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ON THE DISTRIBUTION OF RANDOM LINES IN THE GENERAL CASE

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Abstract

L is a line in a plane through the origin, with an angle α to the x-axis, $0<\alpha<\pi$. M is a point process on positive x-axis. Through the nth point of M draw a line with a random angle θ_n to x-axis, φ^+ is the set of intersections of those lines with L^+. Let $m=EM$. If, for every $c>0$, $Em(c\theta^{-1})<\infty$, then φ^+ is locally finite on L, and let $\tilde{M}$ be the point process constructed by φ^+, then $E\tilde{M}$ exists. If, for all interval $L\subset L^+$, $\int_0^\infty r(I, x)M\ (dx)=\infty$ a. s., then φ^+ is dense on L^+. If L is drawn parallel to x-axis, the same results can be got, and this time $\tilde{M}$ is a cluster point process with cluster center M.

Draw a line L through the origin of a plane, with an angle α to the positive direction of X-axis, $0<\alpha<\pi$. L^+ is the positive side of L(all points along L^+ have their Y-coordinate positive). Place a point process M along the positive X-axis, and through the n-th point draw a line at a random angle θ_n to the negative direction of X-axis. Let φ^+ be the set of intersections of those random lines with the L^+. Suppose the $\{\theta_n;\ n=1, 2,\cdots\}$ are independent and identically distributed as θ, $0<\theta<\pi$, $F(\cdot)$ is the distribution function of θ, and $\{\theta_n\}$ are independent of M. When M is Poisson process, the properties of φ^+ are discussed in [1]. In this article, we investigate the properties of φ^+ without assumption of what process M is. We shall give some necessary and sufficient conditions for φ^+ to be a.s. dense in L^+ and for φ^+ to be a.s. locally finite (that is, there are only finite number of points of φ^+ in a finite interval). When φ^+ is a.s. locally finite, let $\tilde{M}$ be a point process on L^+, $\tilde{M}(I)$ be the number of points of φ^+ in the interval I. We also investigate the properties of $\tilde{M}$.

§ 1.

It is seen in [1] that if M is homogeneous Poisson process and $E\,\theta^{-1}<\infty$, the φ^+ is locally finite. We will show that it holds for any stationary point process M. Also it is shown that if there is a function $\Lambda(\cdot)\geqslant 0$ such that $ME(B)=\int_B \Lambda(t)\,dt$,

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$B\in\mathscr{B}(R^+)$, then there exists a function $\tilde{\Lambda}(\cdot)\geqslant 0$ such that $EM(I)=\int_I\tilde{\Lambda}(l)\,dl$, $I\in\mathscr{B}(L^+)$. This includes the calculation of intensity of $\tilde{M}$ in [1].

If L is drawn parallel to X-axis, and M is on R (both on $(-\infty, 0)$ and $[0, \infty)$), then if $\tilde{M}$ exists, it must be a cluster process with cluster center M.

In § 4, we let M be a Poisson cluster process, and investigate the conditions for the existence of $\tilde{M}$ and the properties of $\tilde{M}$.

In this article we assume that M is a simple process (only one point occurs at a time). But this assumption can be removed by treating it as K points if K points occur at the same time.

D. J. Daley (professor of Australian National University) suggested that I work on this title. To him I am grateful.

§ 2.

Sometimes the φ^+ is locally finite, but sometimes the φ^+ is even dense in L^+, although M is always locally finite. The following theorem is a necessary and sufficient condition for this. Let I be an interval $(A, B]$ on L^+ (See Fig.), the probability that the random line through the point R on X-axis intersects I is $r(I, X)=F(\beta+\delta\beta)-F(\beta)$, X is the X-coordinate of R; β, $\delta\beta$ depend on X.

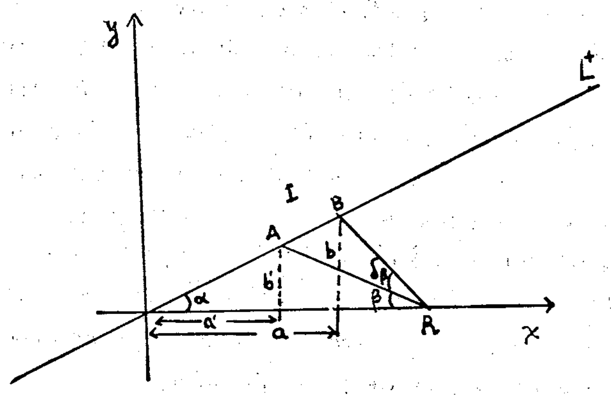

Fig.

Theorem 2.1 *M is a point process on positive X-axis. φ^+ is a.s. dense in L^+ iff for all interval $I\subset L^+$, $\int_0^\infty r(I, X)\,M(dx)=\infty$ a.s.; and φ^+ is a.s. locally finite iff for all interval $I\subset L^+$, $\int_0^\infty r(I, X)\,M(dx)<\infty$ a.s..*

Proof Let E_I be the event "there is at least one point of φ^+ in I". Then φ^+ is a.s. dense in L^+ iff for all I, $P(E_I)=1$. As $P(E_I)=EP(E_I|M)$, $P(E_I)=1$ is equivalent to $P(E_I|M)=1$ a.s.. As $\{\theta_n\}$ are independent of the M, for a certain ω_0, let $\{x_1, x_2, \cdots\}$ be the points of $M(\omega_0)$. The probability that the random lines through the point x_i do not intersect I is $1-r(I, x_i)$. So $P(E_I|M(\omega_0))=1-\prod_{i=1}^\infty(1-r(I, x_i))$; $P(E_I|M(\omega_0))=1$ is equivalent to $\prod_{i=1}^\infty(1-r(I, x_i))=0$, that is $\sum_{i=1}^\infty r(I, x_i)=\infty$. So it is equivalent to $\int_0^\infty r(I, x)M(dx)=\infty$ a.s..

Now Let A_I be the event "there are only finite number of points of φ^+ in I", $P(A_I) = EP(A_I | M)$, so φ^+ is a.s. locally finite iff $P(A_I | M) = 1$ a.s.. Let $\{x_1, x_2, \cdots\}$ be the points of $M(\omega_0)$, and K_i be the event "the line drawn through x_i intersects I". Then $K_i (i=1, 2, \cdots)$ are independent and we see that the following are all equivalent: "$P(A_I | M(\omega_0)) = 1$"; "$\sum_{i=1}^{\infty} 1_{K_i}(\omega) < \infty$ a.s.;" "$\lim_{n\to\infty} P\left(\bigcup_{i=n}^{\infty} k_i\right) = 0$"; "$1 - \lim_{n\to\infty} \prod_{i=n}^{\infty} (1 - r(I, x_i)) = 0$"; "$\sum_{i=1}^{\infty} r(I, x_i) < \infty$".

So φ^+ being locally finite is equivalent to $\int_0^\infty r(I, x) M(dx) < \infty$. The theorem is proved.

Remak 1. It is also obtained that the intersections of random lines through a certain set $\{x_1, x_2, \cdots\}$ with an interval $I \subset L^+$ are either a.s. finite or a.s. infinite. This is just the 0–1 Law ([2] p. 243) to $\sum_{i=1}^{\infty} 1_{K_i}(\omega)$.

The results of the theorem above is in stochastic form and is therefore of limited value in applications. However, a sufficient condition that is easy to use follows. (In the following $\int_a^b$ denotes integration over the interval (a, b].)

Theorem 2.2. *Suppose EM exists, and let $m = EM$, $m(x) = EM\,(0, x]$. If for all $c > 0$, $Em(c\theta^{-1}) < \infty$, then φ^+ is a.s. locally finite.*

Proof Let $I = (0, l] \subset L^+$. If for those intervals I, $E\int_0^\infty r(I, x) M(dx) < \infty$, then $\int_0^\infty r(I, x) M(dx) < \infty$ a.s., that means φ^+ is a.s. locally finite. But then

$$E\int_0^\infty r(I, x) M(dx) = \int_0^\infty r(I, x) m(dx) = \int_0^\infty F(\delta\beta) m(dx)$$

$$= \int_0^\infty \int_0^{\delta\beta} dF(u) m(dx) = \int_0^\infty \int_0^{\frac{\pi}{2} - \mathrm{tg}^{-1}\left(\frac{x-a}{b}\right)} dF(u) m(dx)$$

$$= \int_0^{\pi-\alpha} dF(u) \int_0^{a+b\,\mathrm{ctg}\,u} m(dx) = \int_0^{\pi-\alpha} dF(u) m(a + b\,\mathrm{ctg}\,u).$$

Let $m(x) = 0$ when $x \leqslant 0$. $\int_0^{\pi-\alpha} dF(u) m(a + b\,\mathrm{ctg}\,u)$ being infinite is equivalent to $Em(a + b\,\mathrm{ctg}\,\theta)$ being infinit. As $m(X)$ is non-decreasing, "for all I, $Em(a + b\,\mathrm{ctg}\,\theta) < \infty$" is equivalent to "for all $c > 0$, $Em(c\theta^{-1}) < \infty$". This is because for every $b > 0$ there is $c > b > 0$, so $m(a + b\,\mathrm{ctg}\,\theta) \geqslant m(c\theta^{-1})$ when θ is small enough, and from $Em(c\theta^{-1}) < \infty$ we get $Em(a + b\,\mathrm{ctg}\,\theta) < \infty$. On the orther hand, for every $c > 0$, there exists an I such that $b > c$, and $Em(C\theta^{-1}) < \infty$ is got from $Em(a + b\,\mathrm{ctg}\,\theta) < \infty$. This completes the proof.

Remark 2. Let $M(x) = M((0, x])$. From the proof of the theorem above, it is seen that the condition "$\int_0^\infty r(I, x) M(dx) < \infty$ a.s. for all I" is equivalent to "for

$c>0$, $E_\theta M(C\theta^{-1})<\infty$ a.s. ", where E_θ means the integral for θ only; and condition "$\int_0^\infty r(I, x)M(dx)=\infty$ a.s. for all I" is equivalent to "for any $c_1>c_2\geqslant 0$, $E_\theta[M(c_1\theta^{-1})-M(c_2\theta^{-1})]=\infty$ a.s.".

Corollary 2.3. *Suppose for every $c>0$ there exists a constant $K(c)>0$ such that for u small enough $F(cu)\leqslant K(c)F(u)$; or , for every $c>0$ there exists $K'(c)>0$ such that $m(cx)\leqslant K'(c)m(x)$ for x large enough. If $Em(\theta^{-1})<\infty$, then φ^+ is a.s. locally finite.*

Proof If for every $c>0$ there is $K'(c)>0$, $m(cx)\leqslant K'(c)m(x)$, then $m(c\theta^{-1})\leqslant K'(c)m(\theta^{-1})$ when θ is small enough. So $Em(\theta^{-1})<\infty$ means $Em(c\theta^{-1})<\infty$, $c>0$. On the other hand, $Em(c\theta^{-1})=\int_0^\pi dF(u)\int_0^{c/u} m(dx)<\infty$ is equivalent to $\int_A^\infty\left(\int_0^{c/x} dF(u)\right)m(dx)=\int_A^\infty F\left(\frac{c}{x}\right)m(dx)<\infty$ for some $A>0$. If $F(cu)\leqslant K(c)F(u)$, then $\int_A^\infty F\left(\frac{c}{x}\right)m(dx)<\infty$ follows from $Em(\theta^{-1})<\infty$.

There are many F which fulfil the condition of Corollary 2.3. For example, θ is uniform on $(0, \pi)$; or $F(x)$ is $a\sin x+b\cos x$ on some interval $B\subset(0, \pi)$; or $F(x)=ax^k$, $K>0$, on $(0, \pi)$. In particular, if there exists a $c>0$ such that $P(\theta<c)=0$, then F fulfils the condition and meanwhile $Em(\theta^{-1})<\infty$. So in this case, φ^+ is a.s. locally finite.

Using Corollary 2.3 to stationary point processes, we get the following corollary.

Corollary 2.4. *Suppose M is a stationary point process, EM exists. If $E\theta^{-1}<\infty$, then φ^+ is a.s. locally finite.*

For $x\geqslant 0$, let $\xi(x; \cdot)$ be a point process on L constructed by intersecting of L^+ with a line through X at a random angle θ with distribution function F. Then $\xi(x,L^+)\leqslant 1$. Let $\{\xi(x,\cdot); x\in R^+\}$ be independent and independent of M. If the points of $M(\omega_0)$ are $\{x_1, x_2, \cdots\}$, then, conditioning on $M(\omega_0)$, $\widetilde{M}(\cdot)=\sum_{i=1}^\infty\xi(x_i, \cdot)$. So $\widetilde{M}(\cdot)=\int_{R^+}\xi(x, \cdot)M(dx)$ (because, conditioning on $M(\omega)$, they are equal in distribution). Using this we get following theorem.

Theorem 2.5. *Suppose $m=EM$ exists. $E\widetilde{M}$ exists iff for every $c>0$ $Em=(c\theta^{-1})<\infty$. In this time, let $I=(l_1, l_2]\subset L^+$.*

$$E\widetilde{M}(I)=E[m(l_2\cos\alpha+l_2\sin\alpha\operatorname{ctg}\theta)-m(l_1\cos\alpha+l_1\sin\alpha\operatorname{ctg}\theta)].$$

Proof

$$\widetilde{M}(I)\overset{d}{=}\int_0^\infty\xi(x, I)M(dx).$$

So

$$E\widetilde{M}(I)=EE\left[\int_0^\infty\xi(x, I)M(dx)\,|\,M\right]=E\left(E\sum_{i=1}^\infty\xi(x_i, I)\right)$$

$$=E\sum_{i=1}^\infty r(I, x_i)=E\int_0^\infty r(I, x)M(dx)=\int_0^\infty r(I, x)m(dx).$$

$\int_0^\infty r(I, x)m(dx)<\infty$ for all I is equivlant to for every $c>0$ $Em(c\theta^{-1})<\infty$ (see the proof of Theorem 2.2). And

$$\int_0^\infty r(I, x)m(dx)=\int_0^\infty\int_\beta^{\delta_\beta} dF(u)m(dx)=\int_0^\infty\int_{\frac{\pi}{2}-\mathrm{tg}^{-1}\left(\frac{x-a'}{b'}\right)}^{\frac{\pi}{2}-\mathrm{tg}^{-1}\left(\frac{x-a}{b}\right)} dF(u)m(dx)$$

$$=\int_0^{\pi-\alpha} dF(u)\,[m(a+b\,\mathrm{ctg}\,u)-m(a'+b'\,\mathrm{ctg}\,u)]$$

$$=E[m(l_2\cos\alpha+l_2\sin\alpha\,\mathrm{ctg}\,\theta)-m(l_1\cos\alpha+l_1\sin\alpha\,\mathrm{ctg}\,\theta)].$$

The theorem is proved.

Theorem 2.6. *Suppose there exists a* $\Lambda(\cdot)\geqslant 0$ *such that* $EM(B)=\int_B\Lambda(t)\,dt=m(B)$, $B\in\mathscr{B}(R^+)$. *Then there exists a* $\widetilde{\Lambda}(\cdot)\geqslant 0$ *such that* $E\widetilde{M}(B)=\int_B\widetilde{\Lambda}(l)\,dl$, $B\in\mathscr{B}(L^+)$, *and* $\widetilde{\Lambda}(l)=\int_0^{\pi-\alpha}\Lambda(l\cos\alpha+l\sin\alpha\,\mathrm{ctg}\,u)\sin(\alpha+u)\,\mathrm{cosec}\,u\,dF(u)$, *a.s.*

Proof For $I=(l_1, l_2]\subset L^+$,

$$\int_I\widetilde{\Lambda}(l)\,dl=\int_{l_1}^{l_2}\widetilde{\Lambda}(l)\,dl=\int_{l_1}^{l_2}dl\int_0^{\pi-\alpha}\Lambda(l\cos\alpha+l\sin\alpha\,\mathrm{ctg}\,u)\sin(\alpha+u)\,\mathrm{cosec}\,u\,dF(u)$$

$$=\int_0^{\pi-\alpha}dF(u)\,\mathrm{cosec}\,u\sin(\alpha+u)\int_{l_1}^{l_2}\Lambda(l\cos\alpha+l\sin\alpha\,\mathrm{ctg}\,u)\,dl$$

$$=\int_0^{\pi-\alpha}dF(u)\,\mathrm{cosec}\,u\sin(\alpha+u)\,(\cos\alpha+\sin\alpha\,\mathrm{ctg}\,u)^{-1}$$

$$\times[m(l_2(\cos\alpha+\sin\alpha\,\mathrm{ctg}\,u))-m(l_1(\cos\alpha+\sin\alpha\,\mathrm{ctg}\,u))]$$

$$=E[m(l_2\cos\alpha+l_2\sin\alpha\,\mathrm{ctg}\,\theta)-m(l_1\cos\alpha+l_1\sin\alpha\,\mathrm{ctg}\,\theta)]$$

$$=E\widetilde{M}(I).$$

This completes the proof.

Remark 3. $\xi(x_i,\cdot)$ has representation: $\xi(x_i, B)=1_B\left(\frac{\sin\theta_i x_i}{\sin(\theta_i+\alpha)}\right)$, $B\in\mathscr{B}(L^+)$ (see Fig.). Conditioning on $M(\omega)$, the jump times of $\widetilde{M}$, μ_1, $\mu_2\cdots$ are order statistic of $\left\{\frac{x_i\sin\theta_i}{\sin(\theta_i+\alpha)}\ i=1, 2, \cdots\right\}$.

§ 3.

Consider the case where L is drawn parallel to the x-axis, intersecting the Y-axis at $Y=b>0$. M is a point process on x-axis (both positive and negative sides). φ is the set of intersections of random lines on L. Like Theorem 2.1, we have the following theorem.

Theorem 3.1. *φ is a.s. dense on L iff for all* $I\subset L$, $\int_R r(I, x)M(dx)=\infty$ a.s.; *φ is a.s. locally finite iff for all* $I\subset L$, $\int_R r(I, x)M(dx)<\infty$ *a.s.*

Theorem 3.2. *Suppose* $EM=m$ *exists. If for all* l_1, l_2, $-\infty<l_1<l_2<\infty$,

$E[m(l_2+b\operatorname{ctg}\theta)-m(l_1+b\operatorname{ctg}\theta)]<\infty$, *then* φ *is a.s. locally finite.*

Proof

$$E\int_R r(I,\ x)M(dx)=\int_R r(I,\ x)m(dx)=\int_{-\infty}^{+\infty}\int_{\frac{\pi}{2}-\operatorname{tg}^{-1}\left(\frac{x-l_1}{b}\right)}^{\frac{\pi}{2}-\operatorname{tg}^{-1}\left(\frac{x-l_2}{b}\right)} dF(u)m(dx)$$
$$=\int_0^\pi dF(u)\int_{l_1+b\operatorname{ctg}u}^{l_2+b\operatorname{ctg}u} m(dx)$$
$$=E[m(l_2+b\operatorname{ctg}\theta)-m(l_1+b\operatorname{ctg}\theta)].$$

So, if $E[m(l_2+b\operatorname{ctg}\theta)-m(l_1+b\operatorname{ctg}\theta)]<\infty$, then $\int_R r(I,\ x)M(dx)<\infty$ a.s..

Suppose that $EM=m$ exists and φ is a.s. locally finite. As that stated in § 2, we see that $\widetilde{M}(\cdot)\overset{d}{=}\int_R \xi(x,\ \cdot)M(dx)$, and $\xi(x,\ B)=1_B(x-b\operatorname{ctg}\theta_x)$, $\{\theta_x;\ x\in R\}$, are independent, indentically distributed as θ. From this it is seen that $\widetilde{M}(\cdot)$is a cluster precess with cluster center M(see[3], § 2), because $\xi(x,\ \cdot)$can be regarded as process $\eta_x(B)=1_B(-b\operatorname{ctg}\theta_x)$ with its center at x. From [3] Corollary 3.2 we know that if M is stationary, then $\widetilde{M}$ exists. But this time $E[m(l_2+b\operatorname{ctg}\theta)-m(l_1+b\operatorname{ctg}\theta)]<\infty$, so $\widetilde{M}$ exists. And it is seen from Theorem 3.2 that if $\sup_t m(I-t)<\infty$, for any interval I, then φ is a.s. locally finite. But this is just the Corollyary 3.3 in [3].

Theorem 3.3. *Suppose* $EM=m$ *exist,*$E\widetilde{M}$ *exists iff for arbitary* $l_1<l_2$,$E[m(l_2+b\operatorname{ctg}\theta)-m(l_1+b\operatorname{ctg}\theta)]<\infty$. *And in this time* $E\widetilde{M}(I)=E[m(l_2+b\operatorname{ctg}\theta)-m(l_1+b\operatorname{ctg}\theta)]$, $I=(l_1,\ l_2]\subset L$. *If there is* $\Lambda(\cdot)\geqslant 0$ *such that* $m(B)=\int_B \Lambda(t)dt$, *then let* $\widetilde{\Lambda}(l)=\int_0^\pi \Lambda(l+b\operatorname{ctg}u)dF(u)$,*we have* $E\widetilde{M}(B)=\int_B \widetilde{\Lambda}(l)dl,\ B\in\mathscr{B}(L)$.

Proof

$$E\widetilde{M}(I)=EE[\widetilde{M}(I)\,|\,M]=E\int_R E\xi(x,\ I)M(dx)$$
$$=\int_R r(I,\ x)m(dx)=E[m(l_2+b\operatorname{ctg}\theta)-m(l_1+b\operatorname{ctg}\theta)]$$

and

$$\int_{l_1}^{l_2}\left(\int_0^\pi \Lambda(l+b\operatorname{ctg}u)\ dF(u)\right)dl=\int_0^\pi\int_{l_1}^{l_2}\Lambda(l+b\operatorname{ctg}\theta)dl\,dF(u)$$
$$=\int_0^\pi[m(l_2+b\operatorname{ctg}u)-m(l_1+b\operatorname{ctg}u)]dF(u)$$
$$=E\widetilde{M}(I).$$

§ 4.

Let M be a Poisson cluster process. We shall investigate the conditions for $\widetilde{M}$ to exist and some properties of $\widetilde{M}$, as an application of the theorems in § 2, § 3. Suppose $\{N_x;\ x\in R^+\}$ is a class of independent, identically distributed point processes, N is a point process on R^+, $\{N_x\}$ is independent of N, cluster process $M(\cdot)=\int_{R^+} M_x(\cdot|x) N(dx)$, where $N_x(\cdot|x)$ is the N_x with its center x; N is called cluster center (we assume that M is locally finite). If N is Poisson process, the M is called Poisson cluster (see[3]).

Suppose that N_x are identically distributed as N' and $N'((-\infty, 0])=0$. Let $\mu=EN$, $\mu(t)=\mu((0, t])$. N is a homogeneous Poisson process with intensity λ. Let $B=(a, b]$. Note that $EM(B)=\lambda\int_{R^+}\mu(B-t)dt\leqslant\lambda\int_{R^+}\mu(b-t)dt\leqslant\lambda\int_0^b\mu(b-t)dt=\mu(b)\cdot\lambda\cdot b<\infty$.

First we suppose that L is drawn at an angle α to X-axis(see § 2).

Theorem 4.1. *M is a Poisson cluster process stated above. If for all $c>0$, $E\int_0^{c\theta^{-1}}\mu(t)dt<\infty$, then φ^+ is a.s. locally finite. This time $E\widetilde{M}$ exists and $E\widetilde{M}(I)=\int_{l_1}^{l_2}\tilde{\mu}(l)dl$, $I=(l_1, l_2]$, where*

$$\tilde{\mu}(l)=\int_0^{\pi-\alpha}\lambda\mu(l\cos\alpha+l\sin\alpha\operatorname{ctg}u)\sin(\alpha+u)\operatorname{cosec}u\,dF(u).$$

(That is, conditions for $\widetilde{M}$ and $E\widetilde{M}$ are the same as the case where M is non-homogeneous Poisson process with intensity function $\lambda\mu(t)$).

Proof Let $m(\cdot)=EM(\cdot)$, $m(x)=EM((0, x])$. But

$$m(x)=EM((0, x])=\lambda\int_{R^+}\mu((0, x]-t)dt=\lambda\int_0^x\mu(x-t)dt=\lambda\int_0^x\mu(t)dt.$$

So if, for every $c>0$, $Em(c\theta^{-1})=E\int_0^{c\theta^{-1}}\mu(t)dt<\infty$, then φ^+ is a.s. locally finite (see Theorem 2.2). Also we have

$$m((x_1, x_2])=\lambda\int_{x_1}^{x_2}\mu(t)dt.$$

So from Theorem 2.6 we know that there exist $\tilde{\mu}\geqslant 0$, $E\widetilde{M}(I)=\int_{l_1}^{l_2}\tilde{\mu}(l)dl$, and $\tilde{\mu}(l)=\int_0^{\pi-\alpha}\lambda\mu(l\cos\alpha+l\sin\alpha\operatorname{ctg}u)\sin(u+\alpha)\operatorname{cosec}u\,dF(u)$.

The theorem is proved.

Then we suppose that L is drawn parallel to X-axis (see § 3), N, N_x, $x\in R$ all are the Point processes on$(-\infty, +\infty)$.

Theorem 4. 2 *Suppose M is a Poisson cluster process stated above, and $EM=m$ exists. Then φ is a.s. locally finite, and $\widetilde{M}$ is also a Poisson cluster process directed by N, $\widetilde{N}_x$, $x\in R$ and $E\widetilde{M}=EM=m$.*

Proof N is a stationary process, so M itself is stationary(see[3] p. 296). Then φ is a.s. locally finite(see § 3). or, let $B=(x_1, x_2]\subset R$,

$$m(B)=EM(B)=\int_R\lambda\mu(B-t)dt=\int_R\lambda[\mu(x_2-t)-\mu(x_1-t)]dt.$$

So $m(B)=m(B+x)$, $x\in R$. Then we get

$$E[m(l_2+b\operatorname{ctg}\theta)-m(l_1+b\operatorname{ctg}\theta)]=m(l_2)-m(l_1)=m((l_1,l_2])<\infty.$$

It follows from Theorem 3.2 that φ is a.s. locally finite. And

$$E\tilde{M}(I)=E[m(l_2+b\operatorname{ctg}\theta)-m(l_1+b\operatorname{ctg}\theta)]=m(I),\ I=(l_1,l_2]\subset L.$$

So $E\tilde{M}=m$.

Let $\tilde{N}_x$ be the point process on L directed by the N_x on R. ($\tilde{N}_x$ is the intersections of random lines through the points of N_x.) It is easily seen that $\tilde{N}_x(\cdot|x)=\tilde{N}_x(\cdot|x)$, $\tilde{N}_x(\cdot|x)$ means $\tilde{N}_x$ with its center x (for every ω, it is equal). Conditional on $N(\omega_0)$, M is $\sum_{i=1}^{\infty}N_{x_i}(\cdot|x_i)$, $\{x_i;\ i=1,2,\cdots\}$ are the points of $N(\omega_0)$. This time $\tilde{M}\overset{d}{=}\sum_{i=1}^{\infty}\tilde{N}_{x_i}(\cdot|x_i)$ $=\sum_{i=1}^{\infty}\tilde{N}_{x_i}(\cdot|x_i)$. So $\tilde{M}(\cdot)\overset{d}{=}\int\tilde{N}_x(\cdot|x)N(dx)$, that is, M is a Poisson cluster, its components are N, $\tilde{N}_x$.

Remark 4. Let M be a cluster process on R with cluster center N, cluster member N_x. If $\tilde{M}$ exists, then it is also a cluster process directed by N, $\tilde{N}_x$ (in the case that L is drawn parallel to X-axis), this is seen by the proof of the theorem above.

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