thus, there exist two positive constants A and M such that

$$\|(\tau+\Delta)^{-1}\|_{L(L^2_0(\Omega),\,L^2(\Omega_b))}\leqslant\frac{M}{|\tau|}\tag{4.7}$$

for all $\tau\in\{D_\tau,\ |\tau|>A\}$, where constants A and M do not depend on τ.

Now, we consider the following integral

$$U(t)-I=\frac{1}{2\pi i}\int_{-d+i\infty}^{-d-i\infty}e^{-\tau t}\left\{(\tau+\Delta)^{-1}-\frac{I}{\tau}\right\}d\tau,\quad d>0.\tag{4.8}$$

Since we have

$$(\tau+\Delta)^{-1}-\frac{I}{\tau}=-\frac{1}{\tau}(\tau+\Delta)^{-1}\Delta,\tag{4.9}$$

we obtain from (4.7)

$$\left\|(\tau+\Delta)^{-1}-\frac{I}{\tau}\right\|_{L(H^2_0(\Omega)\cap H^1_0(\Omega);L^2(\Omega_b))}$$
$$\leqslant\frac{1}{|\tau|}\|(\tau+\Delta)^{-1}\|_{L(L^2_0(\Omega);L^2(\Omega_b))}\|\Delta\|_{L(H^2_0(\Omega)\cap H^1_0(\Omega);L^2_0(\Omega))}\leqslant C|\tau|^{-2}\tag{4.10}$$

for all $\tau\in\{D_\tau,\ |\tau|>\Delta\}$. Therefore, the integral in (4.8) converges absolutely in $L(H^2_a(\Omega)\cap H^1_0(\Omega);\ L^2(\Omega_b))$. By (4.10) and the Cauchy theorem we can shift the contour of the integral in (4.8) into the contour Γ as in Figure 2.

Since

$$\frac{1}{2\pi i}\int_\Gamma e^{-\tau t}\frac{I}{\tau}d\tau=I, \tag{4.11}$$

we obtain from (4.8)

$$U(t)=\frac{1}{2\pi i}\int_\Gamma e^{-\tau t}(\tau+\Delta)^{-1}d\tau. \tag{4.12}$$

Finally, we estimate the integral in (4.12). We denote by Γ^+_A and Γ^-_A the parabolic parts of Γ which are situated on the upper half plane and the lower half plane, respectively. We denote by Γ^+_B and Γ^-_B the straight line parts of Γ which are situated on the upper half plane and the lower half plane, respectively. We denote by Γ_c the circular part of Γ. By Lemma 4.3 we easily see that

$$\lim_{\varepsilon\to 0}\left\|\frac{1}{2\pi i}\int_{\Gamma_c}e^{-\tau t}(\tau+\Delta)^{-1}d\tau\right\|_{L(L^2_a(\Omega),L^2(\Omega_b))}=0. \tag{4.13}$$

By Lemma 4.3 and (4.7) we know that $\|(\tau+\Delta)^{-1}\|_{\Gamma(L^2_a(\Omega),L^2(\Omega_b))}$ is bounded on Γ^+_A and Γ^-_A. Thus, noticing the analytic expression of Γ^+_A and Γ^-_A we obtain

$$\left\|\frac{1}{2\pi i}\int_{\Gamma^+_A\cup\Gamma^-_A}e^{-\tau t}(\tau+\Delta)^{-1}d\tau\right\|_{L(L^2_a(\Omega),L^2(\Omega_b))}\leqslant Ce^{-ct},\quad t\geqslant 1. \tag{4.14}$$

By Lemma 4.3 we can see that if n is odd,

$$(\tau+\Delta)^{-1}=B_1(\tau)+\tau^{\frac{n-2}{2}}B_2(\tau) \tag{4.15}$$

for all $\tau\in\{D,\ |\tau|<\delta_0^2\}$, where $B_1(\tau)$ and $B_2(\tau)$ are $L(L^2_a(\Omega),\ H^2(\Omega_b))$-valued holomorphic functions. If n is even,

$$(\tau+\Delta)^{-1}=B_3(\tau)+B_4\tau^{\frac{n-2}{2}}\ln\sqrt{\tau}+B_5(\tau)\tau^{\frac{n-2}{2}} \tag{4.16}$$

for all $\tau\in\{D,\ |\tau|<\delta_0^2,\ -\pi<\arg\tau<3\pi\}$, where $B_3(\tau)$ is a $L(L^2_a(\Omega),\ H^2(\Omega_b))$-valued holomorphic function. $B_5(\tau)$ is an $L(L^2_a(\Omega),\ H^2(\Omega_b))$-valued bounded continuous function. B_4 is a bounded operator from $L^2_a(\Omega)$ to $H^2(\Omega_b)$.

Therefore, if n is odd, we have

$$\left\|\frac{1}{2\pi i}\int_{\Gamma^+_B\cup\Gamma^-_B}e^{-\tau t}(\tau+\Delta)^{-1}d\tau\right\|_{L(L^2_a(\Omega),L^2(\Omega_b))}$$

$$\leqslant C\int_0^{\delta_0^2}e^{-\xi t}\xi^{\frac{n-2}{2}}d\xi\leqslant Ce^{-Ct}+Ct^{-\frac{n-1}{2}}\int_0^{\delta_0^2}e^{-\xi t}\frac{d\xi}{\sqrt{\xi}}\leqslant Ct^{-\frac{n}{2}},\quad t\geqslant 1. \tag{4.17}$$

If n is even, we have

$$\left\|\frac{1}{2\pi i}\int_{\Gamma^+_B\cup\Gamma^-_B}e^{-\tau t}(\tau+\Delta)^{-1}d\tau\right\|_{L(L^2_a(\Omega),L^2(\Omega_b))}$$

$$\leqslant C\int_0^{\delta_0^2}(e^{-\xi t}\xi^{\frac{n-2}{2}}\ln\sqrt{\xi e^{i2\pi}}-e^{-\xi t}\xi^{\frac{n-2}{2}}\sqrt{\xi})d\xi$$

$$\leqslant C\int_0^{\delta_0^2}e^{-\xi t}\xi^{\frac{n-2}{2}}d\xi\leqslant Ct^{-\frac{n}{2}},\quad t\geqslant 1. \tag{4.18}$$

Noticing (4.12)—(4.14) again we conclude (4.2). The proof of Lemma 4.1 is complete.

§ 5. The Decay Estimate

In this section we shall discuss the time decay estimate of the solution for the following initial boundary value problem for the parabolic equation

$$\begin{cases} u_t-\Delta u=f(t, x), \quad (t, x)\in Q, \\ u(0, x)=\varphi(x), \quad x\in\Omega, \\ u|_{\Sigma}=0, \end{cases} \tag{5.1}$$

where Ω is the same as in problem (1.1), $\varphi(x)$ and $f(t, x)$ are so smooth that all norms of φ and f which will appear in lemmas stated below are bounded. In addtion we assume that φ and f satisfy the compatibility condition.

For the sake of convenience, we introduce the following notations: Let $g(t, x)$ be a function defined on $[0, +\infty)\times\Omega$ and $h(x)$ be a function defined on Ω. For all integers $L\geqslant n+1$, all real numbers $k\geqslant 0$ and all real numbers p with $1\leqslant p\leqslant\infty$, we define $[g, p, k, L, \Omega](t)$ and $Q(h, g, k, L)(t)$ as follows:

$$[g, p, k, L, \Omega](t)=\sup_{0\leqslant\tau\leqslant t}(1+\tau)^k\sum_{|\alpha|+2j\leqslant L}\|D_x^\alpha\partial_t^j g(\tau)\|_{L^p(\Omega)}, \tag{5.2}$$

$$Q(h, g, k, L)(t)=\|h\|_{W^{L,1}(\Omega)}+\|h\|_{H^{L-n-1}(\Omega)}+[g, 1, k, L, \Omega](t) \\ +[g, 2, k, L-n-1, \Omega](t). \tag{5.3}$$

Lemma 5.1. *Let $n\geqslant 3$. Then, for each nonnegative integer L, the solution u of problem* (5.1) *satisfies the following estimates*

$$\left[u, \infty, \frac{n}{2}, 2L, \Omega\right](t)\leqslant CQ\left(\varphi, f, \frac{n}{2}, 2L+2n+4\right)(t), \quad t\geqslant 0, \tag{5.4}$$

$$\left[u, 2, \frac{n}{4}, 2L, \Omega\right](t)\leqslant CQ\left(\varphi, f, \frac{n}{2}, 2L+3\left[\frac{n}{2}\right]+5\right)(t), \quad t\geqslant 0, \tag{5.5}$$

mhere C is a positive constant depending only on n, L and Ω. In addition, for any multi-index α and integer $j\geqslant 0$ with $\frac{|\alpha|}{2}+j\geqslant\frac{n}{2}$, we have

$$\|D_x^\alpha\partial_t^j u(t)\|_{L^2(\Omega)}\leqslant C(1+t)^{-\frac{n}{4}}Q\left(\varphi, f, \frac{n}{4}, |\alpha|+2j+n+3\right)(t), \quad t\geqslant 0, \tag{5.6}$$

where C is a positive constant depending only on n, α, j and Ω.

The method use in the proof of Lemma 5.1 follows[4], where Y. Tsutsumi regarded the problem in an exterior domain as perturbation of problem in entire space. In order to prove Lemma 5.1 we first give several lemmas.

Lemma 5.2. *Let $n\geqslant 3$ and $u(t, x)$ be the smooth solution of the following Cauchy problem*

$$\begin{cases} u_t-\Delta u=f(t, x), \quad (t, x)\in R^+\times R^n, \\ u(0, x)=\varphi(x), \quad x\in R^n. \end{cases} \tag{5.7}$$

Then, for each integer $L\geqslant 0$, the following inequalities hold:

$$\left[u, \infty, \frac{n}{2}, 2L, R^n\right](t)$$
$$\leqslant C\left\{\|\varphi\|_{W^{2L+n+1,1}(R^n)}+\left[f, 1, \frac{n}{2}, 2L+n+1, R^n\right](t)\right\}, \quad t\geqslant 0. \tag{5.8}$$

$$\left[u, 2, \frac{n}{4}, 2L, R^n\right](t)$$
$$\leqslant C\left\{\|\varphi\|_{W^{2L+\left[\frac{n}{2}\right]+1,1}(R^n)}+\left[f, 1, \frac{n}{2}, 2L+\left[\frac{n}{2}\right]+1. R^n\right](t)\right\}, \quad t\geqslant 0, \tag{5.9}$$

where C is a positive constant depending only on n and L. In addition, for any multi-index α and any integer $j\geqslant 0$ with $\frac{|\alpha|}{2}+j\geqslant\frac{n}{2}$, we have

$$\|D_x^\alpha\partial_t^j u(t)\|_{L^2(R^n)}\leqslant C(1+t)^{-\frac{n}{4}}\left\{\|\varphi\|_{H^{|\alpha|+2j}(R^n)}+\left[f, 2, \frac{n}{4}, |\alpha|+2j, R^n(t)\right\}, \quad t\geqslant 0,\right. \tag{5.10}$$

where C is a positive constant depending only on n, α, j.

Proof Let $W(t)$ denote the evolution operator associated with the following heat equation

$$\begin{cases} u_t=\Delta u, & (t, x)\in R^+\times R^n, \\ u(0, x)=v(x), & x\in R^n. \end{cases} \tag{5.11}$$

Namely, the solution of problem (5.11) is expressed as

$$u(t, x)=W(t)v\triangleq\left(\frac{1}{2\sqrt{\pi t}}\right)^n\int_{R_\xi^n} e^{-\frac{|x-\xi|^2}{4t}}v(\xi)d\xi. \tag{5.12}$$

We easily see that for any multi-index α the following inequalities hold:

$$\|D_x^\alpha(W(t)v)\|_{L^\infty(R^n)}\leqslant Ct^{-\left(\frac{n}{2}+\frac{|\alpha|}{2}\right)}\|v\|_{L^1(R^n)}, \quad t\geqslant 1, \tag{5.13}$$
$$\|D_x^\alpha(W(t)v)\|_{L^1(R^n)}\leqslant Ct^{-\frac{|\alpha|}{2}}\|v\|_{L^1(R^n)}, \quad t\geqslant 1, \tag{5.14}$$
$$\|D_x^\alpha(W(t)v)\|_{L^2(R^n)}\leqslant Ct^{-\frac{|\alpha|}{2}}\|v\|_{L^2(R^n)}, \quad t\geqslant 1. \tag{5.15}$$

From (5.13) and (5.14) we obtain

$$\|D_x^\alpha(X(t)v)\|_{L^2(R^n)}\leqslant Ct^{-\left(\frac{n}{4}+\frac{|\alpha|}{2}\right)}\|v\|_{L^1(R^n)}, \quad t\geqslant 1. \tag{5.16}$$

Moreover, by the imbedding theorem we have

$$\|D_x^\alpha(W(t)v)\|_{L^\infty(R^n)}\leqslant C\|D_x^\alpha v\|_{L^\infty(R^n)}\leqslant C\|v\|_{W^{|\alpha|+n+1,1}(R^n)}, \quad t>0. \tag{5.13'}$$
$$\|D_x^\alpha(W(t)v)\|_{L^2(R^n)}\leqslant\|D_x^\alpha v\|_{L^2(R^n)}, \quad t>0, \tag{5.15'}$$
$$\|D_x^\alpha(W(t)v)\|_{L^2(R^n)}\leqslant C\|v\|_{W^{|\alpha|+\left[\frac{n}{2}\right]+1,1}(R^n)}, \quad t>0. \tag{5.16'}$$

Noticing equality (5.11) we conclude

$$\|D_x^\alpha\partial_t^j(W(t)v)\|_{L^\infty(R^n)}\leqslant C(1+t)^{-\left(\frac{n}{2}+\frac{|\alpha|}{2}+j\right)}\|v\|_{W^{|\alpha|+2j+n+1,1}(R^n)}, \quad t>0. \tag{5.17}$$
$$\|D_x^\alpha\partial_t^j(W(t)v)\|_{L^2(R^n)}\leqslant C(1+t)^{-\left(\frac{n}{4}+\frac{|\alpha|}{2}+j\right)}\|v\|_{W^{|\alpha|+2j+\left[\frac{n}{2}\right]+1,1}(R^n)}, \quad t>0. \tag{5.18}$$

$$\|D_x^\alpha\partial_t^j(W(t)v)\|_{L^2(R^n)}\leqslant C(1+t)^{-\left(\frac{|\alpha|}{2}+j\right)}\|v\|_{H^{|\alpha|+2j}(R^n)}, \quad t>0. \tag{5.19}$$

On the other hand, by Duhamel's principle for the solution we have

$$D_x^\alpha \partial_t^j u(t, x) = D_x^\alpha \partial_t^j (W(t)\varphi) + \int_0^t D_x^\alpha \partial_t^j (W(t-s) f(s)) ds. \tag{5.20}$$

Therefore, for all $t \geqslant 0$ the following inequalities hold:

$$\begin{aligned} &\| D_x^\alpha \partial_t^j u(t) \|_{L^\infty(R^n)} \\ &\leqslant C(1+t)^{-\left(\frac{n}{2}+\frac{|\alpha|}{2}+j\right)} \|\varphi\|_{W^{|\alpha|+2j+n+1,1}(R^n)} + C\int_0^t (1+t-s)^{-\left(\frac{n}{2}+\frac{|\alpha|}{2}+j\right)} (1+s)^{-\frac{n}{2}} ds \\ &\quad \cdot \left[f, 1, \frac{n}{2}, |\alpha|+2j+n+1, R^n\right](t), \end{aligned} \tag{5.21}$$

$$\begin{aligned} &\| D_x^\alpha \partial_t^j u(t) \|_{L^2(R^n)} \\ &\leqslant C(1+t)^{-\left(\frac{n}{4}+\frac{|\alpha|}{2}+j\right)} \|\varphi\|_{W^{|\alpha|+2j+\left[\frac{n}{2}\right]+1,1}(R^n)} + C\int_0^t (1+t-s)^{-\left(\frac{n}{4}+\frac{|\alpha|}{2}+j\right)} (1+s)^{-\frac{n}{2}} ds \\ &\quad \cdot \left[f, 1, \frac{n}{2}, |\alpha|+2j+\left[\frac{n}{2}\right]+1, R^n\right](t), \end{aligned} \tag{5.22}$$

$$\begin{aligned} &\| D_x^\alpha \partial_t^j u(t) \|_{L^2(R^n)} \\ &\leqslant C(1+t)^{-\left(\frac{|\alpha|}{2}+j\right)} \|\varphi\|_{H^{|\alpha|+2j}(R^n)} + C\int_0^t (1+t-s)^{-\left(\frac{|\alpha|}{2}+j\right)} (1+s)^{-\frac{n}{4}} ds \\ &\quad \cdot \left[f, 2, \frac{n}{4}, |\alpha|+2j, R^n\right](t). \end{aligned} \tag{5.23}$$

Noticing $n \geqslant 3$, we immediately conclude (5.8)—(5.10). The proof of Lemma 5.2 is complete.

Lemma 5.3. *Let Ω be the same as in problem (1.1) and $u(x)$ be a solution in $H^1(\Omega)$ of*

$$\begin{cases} \Delta u = f, & x \in \Omega. \\ u|_{\partial\Omega} = 0. \end{cases} \tag{5.24}$$

Let L be an arbitrary nonnegative integer and r_1 and r_2 be two positive constants such that $r_2 < r_1$ and $\partial\Omega \subset \{x \in R^n, |x| < r_2\}$. Then, if $f \in H^L(\Omega)$, $u(x)$ satisfies

$$\sum_{|\alpha| \leqslant L+2} \|D_x^\alpha u\|_{L^2(\Omega_{r_2})} \leqslant C\Big(\|u\|_{L^2(\Omega_{r_1})} + \sum_{|\alpha| \leqslant L} \|D_x^\alpha f\|_{L^2(\Omega_{r_1})}\Big), \tag{5.25}$$

where C is a positive constant depending only on L, r_1, r_2, n and Ω.

For the proof of this lemma refer to D. Gilberg & N. S. Tradinger[2].

Now, we prove Lemma 5.1.

Proof of Lemma 5.1. We extend $\varphi(x)$ and $f(t, x)$ from Ω to R^n, in the same regular class. Note that relevent Sobolev norms of extensions of $\varphi(x)$ and $f(t, \cdot)$ can be bounded by corresponding Sobolev norms of $\varphi(x)$ and $f(t, \cdot)$ multiplied by a constant independent of φ, $f(t, x)$ and t. We also denote extended functions by $\varphi(x)$ and $f(t, x)$, respectively. Now let $u_1(t, x)$ be the solution of the Cauchy problem

$$\begin{cases} \partial_t u_1 = \Delta u_1 + f(t, x), & (t, x) \in R^+ \times R^n, \\ u_1(0, x) = \varphi(x). \end{cases} \tag{5.26}$$

By Lemma 5.2 and the property of the extension we obtain

$$\left[u_1,\ \infty,\ \frac{n}{2},\ 2L,\ R^n\right](t) \leqslant CQ\left(\varphi,\ f,\ \frac{n}{2},\ 2L+n+1\right)(t),\quad t\geqslant 0. \tag{5.27}$$

$$\left[u_1,\ 2,\ \frac{n}{4},\ 2L,\ R^n\right](t) \leqslant CQ\left(\varphi,\ f,\ \frac{n}{2},\ 2L+\left[\frac{n}{2}\right]+1\right)(t),\quad t\geqslant 0. \tag{5.28}$$

In addition, for $\frac{|\alpha|}{2}+j\geqslant\frac{n}{2}$ we have

$$\|D_x^\alpha\partial_t^j u_1(t)\|_{L^2(R^n)} \leqslant C(1+t)^{-\frac{n}{4}}\left\{\|\varphi\|_{H^{|\alpha|+2j}(\Omega)}+\left[f,\ 2,\ \frac{n}{4},\ |\alpha|+2j,\ \Omega\right](t)\right\},\quad t\geqslant 0. \tag{5.29}$$

Now we fix a positive constant γ such that $\partial\Omega\subset\{x\in R^n,\ |x|<\gamma\}$. We choose a function $\tilde{\varphi}(x)\in C^\infty(R^n)$ such that $\tilde{\varphi}(x)=1$ for $|x|>\gamma+2$ and $\tilde{\varphi}(x)=0$ for $|x|<\gamma+1$. We define $u_1(t,x)$ and $u_2(t,x)$ by

$$u_2(t,\ x)=\tilde{\varphi}(x)u_1(t,\ x), \tag{5.30}$$

$$u_3(t,\ x)=u(t,\ x)-u_2(t,\ x). \tag{5.31}$$

From (5.27) and (5.28) we easily know

$$\left[u_2,\ \infty,\ \frac{n}{2},\ 2L,\ \Omega\right](t) \leqslant CQ\left(\varphi,\ f,\ \frac{n}{2},\ 2L+n+1\right)(t),\quad t\geqslant 0, \tag{5.32}$$

$$\left[u_2,\ 2,\ \frac{n}{4},\ 2L,\ \Omega\right](t) \leqslant CQ\left(\varphi,\ f,\ \frac{n}{2},\ 2L+\left[\frac{n}{2}\right]+1\right)(t),\quad t\geqslant 0. \tag{5.33}$$

Moreover, we have for any multi-index α and any integer $j\geqslant 0$

$$\begin{aligned}\|D_x^\alpha\partial_t^j u_2(t)\|_{L^2(\Omega)} &= \|D_x^\alpha\partial_t^j u_2(t)\|_{L^2(R^n)}\\ &\leqslant \|\tilde{\varphi}(D_x^\alpha\partial_t^j u_1)(t)\|_{L^2(R^n)}+C\sum_{\substack{|\alpha_1|+|\alpha_2|\leqslant|\alpha|\\ |\alpha_1|>0}}\|(D_x^{\alpha_1}\tilde{\varphi})(D_x^{\alpha_2}\partial_t^j u_1)(t)\|_{L^2(\Omega_{\gamma+2})}\\ &\leqslant C(\|D_x^\alpha\partial_t^j u_1(t)\|_{L^2(R^n)}+\|D_x^\alpha\partial_t^j u_1(t)\|_{L^2(\Omega_{\gamma+2})})\leqslant C\|D_x^\alpha\partial_t^j u_1(t)\|_{L^2(R^n)}.\end{aligned}$$

Therefore, from (5.29) and (5.34) we see that when $\frac{|\alpha|}{2}+j\geqslant\frac{n}{2}$, for all $t\geqslant 0$, we have

$$\|D_x^\alpha\partial_t^j u_2(t)\|_{L^2(\Omega)} \leqslant C(1+t)^{-\frac{n}{4}}\left\{\|\varphi\|_{B^{|\alpha|+2j}(\Omega)}+\left[f,\ 2,\ \frac{n}{4},\ |\alpha|+2j,\ \Omega\right](t)\right\},\quad t\geqslant 0. \tag{5.35}$$

Next, we shall estimate $u_3(t,x)$. It is easily known that $u_3(t,x)$ satisfies

$$\begin{cases}\partial_t u_3=\Delta u_3+h(t,\ x), & (t,\ x)\in Q,\\ u_3(0,\ x)=(1-\tilde{\varphi}(x))\varphi(x), & x\in\Omega,\\ u_3|_\Sigma=0,\end{cases} \tag{5.36}$$

where

$$h(t,\ x)=g(t,\ x)+(1-\tilde{\varphi}(x))f(t,\ x), \tag{5.37}$$

$$g(t,\ x)=2\nabla\tilde{\varphi}\cdot\nabla u_1+u_1\Delta\tilde{\varphi}. \tag{5.38}$$

Noticing (5.27) we have

$$\operatorname{supp} g\subset[0,\ \infty)\times\{x\in R^n,\ \gamma+1\leqslant|x|\leqslant\gamma+2\}, \tag{5.39}$$

$$\left[g,\ \infty,\ \frac{n}{2},\ 2L,\ \Omega\right](t) \leqslant CQ\left(\varphi,\ f,\ \frac{n}{2},\ 2L+n+2\right)(t),\quad t\geqslant 0. \tag{5.40}$$

Moreover, using the method by which we get (5.27), we see that for the solution $u_1(t, x)$ the following inequality holds:

$$\left[u_1, \infty, \frac{n}{4}, 2L, R^n\right](t) \leqslant CQ\left(\varphi, f, \frac{n}{4}, 2L+n+1\right)(t), \quad t \geqslant 0. \tag{5.41}$$

Thus, from (5.38) we obtain

$$\left[g, \infty, \frac{n}{4}, 2L, \Omega\right](t) \leqslant CQ\left(\varphi, f, \frac{n}{4}, 2L+n+2\right)(t), \quad t \geqslant 0. \tag{5.42}$$

On the other hand, from (5.36) we know that for all integers $L \geqslant 1$, $\partial_t^L u_3(t, x)$ satisfies

$$\begin{cases} \partial_t(\partial_t^L u_3) = \Delta(\partial_t^L u_3) + \partial_t^L h, \quad (t, x) \in Q, \\ (\partial_t^L u_3)(0, x) = \Delta^L((1-\tilde{\varphi})\varphi) + \sum\limits_{j=0}^{L-1} \Delta^{L-1-j} \partial_t^j h(0, x), \quad x \in \Omega. \\ \partial_t^L u_3 |_\Sigma = 0, \end{cases} \tag{5.43}$$

So, we have

$$\partial_t^L u_3(t, x) = U(t)\left\{\Delta^L((1-\tilde{\varphi})\varphi + \sum_{j=0}^{L-1} \Delta^{L-1-j} \partial_t^j h(0, x)\right\}$$
$$+ \int_0^t U(t-s)(\partial_s^L h)(s) ds, \tag{5.44}$$

where $U(t)$ is an evolution operator defined by (4.2). By Lemma 4.1 (set $a = b = \gamma + 2$), (5.40) and (5.39) we obtain

$$\begin{aligned} &\|\partial_t^L u_3(t)\|_{L^2(\Omega_{\gamma+2})} \\ &\leqslant C(1+t)^{-\frac{n}{2}}\left\{\sum_{|\alpha| \leqslant 2L} \|D_x^\alpha \varphi\|_{L^2(\Omega)} + \sum_{j=0}^{L-1} \sum_{|\alpha| \leqslant 2(L-1-j)} \|D_x^\alpha \partial_t^j h(0, \cdot)\|_{L^2(\Omega)}\right\} \\ &\quad + \int_0^t (1+t-s)^{-\frac{n}{2}} (1+s)^{-\frac{n}{2}} ds \left[h, 2, \frac{n}{2}, 2L, \Omega\right](t) \\ &\leqslant C(1+t)^{-\frac{n}{2}}\left\{\|\varphi\|_{H^{2L}(\Omega)} + \left[g, \infty, \frac{n}{2}, 2L, \Omega\right](t) + \left[f, 2, \frac{n}{2}, 2L, \Omega\right](t)\right\} \\ &\leqslant C(1+t)^{-\frac{n}{2}} Q\left(\varphi, f, \frac{n}{2}, 2L+n+2\right)(t), \quad t \geqslant 0. \end{aligned} \tag{5.45}$$

By the same way we can see that (5.45) holds for $L=0$. Thus, (5.45) holds for all integers $L \geqslant 0$. Next, we shall show that if for an integer J with $0 \leqslant J \leqslant L-1$ we have

$$\sum_{|\alpha| \leqslant 2J} \|D_x^\alpha \partial_t^{L-J} u_3(t)\|_{L^2(\Omega_{\gamma+2-\frac{J}{L}})}$$
$$\leqslant C(1+t)^{-\frac{n}{2}} Q\left(\varphi, f, \frac{n}{2}, 2L+n+2\right)(t), \quad t \geqslant 0, \tag{5.46}$$

the inequality (5.46) also holds with J replaced by $J+1$. In fact, we easily see that for all $t \geqslant 0$ $\partial_t^{L-J-1} u_3(t, x)$ satisfies

$$\begin{cases} \Delta(\partial_t^{L-J-1} u_3) = \partial_t(\partial_t^{L-J-1} u_3) - \partial_t^{L-J-1} h \quad x \in \Omega, \\ \partial_t^{L-J-1} u_3 |_{\partial\Omega} = 0. \end{cases} \tag{5.47}$$

By (5.39), (5.40), (5.45), (5.46) and Lemma 5.3 we obtain

$$\sum_{|\alpha|\leqslant 2(J+1)} \|D_x^\alpha \partial_t^{L-J-1} u_3(t)\|_{L^2(\Omega_{\gamma+2}-\frac{J+1}{L})}$$
$$\leqslant C\{\|\partial_t^{L-J-1} u_3(t)\|_{L^2(\Omega_{\gamma+2}-\frac{J}{L})} + \sum_{|\alpha|\leqslant 2J} \|D_x^\alpha \partial_t^{L-J} u_3(t)\|_{L^2(\Omega_{\gamma+2}-\frac{J}{L})}$$
$$+ \sum_{|\alpha|\leqslant 2J} \|D_x^\alpha \partial_t^{L-J-1} h(t)\|_{L^2(\Omega_{\gamma+2}-\frac{J}{L})}\}$$
$$\leqslant C(1+t)^{-\frac{n}{2}} Q\left(\varphi, f, \frac{n}{2}, 2L+n+2\right)(t). \tag{5.48}$$

From (5.45) we already know that (5.46) holds for $J=0$. Thereby, an induction argument gives

$$\left[u_3, 2, \frac{n}{2}, 2L, \Omega_{\gamma+1}\right](t) \leqslant CQ\left(\varphi, f, \frac{n}{2}, 2L+n+2\right)(t), \quad t\geqslant 0. \tag{5.49}$$

It implies

$$\left[u_3, 1, \frac{n}{2}, 2L, \Omega_{\gamma+1}\right](t) \leqslant CQ\left(\varphi, f, \frac{n}{2}, 2L+n+2\right)(t), \quad t\geqslant 0. \tag{5.50}$$

Moreover, by the imbedding theorem and (5.49) we conclude

$$\left[u_3, \infty, \frac{n}{2}, 2L, \Omega_{\gamma+1}\right](t) \leqslant CQ\left(\varphi, f, \frac{n}{2}, 2L+3\left[\frac{n}{2}\right]+4\right)(t), \quad t\geqslant 0. \tag{5.51}$$

Similarly, by Lemma 4.1 and (5.39), (5.42) we see that for all $t\geqslant 0$, the following inequality holds:

$$\left[u_3, 2, \frac{n}{4}, 2L, \Omega_{\gamma+1}\right](t) \leqslant CQ\left(\varphi, f, \frac{n}{4}, 2L+n+2\right)(t), \quad t\geqslant 0. \tag{5.52}$$

Finally, we shall evaluate $u_3(t, x)$ for $|x|>\gamma+1$. We choose a function $\tilde{\psi}(x)\in C^\infty(R^n)$ such that $\tilde{\psi}(x)=1$ for $|x|>\gamma+\frac{3}{4}$ and $\tilde{\psi}(x)=0$ for $|x|<\gamma+\frac{1}{4}$. Set

$$u_4(t, x)=\tilde{\psi}(x)u_3(t, x). \tag{5.53}$$

It is easily known that $u_4(t, x)$ satisfies

$$\begin{cases} \partial_t u_4 = \Delta u_4 + K(t, x), & (t, x)\in R^+\times R^n, \\ u_4(0, x)=\tilde{\psi}(1-\tilde{\varphi})\varphi, \end{cases} \tag{5.54}$$

where

$$K(t, x) = -2\nabla\tilde{\psi}\cdot\nabla u_3 - u_3\Delta\tilde{\psi} + \tilde{\psi}h. \tag{5.55}$$

Applying Lemma 5.2, from (5.39), (5.40), (5.49)—(5.50) and the property of the extension we have

$$\left[u_4, \infty, \frac{n}{2}, 2L, \Omega\right](t)$$
$$=\left[u_4, \infty, \frac{n}{2}, 2L, R^n\right](t)$$
$$\leqslant \Omega\left\{\|\varphi\|_{W^{2L+n+1,1}(\Omega)} + \left[u_3, 1, \frac{n}{2}, 2L+n+2, \Omega_{\gamma+1}\right](t)\right.$$
$$\left. + \left[g, \infty, \frac{n}{2}, 2L+n+1, \Omega\right](t) + \left[f, 1, \frac{n}{2}, 2L+n+1, \Omega\right](t)\right\}$$
$$\leqslant CQ\left(\varphi, f, \frac{n}{2}, 2L+2n+4\right)(t), \quad t\geqslant 0. \tag{5.56}$$

Similarly, applying Lemma 5.2, from (5.39), (5.40), (5.52) and the property of the extension we obtain

$$\left[u_4,\ 2,\ \frac{n}{4},\ 2L,\ \Omega\right](t)\leqslant CQ\left(\varphi,\ f,\ \frac{n}{2},\ 2L+3\left[\frac{n}{2}\right]+5\right)(t),\quad t\geqslant 0. \tag{5.57}$$

Moreover, from (5.10), (5.42), (5.52) and the property of the extension, we see that when $\frac{|\alpha|}{2}+j\geqslant\frac{n}{2}$, the following inequality holds:

$$\begin{aligned}&\|D_x^\alpha\partial_t^j u_4(t)\|_{L^2(\Omega)}\\ &=\|D_x^\alpha\partial_t^j u_4(t)\|_{L^2(R^n)}\\ &\leqslant C(1+t)^{-\frac{n}{4}}\Big\{\|\varphi\|_{H^{|\alpha|+2j}(\Omega)}+\left[u_3,\ 2,\ \frac{n}{4},\ |\alpha|+2j+1,\ \Omega_{\gamma+1}\right](t)\\ &\quad+\left[g,\ \infty,\ \frac{n}{4},\ |\alpha|+2j,\ \Omega\right](t)+\left[f,\ 2,\ \frac{n}{4},\ |\alpha|+2j,\ \Omega\right](t)\Big\}\\ &\leqslant C(1+t)^{-\frac{n}{4}}Q\left(\varphi,\ f,\ \frac{n}{4},\ |\alpha|+2j+n+3\right)(t).\end{aligned} \tag{5.58}$$

Therefore, from (5.49), (5.51), (5.56) and (5.57) we conclude

$$\left[u_3,\ \infty,\ \frac{n}{2},\ 2L,\ \Omega\right](t)\leqslant CQ\left(\varphi,\ f,\ \frac{n}{2},\ 2L+2n+4\right),\quad t\geqslant 0, \tag{5.59}$$

$$\left[u_3,\ 2,\ \frac{n}{4},\ 2L,\ \Omega\right](t)\leqslant CQ\left(\varphi,\ f,\ \frac{n}{2},\ 2L+3\left[\frac{n}{2}\right]+5\right)(t),\quad t\geqslant 0. \tag{5.60}$$

Moreover, from (5.52) and (5.58) we conclude that when $\frac{|\alpha|}{2}+j\geqslant\frac{n}{2}$, we have

$$\|D_x^\alpha\partial_t^j u_3(t)\|_{L^2(\Omega)}\leqslant C(1+t)^{-\frac{n}{4}}Q\left(\varphi,\ f,\ \frac{n}{4},\ |\alpha|+2j+n+3\right)(t),\quad t\geqslant 0. \tag{5.61}$$

From (5.32), (5.33), (5.35) and (5.59)—(5.61) we come to the conclusion of Lemma 5.1.

§ 6. The A Priori Estimate and The Proof of Theorem 1.1

In order to prove Theorem 1.1 we first establish the following a priori estimate.

Lemma 6.1. *Let the assumptions of Theorem 1.1 hold. Moreover, let u be the solution of problem (1.2) satisfying (2.8) and energy estimate (3.2). Then there exist two positive constants ε_0 and K_N such that if $\varphi(x)$ satisfies*

$$\|\varphi\|_{W^{2N,1}(\Omega)}+\|\varphi\|_{H^{2N}(\Omega)}<\varepsilon_0, \tag{6.1}$$

we have

$$\|u(t)\|_{H^{2N}(\Omega)}\leqslant K_N\|\varphi\|_{H^{2N}(\Omega)}, \tag{6.2}$$

for $0\leqslant t\leqslant T$. Here constants ε_0 and K_N do not depend on T.

Proof For each integer $N\geqslant 3n+3\left[\frac{n}{2}\right]+23$, set

$$M_3(t)=\left[u,\ \infty,\ \frac{n}{2},\ 2\left[\frac{N}{2}\right]+3,\ \Omega\right](t)+\left[u,\ 2,\ \frac{n}{4},\ \bar{s},\ \Omega\right](t), \tag{6.3}$$

where $\bar{s}$ is a positive integer to be determined later. From (5.4) and (1.3) we see that for the local solution u of problem (1.2) the following estimate holds on $[0, T]$:

$$\left[u, \infty, \frac{n}{2}, 2\left[\frac{N}{2}\right]+3, \Omega\right](t)$$
$$\leqslant CQ\left(\varphi, F, \frac{n}{2}, 2\left[\frac{N}{2}\right]+2n+7\right)(t)$$
$$\leqslant C\Bigg\{\|\varphi\|_{W^{2\left[\frac{N}{2}\right]+2n+7,1}(\Omega)}+\|\varphi\|_{H^{2\left[\frac{N}{2}\right]+n+6}(\Omega)}+\left[u, 2, \frac{n}{4}, 2\left[\frac{N}{2}\right]+2n+9, \Omega\right](t)^2$$
$$+\left[u, 2, \frac{n}{4}, 2\left[\frac{N}{2}\right]+n+8\right](t)\cdot\left[u, \infty, \frac{n}{2}, \left[\frac{2\left[\frac{N}{2}\right]+n+6}{2}\right]+2, \Omega\right](t)\Bigg\}. \tag{6.4}$$

Since $N\geqslant 3n+3\left[\frac{n}{2}\right]+23$, we obtain

$$\left[\frac{2\left[\frac{N}{2}\right]+n+6}{2}\right]+2\leqslant 2\left[\frac{N}{2}\right]+3$$

and $2\left[\frac{N}{2}\right]+2n+7\leqslant 2N$. Thereby, from (6.1) and (6.3) we obtain

$$\left[u, \infty, \frac{n}{2}, 2\left[\frac{N}{2}\right]+3, \Omega\right](t)\leqslant C\{\varepsilon_0+M_{\bar{s}}(t)^2\}, \tag{6.5}$$

where we take

$$\bar{s}=2\left[\frac{N}{2}\right]+2n+9. \tag{6.6}$$

Moreover, from (5.5) we know that for $0\leqslant t\leqslant T$ $u(t, x)$ satisfies

$$\left[u, 2, \frac{n}{4}, \bar{s}, \Omega\right](t)$$
$$\leqslant C\Bigg\{\|\varphi\|_{W^{\bar{s}+3\left[\frac{n}{2}\right]+5,1}(\Omega)}+\|\varphi\|_{H^{\bar{s}+\left[\frac{n}{2}\right]+4}(\Omega)}+\left[F, 1, \frac{n}{2}, \bar{s}+3\left[\frac{n}{2}\right]+5, \Omega\right](t)$$
$$+\left[F, 2, \frac{n}{2}, \bar{s}+\left[\frac{n}{2}\right]+4, \Omega\right](t)\Bigg\}. \tag{6.7}$$

Noticing (1.3) we have

$$\left[F, 1, \frac{n}{2}, \bar{s}+3\left[\frac{n}{2}\right]+5, \Omega\right](t)$$
$$\leqslant C\Bigg\{\left[u, 2, \frac{n}{4}, \bar{s}, \Omega\right](t)^2+\left[u, 2, \frac{n}{4}, \left[\frac{\bar{s}+3\left[\frac{n}{2}\right]+5}{2}\right]+2, \Omega\right](t)$$
$$\cdot\sup_{0\leqslant\tau\leqslant t}(1+\tau)^{\frac{n}{4}}\sum_{\bar{s}-1\leqslant|\alpha|+2j\leqslant\bar{s}+3\left[\frac{n}{2}\right]+7}\|D_x^\alpha\partial_\tau^j u(\tau)\|_{L^2(\Omega)}\Bigg\}. \tag{6.8}$$

From (6.6), (6.3) and the fact that $N\geqslant 3n+3\left[\frac{n}{2}\right]+23$, we see that the right side of (6.8) do not exceed

$$C\{M_{\bar{s}}(t)^2+M_{\bar{s}}(t)\sup_{0\le\tau\le t}(1+\tau)^{\frac{n}{4}}\sum_{\bar{s}-1\le|\alpha|+2j\le\bar{s}+3\left[\frac{n}{2}\right]+7}\|D_x^\alpha\partial_\tau^j u(\tau)\|_{L^2(\Omega)}\}. \qquad (6.9)$$

Noticing the fact that $\bar{s}-1>\frac{n}{2}$, from (5.6) and (3.2) we obtain

$$\sup_{0\le\tau\le t}(1+\tau)^{\frac{n}{4}}\sum_{\bar{s}-1\le|\alpha|+2j\le\bar{s}+3\left[\frac{n}{2}\right]+7}\|D_x^\alpha\partial_\tau^j u(\tau)\|_{L^2(\Omega)}$$

$$\le CQ\left(\varphi, F, \frac{n}{4}, \bar{s}+3\left[\frac{n}{2}\right]+n+10\right)(t)$$

$$\le C\Big\{\|\varphi\|_{W^{\bar{s}+3\left[\frac{n}{2}\right]+n+10,1}(\Omega)}+\|\varphi\|_{H^{\bar{s}+3\left[\frac{n}{2}\right]+9}(\Omega)}$$

$$+\left[u, 2, \frac{n}{4}, \left[\frac{\bar{s}+3\left[\frac{n}{2}\right]+n+10}{2}\right]+2, \Omega\right](t)\cdot\|\varphi\|_{H^{\bar{s}+3\left[\frac{n}{2}\right]+n+12}(\Omega)}$$

$$\times\exp\left(C\int_0^t(1+\tau)^{-\frac{n}{2}}d\tau\cdot\left[u, \infty, \frac{n}{2}, 2\left[\frac{\bar{s}+3\left[\frac{n}{2}\right]+n+12}{4}\right]+3, \Omega\right](t)\right)$$

$$+\left[u, \infty, \frac{n}{2}, \left[\frac{\bar{s}+3\left[\frac{n}{2}\right]+9}{2}\right]+2, \Omega\right](t)\cdot\|\varphi\|_{H^{\bar{s}+3\left[\frac{n}{2}\right]+11}(\Omega)}$$

$$\times\exp\left(C\int_0^t(1+\tau)^{-\frac{n}{2}}d\tau\cdot\left[u, \infty, \frac{n}{2}, 2\left[\frac{\bar{s}+3\left[\frac{n}{2}\right]+11}{4}\right]+3, \Omega\right](t)\right)\Big\}. \qquad (6.10)$$

Since $s=2\left[\frac{N}{2}\right]+2n+9$ and $N\ge 3n+3\left[\frac{n}{2}\right]+23$, we get

$$\bar{s}+3\left[\frac{n}{2}\right]+n+12\le 2N,$$

$$\left[\frac{\bar{s}+3\left[\frac{n}{2}\right]+n+10}{2}\right]+2\le\bar{s}, \qquad (6.11)$$

$$2\left[\frac{\bar{s}+3\left[\frac{n}{2}\right]+n+12}{4}\right]+3\le 2\left[\frac{N}{2}\right]+3.$$

Therefore, from (6.8)—(6.10) we conclude

$$\left[F, 1, \frac{n}{2}, \bar{s}+3\left[\frac{n}{2}\right]+5, \Omega\right](t)$$

$$\le C\{M_{\bar{s}}(t)^2+\varepsilon_0 M_{\bar{s}}(t)(1+M_{\bar{s}}(t)e^{CM_{\bar{s}}(t)})\}, \quad 0\le t\le T. \qquad (6.12)$$

Moreover, from (3.2) we obtain

$$\left[F, 2, \frac{n}{2}, \bar{s}+\left[\frac{n}{2}\right]+4, \Omega\right](t)$$

$$\leqslant C\left[u, \infty, \frac{n}{2}, \left[\frac{\bar{s}+\left[\frac{n}{2}\right]+4}{2}\right]+2, \Omega\right](t)\cdot\|\varphi\|_{H^{\bar{s}+\left[\frac{n}{2}\right]+6}(\Omega)}$$

$$\times\exp\left(C\int_0^t(1+\tau)^{-\frac{n}{2}}d\tau\cdot\left[u, \infty, \frac{n}{2}, 2\left[\frac{\bar{s}+\left[\frac{n}{2}\right]+6}{4}\right]+3, \Omega\right](t),\right.$$

$$0\leqslant t\leqslant T. \qquad (6.13)$$

Since $s=2\left[\frac{N}{2}\right]+2n+9$ and $N\geqslant 3n+3\left[\frac{n}{2}\right]+23$, we know that $\bar{s}+\left[\frac{n}{2}\right]+6\leqslant 2N$,

$$2\left[\frac{\bar{s}+\left[\frac{n}{2}\right]+6}{4}\right]+3\leqslant 2\left[\frac{N}{2}\right]+3$$

and

$$\left[\frac{\bar{s}+\left[\frac{n}{2}\right]+4}{2}\right]+2\leqslant 2\left[\frac{N}{2}\right]+3.$$

Thus, we have

$$\left[F, 2, \frac{n}{2}, \bar{s}+\left[\frac{n}{2}\right]+4, \Omega\right](t)$$

$$\leqslant C\varepsilon_0 M_{\bar{s}}(t)e^{CM_{\bar{s}}(t)}\leqslant C(\varepsilon_0+\varepsilon_0 M_{\bar{s}}(t)^2 e^{CM_{\bar{s}}(t)}). \qquad (6.13')$$

From (6.7), (6.12) and (6.13′) we conclude

$$\left[u, 2, \frac{n}{4}, \bar{s}, \Omega\right](t)\leqslant C\{\varepsilon_0+M_{\bar{s}}(t)^2+\varepsilon_0 M_{\bar{s}}(t)(1+M_{\bar{s}}(t)e^{CM_{\bar{s}}(t)})\},\quad 0\leqslant t\leqslant T. \qquad (6.14)$$

Finally, from (6.3), (6.5) and (6.14) we obtain

$$M_{\bar{s}}(t)\leqslant C\{\varepsilon_0+M_{\bar{s}}(t)^2+\varepsilon_0 M_{\bar{s}}(t)(1+M_{\bar{s}}(t)e^{CM(t)})\},\quad 0\leqslant t\leqslant T, \qquad (6.15)$$

where C is a positive constant independent of T.

Set $X=M_{\bar{s}}(t)$, $f(X)=C\{\varepsilon_0+X^2+\varepsilon_0 X(1+Xe^{CX})$ and $g(X)=f(X)-X$. (6.15) expresses $g(X)\geqslant 0$. Because there exists a positive constant b (b depends only on the constant C in (6.15) but not on T) such that when $0\leqslant X\leqslant b$,

$$g'_x(X)=2CX+C\varepsilon_0+C^2\varepsilon_0 X^2 e^{CX}+2C\varepsilon_0 Xe^{CX}-1\leqslant -\frac{1}{2}+C\varepsilon_0$$

and $g(0)=C\varepsilon_0>0$, we can see that when ε_0 is sufficiently small, $g(X)$ has a positive zero point M. Here M do not depend on T. Again because when ε_0 is sufficiently small, $M_{\bar{s}}(0)<M$ and $M_{\bar{s}}(t)$ is a continous function, we obtain

$$M_{\bar{s}}(t)\leqslant M. \qquad (6.16)$$

From (3.2) we conclude

$$\sum_{|\alpha|+2j\leqslant 2N}\|D_x^\alpha\partial_t^j u(t)\|^2_{L^2(\Omega)}\leqslant C_N\|\varphi\|^2_{H^{2N}(\Omega)}e^{C_N\int_0^t(1+\tau)^{-\frac{n}{2}}d\tau M_{\bar{s}}(t)}\leqslant K_N^2\|\varphi\|^2_{H^{2N}(\Omega)}, \qquad (6.17)$$

where $K_N=(C_N e^{4MC_N})^{1/2}$ which is independent of T. It follows that (6.2) holds. The proof of Lemma 6.1 is complete. By the technique of Matsumura and Nishida, taking

$$\varepsilon=\min(\delta E,\ \delta E/K_N,\ \varepsilon_0) \tag{6.18}$$

and applying Theorem 2.1 and Lemma 6.1 repeatedly, we can see that if $\varphi(x)$ satisfies (1.4), problem (1.2) has a unique global solution such that $U=(u, u_t, \cdots, \partial_t^N u)\in X_{\infty,E}$. Namely, the solution u satisfies (1.5). The decay estimate (1.6) follows from (6.3) and (6.16). The proof of Theorem 1.1 is complete.

References

[1] Browder, F. E., On the Spectral Theory of Elliptic Differential Operators, *I. Math. Annalen*, **142** (1961), 22—130.

[2] Gilbarg, D. & Trudinger, N. S., Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, Heidelberg, New York, (1977).

[3] Matsumura, A. & Nishida, T., The Initial Value Problem for the Equations of Motion of Vitcous and Heat-Conductive Gases, *J. Math. Kyoto Univ.*, **20** (1980), 67—104.

[4] Tsutsumi, Y., Global Solution of the Nonlinear Schrödinger Equation in Exterior Domains, *Comm. id P. D. E.*, **8**: 12(1983) 1337—1374.

[5] Tsutsumi, Y., Local Energy Decay of the Solution to the Free Schrödinger Equation in Exterior Domains, (Preprint).

[6] Vainberg, B. R., On the Analytical Propertics of the Resolvent for a Certain Class of Operator-Pencils, *Math. USSR. Sbornic*, **6**(1968), 241—273.

[7] Zheng Songmu & Chen Yunmei, Global Existence for Nonlinear Parabolic Equation (Preprint).

[8] Zheng Songmu, Global Existence of the Initial Boundary Value Problem for Nonlinear Parabolic Equation (Preprint).