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COMPLETELY POSITIVE MAPS AND *-ISOMORPHISM OF C^*-ALGEBRAS

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Abstract

Let A, B be unital C^*-algebras.

$\mathscr{K}_A=\{\varphi|\varphi$ are all completely positive linear maps from $M_n(C)$ to A with $\|a(\varphi)\|\leqslant 1\}$.

$$\left(a(\varphi)=\begin{pmatrix}\varphi(e_{11})\cdots\varphi(e_{1n})\\ \cdots\cdots\\ \varphi(e_{n1})\cdots\varphi(e_{nn})\end{pmatrix}, \text{ where } \{e_{ij}\} \text{ is the matrix unit of } M_n(C).\right)$$

Let α be the natural action of $SU(n)$ on $M_n(C)$.

For $n\geqslant 3$, if Φ is an α-invariant affine isomorphism between $\mathscr{K}_A$ and $\mathscr{K}_B$, $\Phi(0)=0$, then A and B are *-isomorphic.

In this paper a counter example is given for the case $n=2$.

§ 1. Introduction

The relationship between the structure of C^*-algebras and their state spaces has received strong attension from many specialists. Using Kadison's function representation it is proved that if A and B are C^*-algebras with state spaces $\mathfrak{S}(A)$ and $\mathfrak{S}(B)$, and if ψ is a weakly continuous affine isomorphism between $\mathfrak{S}(A)$ and $\mathfrak{S}(B)$, then ψ induces a Jordan isomorphism between A and B.

From now on we assume that all C^*-algebras are unital.

Let A be a C^*-algebra and $\mathscr{K}_A$ be the set of all completely positive maps from $M_n(C)$ to A with $\|a(\varphi)\|\leqslant 1$. Then $\mathscr{K}_A$ is a convex set.

The new idea in this paper is that we can view every φ in $\mathscr{K}_A$ as a "building block" such that $\mathscr{K}_A$ becomes a covering for C^*-algebra A, and if Φ is a certain kind affine isomorphism between $\mathscr{K}_A$ and $\mathscr{K}_B$, we can expect that Φ will induce a *-isomorphism between A and B.

Suppose that $SU(n)$ is the set of all $n\times n$ unimodular unitary matrixes and α is the automorphism group on $M_n(C)$ defined by

$$\alpha_g(x)=gxg^{-1}. \ x\in M_n(C),\ g\in SU(n).$$

By use of $\alpha_g\varphi(x)=\varphi(\alpha_g^{-1}(x)),\varphi\in\mathscr{K}_A,\ x\in M_n(C)$, α induces an action on $\mathscr{K}_A$.

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In this paper we consider $\mathscr{K}_A$ instead of $\mathfrak{S}(A)$ and show the following result:

Theorem 1. *Let A and B be C^*-algebras. If Φ is an α-invariant affine isomorphism between $\mathscr{K}_A$ and $\mathscr{K}_B$ $(n\geqslant 3)$, $\Phi(0)=0$, (α-invariant means that $\Phi\alpha_g=\alpha_g\Phi$), then Φ gives rise naturally to an *-isomorphism.*

If $n=2$, next theorem is a counter example for Theorem 1.

Theorem 2. *Let A and B be C^*-algebras. If $n=2$, there is an α-invariat affine isomorphism between $\mathscr{K}_A$ and $\mathscr{K}_B$, $\Phi(0)=0$, such that Φ does not give rise to a *-isomorphism.*

In view of the Gelfand representation theorem, the complex number field C can be viewed as the fundamental building block of an abelian C^*-algebra. We propose to view $M_n(C)$ $(n\geqslant 3)$ as a candidate for such object. We prove that the structure of $\mathscr{K}_A$ determines the algebraic isomorphism class of A. This is an important first step toward the search of the fundamental building block of a C^*-algebra. One might view an element of $\mathscr{K}_A$ as a "non-commutative singular simplex". This new work is important for classifications of symmetric convex sets and non-commutative K-theory.

§ 2. The Extension of an Affine Isomorphism

Let $M_n(C)$ be the set of all $n\times n$ complex matrixes.

Following the notations in p. 163 [1], let $L(M_n, A)$ be the vector space of linear functions from M_n to A and $L(M_n, A)^{\oplus}$ the cone of all completely positive maps from M_n to A.

Given an element $r=[r_{ij}]\in M_n\otimes A$, we define $\theta(r)$: $M_n\to A$ by

$$\theta(r)(\alpha)=\sum_{i,j=1}^{n}\alpha_{ij}r_{ij},\quad \text{for any } \alpha\in M_n,$$

Lemma 1[[1]p.163]. *If A is a C^*-algebra, then the map*

$$\theta\colon M_n\otimes A\to L(M_n, A)$$

is an order isomorphism (with respect to $L(M_n, A)^{\oplus}$).

For any $\varphi\in L(M_n, A)$, we define

$$a(\varphi)=\begin{pmatrix}\varphi(e_{11})\varphi(e_{12})\cdots\varphi(e_{1n})\\ \varphi(e_{21})\varphi(e_{22})\cdots\varphi(e_{2n})\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \varphi(e_{n1})\varphi(e_{n2})\cdots\varphi(e_{nn})\end{pmatrix}$$

in which $\{e_{ij}\}$ is matrix unit of $M_n(C)$.

By Lemma 1, it follows that φ is completely positive if and only if $a(\varphi)\geqslant 0$ in $M_n\otimes A$.

In the same way, we can define $b(\Psi)$ for any $\Psi\in L(M_n, B)$.

To prove main theorem, we shall use following lemma.

Lemma 2. *Let A and B be C^*-algebras and S_A and S_B closed unit balls in A, B respectively. If Φ is an affine isomorphism between $A^+\cap S_A$ and $B^+\cap S_B$, $\Phi(0)=0$, then Φ can be extended to a Jordan isomorphism from A to B.*

Proof At first, we extend linearly Φ to A^+.

For $a\in A^+$, we define

$$\Phi(a)=\left\{\frac{1}{\lambda}\Phi(\lambda a):\lambda\geqslant 0,\ \lambda a\in A^+\cap S_{A^\circ}\right\}.$$

By the standard procedure, it follows that the extended Φ is a positive map from A onto B. We extend Φ as a linear map from A to B.

Since

$$\|a\|=\operatorname{Inf}\{\lambda\geqslant 0;\ -\lambda I\leqslant a\leqslant \lambda I\}$$

and Φ maps $A^+\cap S_A$ onto $B^+\cap S_B$, Φ is an isometry from A to B.

According to Theorem 7 p. 330 [2], Φ is a Jordan isomorphism.

§ 3. Main Theorem

Now we will show the main theorem in this paper. We put

$\mathscr{K}_A=\{\varphi\,|\,\varphi$ are all completely positive maps from $M_n(C)$ to A with $\|a(\varphi)\|\leqslant 1\}$,

$\mathscr{K}_B=\{\psi\,|\,\psi$ are all completely positive maps from $M_n(C)$ to B with $\|b(\psi)\|\leqslant 1\}$.

Theorem 1. *Let A, B be C^*-algebras. For $n\geqslant 3$, if Φ is an α-invariant affine isomorphism from $\mathscr{K}_A$ to $\mathscr{K}_B$, $\Phi(0)=0$, then A and B are *-isomorphic.*

Proof By Lemma 1,

$$\mathscr{K}_A=\{a\in(M_n\otimes A)^+:\ \|a\|\leqslant 1\}.$$

Applying Lemma 2, we then have an α-invariant positively preserving isometry Φ from $M_n\otimes A$ onto $M_n\otimes B$.

Replacing A and B by their second dual $\tilde{A}$ and $\tilde{B}$ and Φ by $^{tt}\Phi$, we may assume that A and B are Von Neumann algebras. The α-invariance means that

$$\Phi(uxu^*)=u\Phi(x)u^*,\ x\in M_n\otimes A,\ u\in SU(n).$$

By [2], p. 335, let z be the central projection of B such that $x\in M_n\otimes A\mapsto\Phi(x)\,z$ is multiplicative and $x\mapsto\Phi(x)\,z^\perp$ is anti-multiplicative.

We will view A and $SU(n)$ as subsets of $M_n\otimes A$, if it does not cause the danger of confusions.

If $x\in A$, then $uxu^*=x$ for every $u\in SU(n)$ so that

$$\Phi(x)=u\Phi(x)u^*,\ x\in A,\ u\in SU(n).$$

But $M_n'\cap(M_n\otimes B)=B$, so that $\Phi(A)=B$.

Now we have, for any $x\in M_n\otimes A$ and $u\in SU(n)$,

$$\begin{aligned}u\Phi(x)u^* &= \Phi(uxu^*) = \Phi(uxu^*)z+\Phi(uxu^*)z^\perp\\ &= \Phi(u)\Phi(x)\Phi(u^*)z+\Phi(u^*)\Phi(x)\Phi(u)z^\perp\\ &= [\Phi(u)z+\Phi(u^*)z^\perp]\Phi(x)[\Phi(u^*)z+\Phi(u)z^\perp].\end{aligned}$$

Thus, $\rho(u)=u^*[\Phi(u)z+\Phi(u^*)z^\perp]$ belongs to the center $\mathscr{Z}$ of B.

If $u,\ v\in SU(n)$, then we can prove

$$\rho(uv)=\rho(u)\rho(v)$$

by noticing $\rho(u)\in\mathscr{Z}$. Hence ρ is a homomorphism of $SU(n)$ into a commutative group $\mathscr{U}(\mathscr{Z})$, but we know that such a homomorphism must be trivial. Therefore

$$\rho(u)=I,\ u\in SU(n).$$

This means that

$$\begin{cases}\Phi(u)z^\perp=u^*z^\perp,\\ \Phi(u)z=uz,\ u\in SU(n).\end{cases}$$

We will show that $z^\perp=0$.

For $\lambda\in T=\{\lambda\in C: |\lambda|=1\}$, we consider

$$\begin{aligned}&[\lambda\Phi(e_{11})+\Phi(e_{22})+\cdots+\bar\lambda\Phi(e_{ii})+\cdots+\Phi(e_{nn})]z^\perp\\ &=\Phi(\lambda e_{11}+e_{22}+\cdots+\bar\lambda e_{ii}+\cdots+e_{nn})z^\perp\\ &=(\bar\lambda e_{11}+e_{22}+\cdots+\lambda e_{ii}+\cdots+e_{nn})z^\perp\quad(2\leqslant i\leqslant n).\end{aligned}$$

Hence

$$\Phi(e_{11})z^\perp=e_{ii}z^\perp,$$

$$\Phi(e_{ii})z^\perp=e_{11}z^\perp\quad(2\leqslant i\leqslant n).$$

If $n\geqslant 3$, in the same way, we can prove

$$\Phi(e_{22})z^\perp=e_{ii}z^\perp,\quad 3\leqslant i\leqslant n.$$

Therefore, we get

$$\Phi(e_{11})z^\perp=e_{22}z^\perp=e_{33}z^\perp=\cdots=e_{nn}z^\perp=0,$$

$$\Phi(e_{22})z^\perp=e_{11}z^\perp=e_{33}z^\perp=\cdots=e_{nn}z^\perp=0.$$

This means that

$$z^\perp=(e_{11}+e_{22}+\cdots+e_{nn})z^\perp=0.$$

Hence, Φ must be an isomorphism of $M_n\otimes A$ onto $M_n\otimes B$ such that $\Phi(A)=B$.

If we view the set of all $\mathscr{K}_A$ (A is a C^*-algebra) as a category where Hom $(\mathscr{K}_A, \mathscr{K}_B)$ consists of all α-invariant affine isomorphisms from $\mathscr{K}_A$ to $\mathscr{K}_B$, with $\Phi(0)=0$, and the set of all C^*-algebras A as a category where Hom (A, B) consists of all *-isomorphisms from A to B, for $n\geqslant 3$, by Theorem 1 any $\Phi\in$ Hom $(\mathscr{K}_A, \mathscr{K}_B)$ determines $\pi(\Phi)\in$ Hom (A, B) so that π is a functor. We consider the main theorem over this functor.

§ 4. A Counter Example for $n=2$

In this section, we give a counter example for $n=2$.

Theorem 2. *Let A and B be C^*-algebras. If $n=2$, there is an α-invariant affine isomorphism between $\mathscr{K}_A$ and $\mathscr{K}_B$, $\Phi(0)=0$, such that Φ does not give rise to a *-isomorphism.*

Proof In M_2, we define

$$\sigma\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)=\begin{pmatrix}\delta & -\beta\\ -\gamma & \alpha\end{pmatrix}.$$

Then σ is an anti-automorphism of M_2 of order $2'$ such that

$$\sigma(u)=u^*,\quad u\in SU(2).$$

Hence we have

$$\sigma(uxu^*)=u\sigma(x)u^*,\quad x\in M_2,\ u\in SU(2).$$

Therefore, if π is an anti-isomorphism of a C^*-algebra A onto B, then $\Phi=\sigma\otimes\pi$ is an anti-isomorphism of $M_2\otimes A$ onto $M_2\otimes B$ such that

$$\Phi(uxu^*)=u\Phi(x)u^*,\quad x\in M_2\otimes A,\ u\in SU(2).$$

But Φ induces an α-invariant affine isomorphism of

$$\mathscr{K}_A=\{a\in M_2\otimes A:\ a\geqslant 0\ \|a\|\leqslant 1\}$$

onto $\mathscr{K}_B$, with $\Phi(0)=0$.

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