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THE HILBERT BOUNDARY PROBLEM OF DOUBLY PERIODIC ANALYTIC FUNCTIONS*

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Abstract

The doubly periodic Hilbert boundary value problem is discussed in this paper. First, certain kind of integral representations of doubly quasi-periodic analytic functions in multiplication is established so that the Dirichlet problem of such functions is solved. Then, by the method of regularization, the Hilbert boundary value problem is transferred to such a problem, and it is reduced at length to some Fredholm integral equation. The number of solutions and conditions of solvability as well as the form of the general solution are obtained.

We studied the Riemann boundary value problem of doubly periodic analytic functions in [1, 2] and that of doubly quasi-periodic analytic functions in [3]. The corresponding Hilbert problem has not been investigated yet. The special case for the Dirichlet problem of analytic functions, doubly periodic or doubly quasi-periodic in addition, was discussed in [4, 5].

In this paper, the Hilbert boundary value problem of doubly periodic analytic functions (DH problem) is discussed. We first give some integral representation for doubly quasi-periodic analytic functions in multiplication (MQ-function) so as to solve the Dirichlet problem of such functions. Then, by using the method of regularization, the DH problem is solved by reducing it to this solved problem.

§ 1. Definitions and Notations

We shall recall some notations used in [2]. Let the periods be $2\omega_1$, $2\omega_2$ with $\mathrm{Im}\ (\omega_2/\omega_1)>0$ and S_0 be the fundamental cell (the parallelogram with vertices $\pm\omega_1\pm\omega_2$). L_0 is a Liapunov contour in S_0 with usual positive sense, the interior region bounded by which is denoted by S_0^+. We always assume $0\in S_0^+$. Denote $S_0^-=S_0-\overline{S}_0^+$. The union of all the contours congruent to L_0 (mod $2\omega_j$) is denoted by L, the exterior region bounded by which is denoted by S^-.

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Let $w=\omega(z)$ be the function conformally mapping S_0^+ to $|w|<1$ with $\omega(0)=0$, $\omega'(0)>0$.

We shall use the Weierstrass' ζ-function $\zeta(z)$ and σ-function $\sigma(z)$ (cf. [6]). It is well known that $\zeta(z)$ has the single simple pole $z=0$ in S_0 with $1/z$ as its principal part and the property

$$\zeta(z+2\omega_j)=\zeta(z)+2\eta_j, \quad j=1, 2, \tag{1.1}$$

where $\eta_j=\zeta(\omega_j)$, satisfying

$$2\eta_1\omega_2-2\eta_2\omega_1=\pi i. \tag{1.2}$$

$\sigma(z)$ is an entire function which has a unique simple zero at $z=0$ in S_0 with $\sigma'(0)=1$ and has the property

$$\sigma(z+2\omega_j)=-e^{2\eta_j(z+\omega_j)}\sigma(z), \quad j=1, 2. \tag{1.3}$$

A function $\Phi^-(z)$ defined in S^- is called an MQ-function if it is analytic in S^- with the property of double quasi-periodicity

$$\Phi^-(z+2\omega_j)=b_j\Phi^-(z), \quad j=1, 2, \tag{1.4}$$

where b_1, b_2 are two non-zero constants, called the multipliers.

Define two constants λ and z_0 by the following system of linear equations

$$2\omega_j\lambda-2\eta_jz_0=\log b_j, \quad j=1, 2, \tag{1.5}$$

for certain fixed values of $\log b_1$ and $\log b_2$. It is uniquely solvable on account of (1.2). For definiteness, we assume $z_0\in S_0$, which is always possible for suitably chosen $\log b_j$. Note that λ and z_0 are complex in general even if b_1, b_2 are real. It is evident that $e^{\lambda z}$ is an entire MQ-function with multipliers b_1, b_2 if $z_0=0$ (Case I) and

$$q(z)=e^{\lambda z}\frac{\sigma(z-z_0)}{\sigma(z)} \tag{1.6}$$

is a meromorphic MQ-function with multipliers b_1, b_2 if $z\neq 0$ (Case II). $q(z)$ has $z=0$ as its unique simple pole and $z=z_0$ as its unique simple zero in S_0 and hence is holomorphic in S^-.

Case II appears iff

$$\omega_2\log b_1=\omega_1\log b_2 \tag{1.7}$$

is valid for suitably chosen $\log b_1$, $\log b_2$.

§ 2. Integral Representation for MQ-Functions

We gave an integral representation for doubly periodic analytic functions in S^- (cf. [4]), from which we may easily obtain one for MQ-functions by multiplying a factor $e^{\lambda z}$ (Case I) or $q(z)$ (Case II). However, such a representation is inconvenient for our purpose. Here we shall give another kind of representation for MQ-functions in S^-.

Given an MQ-function $\Phi^-(z)=u(z)+iv(z)$ in S^- with multipliers b_1, b_2.

Constants λ and $z_0 \in S_0$ are defined as in § 1. Assume $\Phi^-(t) = u(t) + iv(t) \in H$ (Hölder condition) on L.

Case I: $z_0 = 0$. We shall prove the following lemma.

Lemma 1. *If (1.7) is fulfiled for certain* $\log b_1$, $\log b_2$, *then* $\Phi^-(z)$ *may be represented as*

$$\Phi^-(z) = e^{\lambda z}\left\{\frac{1}{2\pi i}\int_{L_0} \mu(t) e^{-\lambda t}[\zeta(t-z) + \zeta(z)]\,dt + A\right\},\ z \in S^-, \tag{2.1}$$

where $\mu(t) \in H$ *is a real function on* L_0, *uniquely determined up to a term* $\beta_0 + \mathrm{Re}\{C\omega(t)\}$ (β_0 *and* C *being respectivly arbitrary real and complex constants*), *while* A *is a complex constant uniquely determined by* $\Phi^-(z)$.

Proof Suppose that there exist $\mu(t)$ on L_0 and constant A such that (2.1) is valid. Denote the function defined by its right-hand member when $z \in S_0^+$ by $\Phi^+(z)$, which has in general a simple pole at $z=0$. Then, by Plemelj's formulas, we have

$$\Phi^\pm(t_0) = \pm\frac{1}{2}\mu(t_0) + \frac{1}{2\pi i}\int_{L_0} \mu(t) e^{-\lambda(t-t_0)}[\zeta(t-t_0) + \zeta(t_0)]\,dt + Ae^{\lambda t_0}, \quad t_0 \in L_0. \tag{2.2}$$

By the same reasoning as in [4], we have

$$\Phi^+(t) = i(Sv)(t) + \beta_0 + \mathrm{Re}\{C\omega(t)\}, \tag{2.3}$$

where S is the Schwarz operator of S_0^+ (cf. [7]): $(Sv)(z)$ is holomorphic in S_0^+ with the properties

$$\mathrm{Re}\{(Sv)(t)\} = v(t),\ (Sv)^+(t) = (Sv)(t), \quad t \in L_0,$$

and β_0, C are constants as described in the lemma. Moreover, we may set

$$\mu(t) = \mu_0(t) + \beta_0 + \mathrm{Re}\{C\omega(t)\}, \tag{2.4}$$

where

$$\mu_0(t) = i(Sv)(t) - \Phi^-(t) = -\mathrm{Im}\{(Sv)(t)\} - u(t) \tag{2.5}$$

is uniquely determined by $\Phi^-(t)$. Thus, if the representation exists, then $\mu(t)$ must be of the form (2.4). It is easy to verify the term $\beta_0 + \mathrm{Re}\{C\omega(t)\}$ does not effect the value of the integral appeared.

Put

$$\Psi^-(z) = \frac{e^{\lambda z}}{2\pi i}\int_{L_0} \mu_0(t) e^{-\lambda t}[\zeta(t-z) + \zeta(z)]\,dt, \quad z \in S^-.$$

Substituting (2.5) into it, we readily see that

$$\begin{aligned}\Psi^-(z) &= -\frac{e^{\lambda z}}{2\pi i}\int_{L_0} \Phi^-(t) e^{-\lambda t}\zeta(t-z)\,dt \\ &= \Phi^-(z) - \frac{e^{\lambda z}}{2\pi i}\int_\Gamma \Phi^-(\tau) e^{-\lambda\tau}\zeta(\tau-z)\,d\tau, \quad z \in S_0^-,\end{aligned}$$

where Γ is the boundary of S_0 with usual positive sense. We shall show that

$$A = e^{-\lambda z}[\Phi^-(z) - \Psi^-(z)] = \frac{1}{2\pi i}\int_\Gamma \Phi^-(\tau) e^{-\lambda\tau}\zeta(\tau-z)\,d\tau, \quad z \in S_0^-,$$

is actually a constant. In fact, by using the doubly periodic property of $e^{-\lambda\tau}\Phi^-(\tau)$, we may easily obtain

$$A=\frac{\eta_1}{\pi i}\int_{\gamma_2} e^{-\lambda\tau}\Phi^-(\tau)\,d\tau-\frac{\eta_2}{\pi i}\int_{\gamma_1} e^{-\lambda\tau}\Phi^-(\tau)\,d\tau, \tag{2.6}$$

where γ_1 and γ_2 are the directed line-segments from $-\omega_1-\omega_2$ to $\omega_1-\omega_2$ and from $\omega_1-\omega_2$ to $\omega_1+\omega_2$ respectively. The lemma is proved.

Case II: $z_0\neq 0$. In this case, we have the following lemma.

Lemma 2. *If* (1.7) *is not fulfiled for any values of* $\log b_1$, $\log b_2$ *whatever, then* $\Phi^-(z)$ *may be represented as*

$$\Phi^-(z)=\frac{e^{\lambda z}}{\sigma(z_0)}\frac{1}{2\pi i}\int_{L_0}\mu(t)e^{-\lambda t}\frac{\sigma(t-z+z_0)}{\sigma(t-z)}\,dt,\quad z\in S^-, \tag{2.7}$$

where $\mu(t)\in H$ *is a real function uniquely determined by* $\Phi^-(z)$ *up to a real constant term* β_0.

Proof Suppose (2.7) is valid for some real $\mu(t)\in H$. Again define $\Phi^+(z)$ by its right-hand member when $z\in S_0^+$. Then we have

$$\Phi^\pm(t_0)=\pm\frac{1}{2}\mu(t_0)+\frac{e^{\lambda t_0}}{\sigma(t_0)}\frac{1}{2\pi i}\int_{L_0}\mu(t)e^{-\lambda t}\frac{\sigma(t-t_0+z_0)}{\sigma(t-t_0)}\,dt,\quad t_0\in L_0. \tag{2.8}$$

Using the same reasoning as above but noting that $\Phi^+(z)$ is now holomorphic in S_0^+, we must have, instead of (2.4),

$$\mu(t)=\mu_0(t)+\beta_0, \tag{2.9}$$

where β_0 is again a real constant and $\mu_0(t)$ is still given by (2.5). The term β_0 does not effect the value of the integral appeared in (2.7).

We have to prove

$$\Phi^-(z)=\frac{e^{\lambda z}}{\sigma(z_0)}\frac{1}{2\pi i}\int_{L_0}\mu_0(t)e^{-\lambda t}\frac{\sigma(t-z+z_0)}{\sigma(t-z)}\,dt,\quad z\in S^-. \tag{2.10}$$

Substituting (2.5) into it, we see that it is equivalent to

$$\Phi^-(z)=-\frac{e^{\lambda z}}{\sigma(z_0)}\frac{1}{2\pi i}\int_{L_0}\Phi^-(t)e^{-\lambda t}\frac{\sigma(t-z+z_0)}{\sigma(t-z)}\,dt,\quad z\in S^-.$$

When we arbitrarily fix $z\in S_0^-$, the integrand as a function of t is doubly periodic and analytic in S^- with the single simple pole $t=z$ in S_0^-. Therefore its integral taken along Γ must be equal to zero. So, by using the residue theorem in S_0^-, the above equality is valid for $z\in S_0^-$ and hence for $z\in S^-$, i. e., (2.10) is valid. The lemma is proved.

§ 3. The Dirichlet Problem of *MQ*-Functions

Let us now consider the following Dirichlet problem of MQ-functions. Given a real function $f(t)\in H$ on L_0 and two non-zero real constants β_1, β_2, we need to find an MQ-function $\Phi^-(z)$ in S^- with multipliers $b_1=\beta_1$, $b_2=\beta_2$, satisfying the boundary condition

$$\operatorname{Re}\{\Phi^-(t)\}=f(t),\quad t\in L_0. \tag{3.1}$$

For the case $f(t)=0$, the problem was solved in [8] by the following lemma.

Lemma 3. *The Dirichlet problem of MQ-functions, satisfying*

$$\operatorname{Re}\{\Phi^-(t)\}=0,\quad t\in L_0, \tag{3.1}$$

with given real multipliers β_1, β_2, has unique non-trivial solution $\Phi^-(z)$ (up to a real constant coefficient) if $\log\beta_1$, $\log\beta_2$ satisfy two real conditions provided their values are suitably chosen, and otherwise $\Phi^-(z)=0$ in S^-.

The mentioned two conditions were given by (3.4) in [8].

For the general case (3.1), two different cases are divided.

Case I: $z_0=0$. If the problem (3.1) has a solution $\Phi^-(z)$, then it may be represented as (2.1). Then, by (2.2), we have (in the sequel, $\mu(t)$ is always replaced by $2\mu(t)$),

$$-\mu(t_0)+\operatorname{Re}\left\{\frac{1}{\pi i}\int_{L_0}\mu(t)e^{-\lambda(t-t_0)}[\zeta(t-t_0)+\zeta(t_0)]dt\right\}+\operatorname{Re}\{Ae^{\lambda t_0}\}$$
$$=f(t_0),\quad t_0\in L_0. \tag{3.2}$$

It is easy to see

$$k(t_0,t)=e^{-\lambda(t-t_0)}[\zeta(t-t_0)+\zeta(t_0)]-\frac{1}{t-t_0}\in H. \tag{3.3}$$

Thereby (3.2) may be written as

$$K_1\mu\equiv\mu(t_0)-\frac{1}{\pi}\int_{L_0}\mu(t)\frac{\cos(r,n)}{r}ds+\int_{L_0}k_1(t_0,t)\mu(t)ds$$
$$=-f(t_0)+\operatorname{Re}\{Ae^{\lambda t_0}\},\quad t_0\in L_0, \tag{3.4}$$

where $k_1(t_0,t)=\operatorname{Re}\left\{\frac{1}{\pi i}k(t_0,t)\frac{dt}{ds}\right\}\in H$, and

$$\frac{1}{\pi}\int_{L_0}\mu(t)\frac{\cos(r,n)}{r}ds=\operatorname{Re}\left\{\frac{1}{\pi i}\int_{L_0}\frac{\mu(t)}{t-t_0}dt\right\}, \tag{3.5}$$

where $r=|t-t_0|$, (r,n) is the angle between the vector $t-t_0$ and the internal normal n of L_0 at t (Cf. [9], § 61).

(3.4) is a Fredholm equation. To consider its solvability, two subcases should be considered.

1) $(3.1)_0$ has only the trivial solution $\Phi^-(z)=0$. Hence $A=0$ by (2.6). This means that $K_1\mu=0$ has $\beta_0+\operatorname{Re}\{C\omega(t)\}$ as its general solution by Lemma 1.

Thus, $K_1\mu=0$ has exactly three linearly independent (real) solutions

$$1,\ \operatorname{Re}\{\omega(t)\},\ \operatorname{Im}\{\omega(t)\}, \tag{3.6}$$

and so its adjoint equation $K_1'\nu=0$ also has three such solutions $\nu_j(t)$ $(j=1,2,3)$ exactly and the equation (3.4) is solvable iff the following conditions are satisfied:

$$\operatorname{Re}\left\{A\int_{L_0}e^{\lambda t}\nu_j(t)ds\right\}=f_j\left(=\int_{L_0}f(t)\nu_j(t)ds\right),\quad j=1,2,3. \tag{3.7}$$

In this subcase, we also see that (3.4) is unsolvable if $f(t)=0$, $A=1$. Denote

$$\int_{L_j} e^{\lambda t}\nu_j(t)\,ds = I_j + iJ_j, \quad j=1, 2, 3. \tag{3.8}$$

Therefore, I_j, $j=1, 2, 3$, could not be all zero, say, $I_3 \neq 0$. Then we may regularizer $\nu_j(t)$ such that

$$I_1 = I_2 = 0,\ I_3 = 1.$$

Denote $A = \alpha + i\beta$, then (3.6) becomes

$$\beta J_j = -f_j,\ j=1, 2;\ \alpha = f_3 + \beta J_3. \tag{3.9}$$

J_1 and J_2 could not be both zero, since otherwise β may be arbitrary in case $f(t)=0$ so that $f_j=0$ $(j=1, 2, 3)$ and hence $A=\alpha+i\beta\neq 0$, which is a contradiction. Thus, (3.9) means a single condition of solvability and α, β are uniquely determined.

It is convenient to define the generalized degree of (real) freedom of a nonhomogeneous linear problem as the difference $\tau = l - m$ between the number l of the arbitrary (real) constants in its general solution and the number m of the (real) conditions of solvability. Hence, for the problem discussed here, in this subcase, $\tau = -1$ $(l=0, m=1)$.

2) $(3.1)_0$ has a unique non-trivial solution $\Phi_0^-(z)$. Let the constant attached to $\Phi_0^-(z)$ be $A = A_0$.

(i) $A_0 \neq 0$. Then the equation $K_1\mu = \operatorname{Re}\{Ae^{\lambda t}\}$ is solvable iff $A = A_0$ $(\neq 0)$ and so $K_1\mu = 0$ again has exactly three linearly independent solutions (3.6). We may regularize $\nu_j(t)$ as above. The conditions of solvability for (3.4) is still (3.9) for $A=\alpha+i\beta$. Here we must have $J_1 = J_2 = 0$, for, if on the contrary, $(3.1)_0$ would have a solution $\Phi_0^-(z)$ with $A_0 = 0$. Hence, (3.9) reduces to two conditions of solvability $f_1 = f_2 = 0$, β may be arbitrary and α is uniquely determined by β:

$$A = f_3 + \beta(J_3 + i).$$

The general solution of (3.1) is then

$$\Phi^-(z) = \beta\Phi_0^-(z) + \Phi_1^-(z),$$

where β is an arbitrary real constant and $\Phi_1^-(z)$ is a particular solution corresponding to the solution $\mu = \mu_1(t)$ of $K_1\mu = f(t) + \operatorname{Re}\{f_3 e^{\lambda t}\}$.

Thus, in this case, $\tau = -1$ $(l=1, m=2)$.

(ii) $A_0 = 0$. In this case, besides (3.6), there exists another linearly independent solution $\mu_0(t)$ of $K_1\mu = 0$, which gives out a solution $\Phi_0^-(z) \neq 0$ of (3.1). Then $K_1'\nu = 0$ has four linearly independent solutions $\nu_j(t)$ $(j=0, 1, 2, 3)$. Again define I_j, J_j as in (3.8) but with $j=0$ added. The conditions of solvability of (3.4) are still given by (3.7) but with $j=0, 1, 2, 3$. We show that now I_j, $j=0, 1, 2, 3$, could not be all zero. When $f(t)=0$, (3.7) becomes

$$\alpha I_j - \beta J_j = 0, \quad j=0, 1, 2, 3, \tag{3.10}$$

and $K_1\mu = \operatorname{Re}\{Ae^{\lambda t}\}$ is solvable iff $A=0$, i. e., $\alpha=\beta=0$. However, if $I_j=0$, $j=0$ 1, 2, 3, then (3.10) would have solutions $\beta=0$ with α arbitrary, which is a contradiction.

Thus, we may regularize $\nu_j(t)$ as before such that

$$I_j=0,\ j=0,\ 1,\ 2;\ I_3=1,$$

and (3.7) becomes

$$J_j=-f_j,\ j=0,\ 1,\ 2;\ \alpha_3=f_3+\beta J_3. \tag{3.11}$$

In a similar manner, we also know that J_j, $j=0$, 1, 2, could not be all zero, then (3.11) reduces to two conditions of solvability and $A=\alpha+i\beta$ are uniquely determined. When they are satisfied, we may get the general solution of (3.1)

$$\Phi^-(z)=D\Phi_0^-(z)+\Phi_1^-(z), \tag{3.12}$$

where D is an arbitrary real constant and $\Phi_1^-(z)$ is a particular solution of (3.1) corresponding to a particular solution $\mu_1(t)$ of (3.4).

In this case, we still have $r=-1$ ($l=1$, $m=2$).

Case II: $z_0\neq 0$. If the problem is solvable, by using (2.7) and (2.8) ($\mu(t)$ is again replaced by $2\mu(t)$), we obtain

$$-\mu(t_0)+\mathrm{Re}\left\{\frac{e^{\lambda t_0}}{\sigma(z_0)}\frac{1}{\pi i}\int_{L_0}\mu(t)e^{-\lambda t}\frac{\sigma(t-t_0+z_0)}{\sigma(t-t_0)}dt\right\}=f(t_0),\quad t_0\in L_0. \tag{3.13}$$

Note that we may easily prove

$$\frac{e^{-\lambda(t-t_0)}}{\sigma(z_0)}\frac{\sigma(t-t_0+z_0)}{\sigma(t-t_0)}-\frac{1}{t-t_0}\in H.$$

Hence, as above, (3.13) is a Fredholm equation of the form

$$\begin{aligned}K_2\mu\equiv\mu(t_0)-\frac{1}{\pi}\int_{L_0}\mu(t)\frac{\cos(r,\ n)}{r}ds+\int_{L_0}k_2(t_0,\ t)\mu(t)ds\\ =-f(t_0),\quad t_0\in L_0,\end{aligned} \tag{3.14}$$

where $k_2(t_0,\ t)\in H$.

On the analogy of the previous case, we know that $K_2\mu=0$ either has only one linearly independent solution 1 or has another such solution $\mu_0(t)$ according as $(3.1)_0$ has only trivial solution or a nontrivial solution $\Phi_0^-(z)$.

In the first case, $K_2'\nu=0$ has also only one solution $\nu(t)$ and (3.14) is solvable iff

$$\int_{L_0}f(t)\nu(t)ds=0. \tag{3.15}$$

If it is satisfied, then $\Phi^-(z)$ is uniquely determined in spite of $\mu(t)$ is determined up to a real constant term β_0. The generalized degree of freedom of the solutions of the problem in this case is also $r=-1$ ($l=0$, $m=1$).

If $K_2\mu=0$ has two solutions 1 and $\mu_0(t)$, then $K_2'\nu=0$ also has two solutions $\nu_1(t)$, $\nu_2(t)$. Then (3.14) is solvable iff

$$f_j=\int_{L_0}f(t)\nu_j(t)ds=0,\quad j=1,\ 2. \tag{3.16}$$

If they are satisfied, then (3.14) has a particular solution $\mu_1(t)$ which corresponds to a solution $\Phi_1(z)$ of (3.1). Its general solution is again given in the form of (3.12). In this case, $r=-1$ ($l=1$, $m=2$).

Hence we obtain the following theorem.

Theorem 1. *The generalized degree of freedom of solutions for the Dirichlet problem of MQ-functions is* -1 *with at most one arbitrary real constant in its general solution.*

§ 4. Method of MQ-Regularization

Before we study the DH problem, we shall solve the problem of MQ-regularization. That is, given $\gamma(t)=a(t)+ib(t)\in H$, $\neq 0$ on L_0, with the index

$$\varkappa=\frac{1}{2\pi}[\arg\gamma(t)]_{L_0}, \tag{4.1}$$

we need to find a real function $p(t)\in H$ on L_0 such that

$$\psi^-(t)=p(t)\gamma(t), \quad t\in L_0, \tag{4.2}$$

is the boundary value of an MQ-function $\psi^-(z)$ in S^- with certain real multipliers β_1, β_2. But now we allow $\psi^-(z)$ may have some poles in S^-. $p(t)$ is called the factor of MQ-regularization of $\gamma(t)$.

Without loss of generality, we may assume $|\gamma(t)|=1$ and write $\gamma(t)=e^{i\theta(t)}$, where $\theta(t)$ is multi-valued unless $\varkappa=\frac{1}{2\pi}[\theta(t)]_{L_0}=0$.

We discuss the following cases for different values of $\varkappa$:

1° $\varkappa=0$. $\theta(t)\in H$ is single-valued in this case. Let us solve the Dirichlet problem of doubly quasi-periodic holomorphic function $\Omega^-(z)$ in addition in S^- satisfying

$$\operatorname{Re}\{-i\Omega^-(t)\}=\theta(t), \quad t\in L_0, \tag{4.3}$$

with real addenda α_j:

$$\Omega^-(z+2\omega_j)=\Omega^-(z)+\alpha_j, \quad j=1, 2. \tag{4.4}$$

It is known from [5] that this problem has a unique solution $-i\Omega^-(z)=U(z)+iV(z)$, where $U(t)=\theta(t)$. Put $\psi^-(z)=e^{\Omega^-(z)}$. Then $\psi^-(z)$ is a holomorphic MQ-function in S^- with real multipliers $\beta_j=e^{\alpha_j}$ $(j=1, 2)$ and $\psi^-(t)=e^{-V(t)+i\theta(t)}$ on L_0. The factor of MQ-regularization is $p(t)=e^{-V(t)}\in H$. Here, α_j and hence β_j are uniquely determined.

We note that $\psi^-(z)$ in this case has neither zeros nor poles in S^-.

2° $\varkappa\geqslant 2$. Let

$$h_\varkappa(z)=\Pi(z)/\sigma^\varkappa(z), \ \Pi(z)=\prod_{j=1}^{\varkappa}\sigma(z-z_j), \tag{4.5}$$

where $z_1, \cdots, z_\varkappa\in S_0^-$ are fixed and arbitrary but $\sum_{k=1}^{\varkappa}z_k=0$. $h_\varkappa(z)$ is an elliptic function of order $\varkappa$ with simple zeros $z_1, \cdots, z_\varkappa$ and without poles in S_0^-. Its index on L_0 is $-\varkappa$. Let $\delta(t)=\arg h_\varkappa(t)$. Then

$$\theta_0(t)=\theta(t)+\delta(t) \tag{4.6}$$

is single-valued on L_0. Constructing the factor of MQ-regularization $p_0(t)$ of $e^{i\theta_0(t)}$ as in 1°, we obtain a holomorphic MQ-function $\psi_0^-(z)$ in S^- with real multipliers β_1, β_2, satisfying

$$\psi_0^-(t)=p_0(t)e^{i\theta_0(t)}, \quad t\in L_0. \tag{4.7}$$

Then

$$\psi^-(z)=\psi_0^-(z)/h_\varkappa(z) \tag{4.8}$$

is a meromorphic MQ-function with the same multipliers and

$$\psi^-(t)=\frac{p_0(t)e^{i\theta_0(t)}}{h_\varkappa(t)}=\frac{p_0(t)}{|h_\varkappa(t)|}e^{i\theta(t)}.$$

Hence $p(t)=p_0(t)/h_\varkappa(t)$ is a factor of MQ-regularization of $\gamma(t)$.

We note that, in this case, $\psi^-(z)\neq 0$ has exactly $\varkappa$ simple poles $z_1, \cdots, z_\varkappa$ in S_0^-.

3° $\varkappa=1$. In place of $h_\varkappa(t)$ given by (4.5), we take

$$h_1(z)=\frac{\sigma(z-c_1)\sigma(z-z_1)}{\sigma(z)\sigma(z-c_2)}, \tag{4.9}$$

where c_1, $c_2\in S_0^+$ and $z_1=c_2-c_1\in S_0^-$ are fixed and arbitrarily chosen. The remaining discussions are the same as in 2°. Now $\psi^-(z)\neq 0$ has a single simple pole z_1 in S_0^-.

4° $\varkappa\leqslant -2$. In this case, we may take $h_\varkappa(z)=1/h_{-\varkappa}(z)$, where $h_{-\varkappa}(z)$ is defined by (4.5). Then we obtain an MQ-function $\psi^-(z)$, holomorphic in S^-, with simple zeros $z_1, \cdots, z_{-\varkappa}$ in S_0^-.

5° $\varkappa=-1$. Taking $h_1(z)=1/h_{-1}(z)$ where $h_1(z)$ is given by (4.9), we get an MQ-function $\psi^-(z)$ holomorphic in S^- with a simple zero z_1 in S_0^-.

Thus, we obtain the following theorem.

Theorem 2. *For given $\gamma(t)\in H$, $\neq 0$ on L_0 with index $\varkappa$, a factor of* MQ-*regularization $p(t)$ with real multipliers in S^- always exists. The result* MQ-*function $\psi^-(z)$ has neither zeros nor poles in S_0^- if $\varkappa=0$, has $\varkappa$ simple poles and no zeros if $\varkappa>0$, has-$\varkappa$ simple zeros and no poles if $\varkappa<0$.*

Remark. We did not attempt to obtain the general solution of the problem of MQ-regularization which is of no use for our purpose. However, if the locations of the possibly appeared zeros and poles of $\psi^-(z)$ in S_0^- are prescribed, it is not difficult to get it.

§ 5. The DH Problem

It is easy now to solve the DH problem on the basis of the foregoing discussions. Given $\gamma(t)$ as in § 4, we need to find a doubly periodic holomorphic function $\Psi^-(z)$ in S^- such that

$$\mathrm{Re}\{\gamma(t)\Psi^-(t)\}=f(t), \quad t\in L_0, \tag{5.1}$$

where $f(t)\in H$ is a given function on L_0. As usual, we call (4.1) the index of the problem (5.1).

According to the results in § 4, we may construct a factor of MQ-regularization $p(t)$ for $\overline{\gamma(t)}$ such that $\psi^-(t)=p(t)\overline{\gamma(t)}$ is the boundary value of an MQ-function $\psi^-(z)$ in S^-, having poles or zeros according as $\varkappa>0$ or $\varkappa<0$ as described in Theorem 2, where $\mathrm{Ind}_{L_0}\gamma(t)=\varkappa$.

Put $\Phi^-(z)=\psi^-(z)\Psi^-(z)$. Then the problem (5.1) is reduced to the Dirichlet problem of MQ-function $\Phi^-(z)$ with known real multipliers:

$$\mathrm{Re}\{\Phi^-(t)\}=p(t)f(t),\quad t\in L_0. \tag{5.2}$$

The multipliers β_1, β_2 of $\Phi^-(z)$ are the same as those of $\psi^-(z)$.

Case I: $z_0=0$. Consider different cases of $\varkappa$.

1° $\varkappa=0$. Using the corresponding results in § 3, we have the conditions (3.9) or (3.11) of solvability and for determining α, β with

$$f_j=\int_{L_0}p(t)f(t)\nu_j(t)\,ds, \tag{5.3}$$

$j=1,\ 2,\ 3$, or $j=0,\ 1,\ 2,\ 3$ for different cases. When they are satisfied, $\Psi^-(z)=\Phi^-(z)/\chi^-(z)$ is the general solution of (5.1), where $\Phi^-(z)$ is the general solution of (5.2). From the discussions made in § 3, the generalized degree of freedom of the DH problem is $r=-1$ and the number of arbitrary constants in its general solution is $l\leqslant 1$.

2° $\varkappa<0$. The situation is the same as in 1° except that $\Phi^-(z)$ ought to have $-\varkappa$ zeros at $z_1,\ \cdots,\ z_{-\varkappa}$ in S_0^-. Thereby, besides (3.9) or (3.11), the following conditions of solvability

$$\int_{L_0}\mu(t)e^{-\lambda t}[\zeta(t-z_k)+\zeta(z_k)]\,dt+A=0,\quad k=1,\ \cdots,-\varkappa, \tag{5.4}$$

must be added, where $\mu(t)$, by (3.4), is any solution of

$$K_1\mu=-p(t_0)f(t_0)+\mathrm{Re}\{Ae^{\lambda t_0}\},\quad t_0\in L_0.$$

(5.4) consists of $-2\varkappa$ real conditions total in number.

By the discussions in § 3, we have, in this case, $r=2\varkappa-1$, $l\leqslant 1$.

3° $\varkappa>0$. The corresponding MQ-function $\Phi^-(z)$ may have simple poles at $z_1,\ \cdots,\ z_\varkappa$ in S_0^-. Let

$$\chi_k(z)=e^{\lambda z}[\zeta(z-z_k)+\zeta(z)],\quad k=1,\ \cdots,\ \varkappa, \tag{5.5}$$

which is an MQ-function in S^- with single simple pole z_k in S_0^- and with the same multipliers as those of $\Phi^-(z)$. Therefore, in this case, $\Phi^-(z)$ may be represented as

$$\Phi^-(z)=e^{\lambda z}\left\{\frac{1}{\pi i}\int_{L_0}\mu(t)e^{-\lambda t}[\zeta(t-z)+\zeta(z)]\,dt+A\right\}+\sum_{k=1}^{\varkappa}C_k\chi_k(z),\quad z\in S^-, \tag{5.6}$$

where $C_k=\gamma_k+i\delta_k$, $k=1,\ \cdots,\ \varkappa$, are arbitrary complex constants, and $\mu(t)$ is any solution of

$$K_1\mu=-p(t_0)f(t_0)-\mathrm{Re}\{Ae^{\lambda t_0}\}-\sum_{k=1}^{\varkappa}\mathrm{Re}\{C_k\chi_k(t_0)\},\quad t_0\in L_0. \tag{5.7}$$

Then the conditions (3.9) or (3.11) become now

$$\left.\begin{aligned}&-\beta J_j+\sum_{k=1}^{\varkappa}\mathrm{Re}\left\{C_k\int_{L_0}\chi_k(t)\nu_j(t)ds\right\}=-f_j,\quad j=1,\ 2\ \text{or}\ j=0,\ 1,\ 2;\\&\alpha+\beta J_3+\sum_{k=1}^{\varkappa}\mathrm{Re}\left\{C_k\int_{L_0}\chi_k(t)\nu_3(t)ds\right\}=-f_3,\end{aligned}\right\} \tag{5.8}$$

which contain $2\varkappa+2$ real constants γ_k, δ_k $(k=1, \cdots, \varkappa)$ in addition to α, β. Hence, in this case, we have, $r=2\varkappa-1$, $l\leqslant 2\varkappa+1$.

Case II: $z_0\neq 0$. The discussions are only a little different from Case I and will be stated briefly.

1° $\varkappa=0$. The conditions of solvability (3.15) or (3.16) now become

$$\int_{L_0}p(t)f(t)\nu(t)ds=0 \tag{5.9}$$

or

$$f_j=\int_{L_0}p(t)f(t)\nu_j(t)ds=0,\quad j=1,\ 2, \tag{5.9$'$}$$

respectively. According to the results in § 3, we have, in this case, $r=-1$, $l\leqslant 1$.

2° $\varkappa<0$. Besides (5.9) or (5.9)$'$, we have additional conditions of solvability

$$\int_{L_0}\mu(t)e^{-\lambda t}\frac{\sigma(t-z_k+z_0)}{\sigma(t-z_k)}dt=0,\quad k=1, \cdots, -\varkappa, \tag{5.10}$$

where $\mu(t)$ is any solution of $K_2\mu=-p(t_0)f(t_0)$. Here, we have $r=2\varkappa-1$, $l\leqslant 1$.

3° $\varkappa>0$. Now we should solve the integral equation

$$K_2\mu=-p(t_0)f(t_0)+\sum_{k=1}^{\varkappa}\mathrm{Re}\{C_kq(t_0)\chi_k(t_0)\},\quad t_0\in L_0, \tag{5.11}$$

where $q(z)$ is given by (1.6). The conditions of solvability (3.15) or (3.16) become

$$\sum_{k=1}^{\varkappa}\mathrm{Re}\left\{\int_{L_0}C_kq(t)\chi_k(t)ds\right\}=\int_{L_0}p(t)f(t)\nu(t)ds, \tag{5.12}$$

or

$$\sum_{k=1}^{\varkappa}\mathrm{Re}\left\{\int_{L_0}C_kq(t)\chi_k(t)ds\right\}=\int_{L_0}p(t)f(t)\nu_j(t)ds,\quad j=1,\ 2. \tag{5.12$'$}$$

Here $r=2\varkappa-1$, $l\leqslant 2\varkappa+1$.

In conclusion, for any case whatever, we have the following theorem.

Theorem 3. *If the index of the DH problem (5.1) is $\varkappa$, then its generalized degree of freedom is $2\varkappa-1$, and when the conditions of its solvability are satisfied, the number of arbitrary (real) constants in its general solution is not greater than $2\varkappa+1$ if $\varkappa\geqslant 0$ and 1 if $\varkappa<0$.*

We mention that our method of solution is constructive, which reduces the problem to solving certain definite Fredholm integral equation.

Remark. Of course we may formulate the Hilbert problem (5.1) for MQ-functions. Since the process of MQ-regularization is the same as that given in § 4 and the quasi-periodicities may be united together, there is nothing new to be discussed. We may also formulate the Hilbert problem for doubly quasi-periodic analytic functions in addition. However, it may be easily solved by combining the method used here and that in § 5.

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