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MINIMAL SUBMANIFOLDS IN A RIEMANNIAN MANIFOLD OF QUASI CONSTANT CURVATURE

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Abstract

A Riemannian manifold V^m which admits isometric imbedding into two spaces V^{m+p} of different constant curvatures is called a manifold of quasi constant curvature[2]. The Riemannian curvature of V^m is expressible in the form

$K_{ABCD}=a(g_{AC}g_{BD}-g_{AD}g_{BC})+b(g_{AC}\lambda_B\lambda_D+g_{BD}\lambda_A\lambda_C-g_{AD}\lambda_B\lambda_C-g_{BC}\lambda_A\lambda_D)$, $(1=\sum g_{AB}\lambda_A\lambda_B)$

and conversely. In this paper it is proved that if M^n is any compact minimal submanifold without boundary in a Riemannian manifold V^{n+p} of quasi constant curvature, then

$$\int_{M^n}\left\{\left(2-\frac{1}{p}\right)\sigma^2-\left[na+\frac{1}{2}(b-|b|)(n+1)\right]\sigma+n(n-1)b^2\right\}*1\geq 0,$$

where σ is the square of the norm of the second fundamental form of M^n. When V^{n+p} is a manifold of constant curvature, $b=0$, the above inequality reduces to that of Simons.

It is well known that for any compact minimal submanifold M^n without boundary in a space S^{n+p} of constant curvature we have the inequality of Simons which is expressed in terms of a constant a and the invariant σ, where a is the curvature of S^{n+p} and σ is the square of the norm of the second fundamental form of M^n. In this paper we shall extend this formula to the minimal submanifolds in a space V^{n+p} of quasi constant curvature. The inequality (8) which we shall prove is expressed in terms of the invariant σ and the functions a and b which are invariants of the space V^{n+p}.

1. Let V^{n+p} be a C^∞ Riemannian manifold of $n+p$ dimensions. In what follows we shall make use of the following convention on the range of indices:

$$1\leq A,\ B,\ C,\ \cdots\leq n+p,$$
$$1\leq i,\ j,\ k,\ \cdots\leq n,$$
$$n+1\leq\alpha,\ \beta,\ \gamma,\ \cdots\leq n+p.$$

As usually we have for V^{n+p}:

$$d\omega_A=\sum_B\omega_{AB}\wedge\omega_B,\ \omega_{AB}+\omega_{BA}=0, \tag{1}$$

$$d\omega_{AB}=\sum_C\omega_{AC}\wedge\omega_{CB}-\frac{1}{2}\sum_{C,E}K_{ABCE}\,\omega_C\wedge\omega_E. \tag{2}$$

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Let a C^∞ submanifold M^n of n dimensions of V^{n+p} be given as follows:

$$\omega_\alpha=0, \quad (\alpha=n+1, \cdots, n+p). \tag{3}$$

If TV^{n+p} denotes the tangent bundle of V^{n+p}, its induced bundle over M^n splits into a direct sum

$$TV^{n+p}=TM^n \oplus (TM^n)^\perp,$$

where TM^n and $(TM^n)^\perp$ are respectively the tangent bundle and normal bundle of M^n. Let θ_A, θ_{AB} be the forms previously denoted by ω_A, ω_{AB} relative to this particular frame field. Then we have

$$\theta_\alpha=0,$$

and

$$\theta_{i\alpha}=\sum_j h_{i\alpha j}\theta_j, \tag{4}$$

where

$$h_{i\alpha j}=h_{j\alpha i}.$$

The mean curvature vector of M^n is defined by

$$M=\sum_{i,\alpha} h_{i\alpha i}e_\alpha,$$

and M^n is called a minimal submanifold if $M=0$.

We define the covariant derivatives of $h_{i\alpha j}$ by

$$Dh_{i\alpha j}=\sum_k h_{i\alpha jk}\theta_k=dh_{i\alpha j}+\sum_l (h_{l\alpha j}\theta_{li}+h_{i\alpha l}\theta_{lj})+\sum_\beta h_{i\beta j}\theta_{\beta\alpha}, \tag{5}$$

and the Laplacian of $h_{i\alpha j}$ is defined to be

$$\Delta h_{i\alpha j}=\sum_l h_{i\alpha jll}.$$

It is well known that when M^n is compact and without boundary, we have for M^n the integral formula

$$\int_{M^n}\sum_{i,\alpha,j} h_{i\alpha j}\Delta h_{i\alpha j}{}^*1=-\int_{M^n}\sum h^2_{i\alpha jk}{}^*1\leqslant 0, \tag{6}$$

where ^{*1} is the element of volume of M^n.

A Riemannian manifold V^{n+p} is said to be of quasi constant curvature when its Riemannian curvature tensor is expressible in the form [2]:

$$K_{ABCD}=a(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC})+b(\delta_{AC}\lambda_B\lambda_D+\delta_{BD}\lambda_A\lambda_C-\delta_{AD}\lambda_B\lambda_C-\delta_{BC}\lambda_A\lambda_D), \tag{7}$$

where a and b are arbitrary functions and λ_A is an arbitrary unit vector.

The purpose of the present paper is to establish the following theorem.

Theorem. *If M^n is any compact minimal submanifold without boundary in a Riemannian manifold V^{n+p} of quasi constant curvature, we have*

$$\int_{M^n}\left\{\left(2-\frac{1}{p}\right)\sigma^2-\left[na+\frac{1}{2}(b-|b|)(n+1)\right]\sigma+n(n-1)b^2\right\}{}^*1\geqslant 0. \tag{8}$$

2. We proceed to prove this theorem. From (2), (3) it follows that

$$d\theta_{i\alpha}=\sum_c \theta_{ic}\wedge\theta_{c\alpha}-\frac{1}{2}\sum_{k,l}K_{i\alpha kl}\theta_k\wedge\theta_l,$$

and from (1), (4) we have

$$d\theta_{i\alpha}=\sum_j dh_{i\alpha j}\wedge\theta_j+\sum_{k,j}h_{i\alpha j}\theta_k\wedge\theta_{kj}.$$

Hence from (5), we have

$$\sum_{k,j}\left(h_{i\alpha jk}+\frac{1}{2}K_{i\alpha kj}\right)\theta_k\wedge\theta_j=0,$$

that is,

$$h_{i\alpha jk}-h_{i\alpha kj}=K_{i\alpha jk}. \tag{9}$$

If we make use of the Ricci identity and apply the equation (9) repeatedly, we have

$$\begin{aligned}h_{i\alpha jkk}&=h_{i\alpha kjk}+K_{i\alpha jkk}=(h_{i\alpha kjk}-h_{i\alpha kkj})+h_{i\alpha kkj}+K_{i\alpha jkk}\\&=(h_{i\alpha kjk}-h_{i\alpha kkj})+h_{k\alpha ikj}+K_{i\alpha jkk}\\&=(h_{i\alpha kjk}-h_{i\alpha kkj})+h_{k\alpha kij}+K_{k\alpha ikj}+K_{i\alpha jkk}\\&=\sum_l h_{l\alpha k}R_{lijk}+\sum_\beta h_{i\beta k}R_{\beta\alpha jk}+\sum_l h_{i\alpha l}R_{lkjk}+h_{k\alpha kij}+K_{k\alpha ikj}+K_{i\alpha jkk},\end{aligned}$$

and consequently

$$\begin{aligned}h_{i\alpha j}\,\Delta h_{i\alpha j}=&\sum_{k,l}(h_{i\alpha j}h_{l\alpha k}R_{lijk}+h_{i\alpha j}h_{i\alpha l}R_{lkjk})\\&+\sum_{k,\beta}h_{i\alpha j}h_{i\beta k}R_{\beta\alpha jk}+\sum_k h_{i\alpha j}(h_{k\alpha kij}+K_{k\alpha ikj}+K_{i\alpha jkk}).\end{aligned} \tag{10}$$

On the other hand, we can prove readily the following equations:

$$d(\sum h_{i\alpha j}K_{i\alpha jk}{}^*\theta_k)=\sum(h_{i\alpha j}K_{i\alpha jkk}+h_{i\alpha jk}K_{i\alpha jk})\theta_1\wedge\theta_2\wedge\cdots\wedge\theta_n, \tag{11}$$

$$d(\sum h_{i\alpha j}h_{k\alpha ki}{}^*\theta_j)=\sum(h_{i\alpha j}h_{k\alpha kij}+h_{k\alpha ki}h_{i\alpha jj})\theta_1\wedge\theta_2\wedge\cdots\wedge\theta_n. \tag{12}$$

$$d(\sum h_{i\alpha j}K_{k\alpha ik}{}^*\theta_j)=\sum(h_{i\alpha j}K_{k\alpha ikj}+h_{i\alpha jj}K_{k\alpha ik})\theta_1\wedge\theta_2\wedge\cdots\wedge\theta_n. \tag{13}$$

Hence for any compact M^n without boundary we have by Stokes' theorem

$$\begin{aligned}&\int_{M^n}\sum h_{i\alpha j}(h_{k\alpha kij}+K_{k\alpha ikj}+K_{i\alpha jkk})^*1\\&=-\int_{M^n}\sum(h_{i\alpha jk}K_{i\alpha jk}+h_{k\alpha ki}h_{i\alpha jj}+h_{i\alpha jj}K_{k\alpha ik})^*1.\end{aligned}$$

By means of the equation (9) we get

$$\sum h_{i\alpha jk}K_{i\alpha jk}=\frac{1}{2}\sum h_{i\alpha jk}K_{i\alpha jk}-\frac{1}{2}\sum h_{i\alpha kj}K_{i\alpha jk}=\frac{1}{2}\sum K^2_{i\alpha jk},$$

$$\sum(h_{k\alpha ki}h_{i\alpha jj}+h_{i\alpha jj}K_{k\alpha ik})=\sum(\sum_j h_{j\alpha ij})^2.$$

From (10)—(13) and (6) it follows that

$$\int_{M^n}\left\{P-\frac{1}{2}\sum K^2_{i\alpha jk}+\sum h^2_{i\alpha jk}-\sum(\sum_l h_{l\alpha jl})^2\right\}^*1=0, \tag{14}$$

where

$$P\equiv\sum(h_{i\alpha j}h_{l\alpha k}R_{lijk}+h_{i\alpha j}h_{i\alpha l}R_{lkjk}+h_{i\alpha j}h_{i\beta k}R_{\beta\alpha jk}). \tag{15}$$

From (2), (4) we have on M^n

$$R_{iljk}=\sum_\alpha(h_{i\alpha j}h_{l\alpha k}-h_{i\alpha k}h_{l\alpha j})+K_{iljk}, \tag{16}$$

$$R_{\alpha\beta jk}=\sum_l(h_{l\alpha j}h_{l\beta k}-h_{l\alpha k}h_{l\beta j})+K_{\alpha\beta jk}. \tag{17}$$

From (16) it follows that

$$\sum h_{i\alpha j}h_{l\alpha k}R_{lijk}=-\frac{1}{2}\sum(h_{i\alpha j}h_{l\alpha k}-h_{i\alpha k}h_{l\alpha j})^2$$
$$-\frac{1}{2}\sum(h_{l\alpha j}h_{i\alpha k}-h_{l\alpha k}h_{i\alpha j})K_{lijk}. \tag{18}$$

Since for any minimal submanifold M^n we have $\sum\limits_{k,\beta}h_{k\beta k}=0$, from (16) we have

$$\sum h_{i\alpha j}h_{i\alpha l}R_{lkjk}=\sum h_{i\alpha j}h_{i\alpha l}K_{lkjk}-\sum_{j,l}(\sum_{i,\alpha}h_{i\alpha j}h_{i\alpha l})^2. \tag{19}$$

When V^{n+p} is a manifold of quasi constant curvature, we have from (7)

$$K_{\alpha\beta jk}=0.$$

Again from (17) it follows that

$$\sum h_{i\alpha j}h_{i\beta k}R_{\beta\alpha jk}=-\frac{1}{2}\sum(h_{i\alpha j}h_{i\beta k}-h_{i\alpha k}h_{i\beta j})^2. \tag{20}$$

Moreover, we have

$$\sum\{\sum_{\alpha}(h_{i\alpha j}h_{l\alpha k}-h_{i\alpha k}h_{l\alpha j})\}^2$$
$$=2\sum_{\alpha,\beta}(\sum_{i,j}h_{i\alpha j}h_{i\beta j})^2-2\sum h_{i\alpha j}h_{l\alpha k}h_{i\beta k}h_{l\beta j},$$

and

$$\sum\{\sum_{i}(h_{i\alpha j}h_{i\beta k}-h_{i\alpha k}h_{i\beta j})\}^2$$
$$=2\sum_{i,l}(\sum_{\alpha,j}h_{i\alpha j}h_{l\alpha j})^2-2\sum h_{i\alpha j}h_{l\alpha k}h_{i\beta k}h_{l\beta j}.$$

Combining the above two equations we have consequently

$$\sum\{\sum_{\alpha}(h_{i\alpha j}h_{l\alpha k}-h_{i\alpha k}h_{l\alpha j})\}^2$$
$$=\sum\{\sum_{i}(h_{i\alpha j}h_{i\beta k}-h_{i\alpha k}h_{i\beta j})\}^2+2\sum_{\alpha,\beta}(\sum_{i,j}h_{i\alpha j}h_{i\beta j})^2-2\sum(\sum_{\alpha,j}h_{i\alpha j}h_{l\alpha j})^2. \tag{21}$$

If we substitute (18)—(21) in (15), we have

$$-P=\sum\{\sum_{i}(h_{i\alpha j}h_{i\beta k}-h_{i\alpha k}h_{i\beta j})\}^2$$
$$+\sum_{\alpha,\beta}(\sum_{i,j}h_{i\alpha j}h_{i\beta j})^2-\sum h_{i\alpha j}h_{i\alpha l}K_{lkjk}$$
$$+\frac{1}{2}\sum(h_{l\alpha j}h_{i\alpha k}-h_{l\alpha k}h_{i\alpha j})K_{lijk}. \tag{22}$$

3. We put

$$\sigma_{\alpha\beta}\equiv\sum_{i,j}h_{i\alpha j}h_{i\beta j}, \tag{23}$$
$$\sigma\equiv\sum_{\alpha}\sigma_{\alpha\alpha}=\sum_{i,j,\alpha}h^2_{i\alpha j}, \tag{24}$$
$$H_\alpha\equiv(h_{i\alpha j}), \tag{25}$$

where H_α indicates a matrix. For any matrix M we denote by $N(M)$ the sum of the square of all its elements, thus

$$N(H_\alpha)\equiv\sum_{i,j}h^2_{i\alpha i}=\sigma_{\alpha\alpha}. \tag{26}$$

Lemma.

$$\sum_{\alpha,\beta}N(H_\alpha H_\beta-H_\beta H_\alpha)+\sum_{\alpha,\beta}N(\sigma_{\alpha\beta})\leqslant\left(2-\frac{1}{p}\right)\sigma^2. \tag{27}$$

In proving this Lemma, we can suppose H_α to be diagonal. We have in this case

$$N(H_\alpha H_\beta - H_\beta H_\alpha) = \sum_{j\neq k}\{\sum_i (h_{j\alpha i}h_{i\beta k} - h_{k\alpha i}h_{i\beta j})\}^2$$

$$= \sum_{j\neq k}(h_{j\alpha j}h_{j\beta k} - h_{j\beta k}h_{k\alpha k})^2 = \sum_{j\neq k} h^2_{j\beta k}(h_{j\alpha j} - h_{k\alpha k})^2$$

$$\leqslant 2\sum_{j\neq k} h^2_{j\beta k}(h^2_{j\alpha j} + h^2_{k\alpha k}) \leqslant 2N(H_\alpha)N(H_\beta)$$

$$= 2\sigma_{\alpha\alpha}\sigma_{\beta\beta}.$$

Since $N(H_\alpha)$ is an invariant under any orthogonal transformations in TM^n, we have for any symmetric matrix H_α (see [3])

$$N(H_\alpha H_\beta - H_\beta H_\alpha) \leqslant 2N(H_\alpha)N(H_\beta). \tag{28}$$

By means of certain orthogonal transformations in the normal space $(TM^n)^\perp$, we can suppose the matrix $(\sigma_{\alpha\beta})$ to be diagonal. Then we have

$$\sum_{\alpha,\beta}\{N(H_\alpha H_\beta - H_\beta H_\alpha) + N(\sigma_{\alpha\beta})\}$$

$$\leqslant 2\sum_{\alpha\neq\beta}\sigma'_{\alpha\alpha}\sigma'_{\beta\beta} + \sum_\alpha \sigma'^2_{\alpha\alpha} = 2\sum_{\alpha,\beta}\sigma'_{\alpha\alpha}\sigma'_{\beta\beta} - 2\sum_\alpha \sigma'^2_{\alpha\alpha} + \sum_\alpha \sigma'^2_{\alpha\alpha}$$

$$= 2(\sum_\alpha \sigma'_{\alpha\alpha})^2 - \sum_\alpha \sigma'^2_{\alpha\alpha} = 2\sigma'^2 - \sum_\alpha \sigma'^2_{\alpha\alpha}. \tag{29}$$

Since σ is an invariant under these orthogonal transformations, we have $\sigma' = \sigma$, and

$$\sigma^2 = (\sum_\alpha \sigma_{\alpha\alpha})^2 = \sum_\alpha \sigma^2_{\alpha\alpha} + 2\sum_{\alpha<\beta}\sigma_{\alpha\alpha}\sigma_{\beta\beta}.$$

On the other hand, we have

$$0 \leqslant \sum_{\alpha<\beta}(\sigma_{\alpha\alpha} - \sigma_{\beta\beta})^2 = (p-1)\sum_\alpha \sigma^2_{\alpha\alpha} - 2\sum_{\alpha<\beta}\sigma_{\alpha\alpha}\sigma_{\beta\beta}.$$

Hence

$$\sum_\alpha \sigma^2_{\alpha\alpha} \geqslant \frac{1}{p}\sigma^2. \tag{30}$$

From (30) it follows that the right hand member of (29)

$$\leqslant 2\sigma'^2 - \frac{1}{p}\sigma'^2 = \left(2 - \frac{1}{p}\right)\sigma^2.$$

From (22) and (27) it follows that any minimal submanifold M^n in a V^{n+p} of quasi constant curvature satisfies the following inequality

$$-P \leqslant \left(2 - \frac{1}{p}\right)\sigma^2 + \frac{1}{2}\sum(h_{la j}h_{i\alpha k} - h_{l\alpha k}h_{i\alpha j})K_{lijk} - \sum h_{i\alpha j}h_{i\alpha l}K_{lkjk}. \tag{31}$$

From (7) we have

$$K_{lijk} = a(\delta_{lj}\delta_{ik} - \delta_{lk}\delta_{ij}) + b(\delta_{lj}\lambda_i\lambda_k + \delta_{ik}\lambda_l\lambda_j - \delta_{lk}\lambda_i\lambda_j - \delta_{ij}\lambda_l\lambda_k). \tag{32}$$

When we put $i = k$ in (32) and sum for k, we have

$$\sum_k K_{lkjk} = [(n-1)a + b]\delta_{lj} + b(n-2)\lambda_l\lambda_j. \tag{33}$$

Substituting (32), (33) in (31) we have

$$-P \leqslant \left(2 - \frac{1}{p}\right)\sigma^2 - [na+b]\sigma - bn\sum(\sum_l \lambda_l h_{i\alpha l})^2. \tag{34}$$

Also from (7), we have

$$K_{l\alpha jk} = b\lambda_\alpha(\delta_{lj}\lambda_k - \delta_{lk}\lambda_j),$$

and consequently

$$\sum K^2_{l\alpha jk}=2(n-1)b^2\sum\lambda^2_\alpha. \tag{35}$$

From (9)

$$h_{l\alpha jl}-h_{l\alpha lj}=K_{l\alpha jl},$$

we have for any minimal M^n

$$\sum\Big(\sum_l h_{l\alpha jl}\Big)^2=\sum\Big(\sum_l K_{l\alpha jl}\Big)^2=(n-1)^2b^2\sum\lambda^2_\alpha. \tag{36}$$

If we substitute (34), (35), (36) in (14), we get

$$\int_{M^n}\Big\{\Big(2-\frac{1}{p}\Big)\sigma^2-[na+b]\sigma-nb\sum_{i,\alpha}\Big(\sum_l\lambda_l h_{i\alpha l}\Big)^2+n(n-1)b^2\sum\lambda^2_\alpha\Big\}^*1\geqslant 0. \tag{37}$$

If we observe that

$$\sum\lambda^2_\alpha\leqslant 1,\ \sigma\geqslant 0,$$

we have when $b\geqslant 0$,

$$\int_{M^n}\Big\{\Big(2-\frac{1}{p}\Big)\sigma^2-na\sigma+n(n-1)b^2\Big\}^*1\geqslant 0; \tag{38}$$

and when $b\leqslant 0$, since

$$\sum_{i,\alpha}\Big(\sum_l\lambda_l h_{i\alpha l}\Big)^2\leqslant\sum_{i,\alpha}\Big(\sum_l\lambda^2_l\sum_k h^2_{i\alpha k}\Big)\leqslant\sigma,$$

we have

$$\int_{M^n}\Big\{\Big(2-\frac{1}{p}\Big)\sigma^2-na\sigma-(n+1)b\sigma+n(n-1)b^2\Big\}^*1\geqslant 0. \tag{39}$$

Combining equations (38) and (39) we obtain the required formula (8).

When V^{n+p} is a manifold of constant curvature, $b=0$, and (8) reduces to the inequality of Simons.

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