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THE CLASSES OF SOME POSITIVE DEFINITE UNIMODULAR LATTICES OVER $Z[\sqrt{3}]$ AND $Z[\sqrt{6}]$

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Abstract

In this paper, the author generalizes Kneser's method which was used by Kneser, R. Salamon and Y. Minura, and applies this method to determine the classes of some positive definite unimodular lattices over $Z[\sqrt{3}]$ and $Z[\sqrt{6}]$.

§ 1. Introduction

Let $K=Q(\sqrt{d})$ be a real quadratic field over rational field Q with a square-free rational integer d and ε be the fundamental unit of k. We have following three unimodular lattices:

(1) $I_n=\langle 1\rangle\perp\langle 1\rangle\perp\cdots\perp\langle 1\rangle$;

(2) $E_n=\langle \varepsilon\rangle\perp\langle \varepsilon\rangle\perp\cdots\perp\langle \varepsilon\rangle$;

(3) $E_r\perp I_{n-r}\quad(1<r<n)$.

We call I_n an I-type lattice, E_n an E-type lattice and $E_r\perp I_{n-r}$ a mixed-type lattice. I_n is also called unit lattice. It is clear that I_n is a positive lattice, E_n is a definite lattice and $E_r\perp I_{n-r}$ is a positive definite lattice if $\varepsilon\gg 0$ (i.e., ε is totally positive).

We have the following two problems to consider:

(1) For given field k and a lattice L over k, determine the class number $h_k(L)$ in gen L and the set of all representatives of classes in gen L

(2) For given natural number n, determine all totally real algebraic number fields k and all lattices L over k such that $h_k(L)=n$

The main results about the problems are the following

(1) C. L. Siegel proved in 1935 "There are finite totally real algebraic number fields k such that $h_k(I_4)=1$". He has conjectured that there are finite number of totally real algebraic number fields k such that the class number in genus of a positive definite integer quadratic form Q over k is exactly a given number h. The

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conjecture was proved by H. Pfeuffer[9] in 1979.

(2) J. Dzewas[1], O. Neblung[3], H. Pfeuffer[9,7] and M. Peter[5] have studied the class number one problem. They have proved: if $n\geqslant 3$, then $h_k(I_n)=1$ if and only if $k=Q$, $n\leqslant 8$; $k=Q(\sqrt{2})$, $n\leqslant 4$; $k=Q(\sqrt{5})$, $n\leqslant 4$; $k=Q(\sqrt{17})$, $n=3$; $k=k^{(49)}$, $n=3$ and $k=k^{(148)}$, $n=3$, where Q is the rational number field and $k^{(49)}$ (resp. $k^{(148)}$) is the unique totally real cubic number field with discriminant 49 (resp. 148). The class number two problem has been studied by M. Pohst[12], who gets a nearly complete result for $n\geqslant 4$.

(3) In 1968, R. Salamon[10] proved $h_{Q(\sqrt{3})}\ I_3=2$ and $h_{Q(\sqrt{3})}\ I_4=3$ by Kneser's method. He also gave the representative of each class in gen I_3 and gen I_4.

(4) In 1978, H. Pfeuffer[8] proved $h_{Q(\sqrt{5})}I_6=3$.

(5) In 1983, Y. Mimura[11] proved the result by Kneser's method: "In the case of real quadratic field, $h_k(I_n)=2$ $(n\geqslant 2)$ if and only if $K=Q(\sqrt{2})$, $n=5$; $k=Q(\sqrt{3})$, $n=3$; $k=Q(\sqrt{5})$, $n=5$; $k=Q(\sqrt{13})$, $n=3$; $k=Q(\sqrt{33})$, $n=3$ and $k=Q(\sqrt{41})$, $n=3$".

In this paper, we shall work at mixed-type lattices which have not been studied. We shall determine the class numbers and give the representative of each class in genus of all mixed-type lattices of rank 3 and rank 4 over $Z[\sqrt{3}]$ and $Z[\sqrt{6}]$ by Kneser's method.

In § 2 we discuss some properties of adjacent lattices and genus, and the ralations between them. We shall see this part generalizes Kneser's method over real quadratic fields which was applied by Kneser[2], by R. Salamon[10] and by T. Mimura[11]. In § 3 and § 4 we determine the classes in gen $(E_2\perp I_2)$, gen $(E_1\perp I_3)$, gen $(E_1\perp I_2)$ and gen $(E_2\perp I_1)$ by using the generalized Kneser's method.

The notations and the terminology used in this paper will generally be those of O. T. O'Meare[4].

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§ 2. Genus and Adjacent Lattices

Let $D=\{\alpha\in k\,|\,\alpha\gg 0\}$, J be the idèle group of k, P be the principle idèle group of k and P_0 be the subgroup of P generated by D.

It is easy to prove the following proposition.

Proposition 1. *Let the ideal class number C_k of k be one and $\mathfrak{P}=(\pi)$ be a prime ideal of k, $J^{\mathfrak{P}}=\{i=(i_q)\in J\,|\,|i_q|_q=1$ for all finite spot $q\neq\mathfrak{P}\}$. Then*

(i) $J=P_DJ^{\mathfrak{P}}\cup\sqrt{d}\,P_D\cdot J^{\mathfrak{P}}$;

(ii) $(J:\,P_DJ^{\mathfrak{P}})=2$ *if and only if* $N_{k/Q}\varepsilon>0$ *and* $N_{k/Q}\pi>0$.

Proposition 2. *Let* $\varepsilon=a+b\sqrt{d}$, a, $b\in Q$.

(i) *$kI_n \cong kE_n$ if and only if* $n\equiv 0 \pmod 2$ *when* $d\not\equiv 1 \pmod 8$ *and* $n\equiv 0 \pmod 4$ *or* $n\equiv 2 \pmod 4$ $a\equiv 1 \pmod 4$ *when* $d\equiv 1 \pmod 8$;

(ii) *$E_n \notin$ class I_n;*

(iii) *if* $0<r\leqslant r'<n$, *then* $E_r \perp I_{n-r} \in \text{gen } (E_{r'} \perp I_{n-r'})$ *if and only if* $kI_{r'-r} \cong kE_{r'-r}$;

(iv) $d\not\equiv 3 \pmod 4$, *then* $E_n \in \text{gen } I_n$ *if and only if* $kI_n \cong kE_n$;

$d\equiv 3 \pmod 4$, *then* $E_n \in \text{gen } I_n$ *if and only if* $kI_n \cong kE_n$ *and* $a\equiv 1 \pmod 2$.

Proof (i) is easy to prove by Hasse–Minkowski Theorem.

(ii) is clear.

(iii) The necessity is clear, we prove the sufficiency. If $\mathfrak{P}$ is a non-dyadic spot, then $(E_r \perp I_{n-r})_{\mathfrak{P}} \cong (E_r \perp I_{n-r})_{\mathfrak{P}}$ since they are unimodular lattices on the same quadratic space over $k_{\mathfrak{P}}$. If $\mathfrak{P}$ is a dyadic spot, then $\mathfrak{G}(E_r \perp I_{n-r})_{\mathfrak{P}} = \mathfrak{G}(EE_{r'} \perp I_{n-r'})_{\mathfrak{P}}$ (cf. [4] § 94 and 93:4). Hence $(E_r \perp I_{n-r})_{\mathfrak{P}} \cong (E_{r'} \perp I_{n-r'})_{\mathfrak{P}}$ by ([4] 93:16). So $E_r \perp I_{n-r} \in \text{gen } E_{r'} \perp I_{n-r'}$.

(iv) It is clear that $kI_n \cong kE_n$ is necessary, so we may prove our assertion under this condition. On the analogy of the proof in (iii) we have

$$E_n \in \text{gen } I_n \Leftrightarrow (E_n)_{\mathfrak{P}} \cong (I_n)_{\mathfrak{P}} \Leftrightarrow \mathfrak{G}(E_n)_{\mathfrak{P}} = \mathfrak{G}(I_n)_{\mathfrak{P}}$$
$$\Leftrightarrow \varepsilon\mathfrak{O}_{\mathfrak{P}}^2 + 2\mathfrak{O}_{\mathfrak{P}} = \mathfrak{O}_{\mathfrak{P}}^2 + 2\mathfrak{O}_{\mathfrak{P}} \Leftrightarrow \mathfrak{d}(\varepsilon) \subseteq 2\mathfrak{O}_{\mathfrak{P}}$$

for any dyadic spot $\mathfrak{P}$ of k.

(a) $d\equiv 2 \pmod 4$. Write $\varepsilon = 1+(a-1+b\sqrt{d})$. Then $2 \mid (a-1+b\sqrt{d})$. Hence $\mathfrak{d}(\varepsilon) \subseteq 2\mathfrak{O}_{\mathfrak{P}}$.

(b) $d\equiv 1 \pmod 8$. Write $\varepsilon = 1+(a-1+b\sqrt{d})$. Then $2 \mid (a-1+b\sqrt{d})$ since $a\equiv 1 \pmod 4$ and $b\equiv 0 \pmod 4$. Hence $\mathfrak{d}(\varepsilon) \subseteq 2\mathfrak{O}_{\mathfrak{P}}$.

(c) $d\equiv 5 \pmod 8$. Then $\mathfrak{P}$ is 2-adic spot. Hence

$$\mathfrak{G}(E_n)_{\mathfrak{P}} = \mathfrak{d}(E_n)_{\mathfrak{P}} = \mathfrak{O}_{\mathfrak{P}} = \mathfrak{d}(I_n)_{\mathfrak{P}} = \mathfrak{G}(I_n)_{\mathfrak{P}}.$$

(d) $d\equiv 3 \pmod 4$. If $a\equiv 1 \pmod 2$, then $b\equiv 0 \pmod 2$, hence $\mathfrak{d}(\varepsilon) \subseteq 2\mathfrak{O}_{\mathfrak{P}}$ anologous to (a); if $a\equiv 0 \pmod 2$, then $b\equiv 1 \pmod 2$. Write $\varepsilon = 1+(\varepsilon-1)$. We have $2 \nmid \varepsilon-1$, but $2 \mid N_{k/Q}(\varepsilon-1)$. Hence $\mathfrak{P} \| (\varepsilon-1)$ (Note there is unique dyadic spot $\mathfrak{P}$). So $\text{Ord}_{\mathfrak{P}}(\varepsilon-1)=1$, and hence $\mathfrak{d}(\varepsilon-1)\mathfrak{O}_{\mathfrak{P}} \not\subseteq 2\mathfrak{O}_{\mathfrak{P}}$ by ([4] 67:5).

Let V_n be a positive definite quadratic space over k of dimension n, L be a unimodular lattice on V_n.

Definition 1. *Let $\mathfrak{A}$ be a non-zero ideal of $\mathfrak{O}$. For $x \in \mathfrak{A}^{-1} L$ such that $Q(x) \in \mathfrak{O}$, we call*

$$L(x) = \mathfrak{O}x + \{y \in L \mid B(x, y) \in \mathfrak{O}\}$$

an adjacent lattice to L.

It is clear that $L(x)$ is a unimodular lattice[11].

Let $x_i \in \mathfrak{A}^{-1}L$ $(i=1, 2)$ such that $Q(x_i) \in \mathfrak{O}$ $(i=1, 2)$. If $L(x_1) \cong L(x_2)$, we write "$x_1 \sim x_2$". Obviously, "$\sim$" is an equivalence ralation.

Proposition 3. *Let L be a unimodular lattice on V, $\mathfrak{A}$ be a nonzero ideal. For $x \in \mathfrak{A}^{-1}L$ such that $Q(x) \in \mathfrak{O}$, we have*

(i) *If $x \in L$, then $L(x) = L$;*

(ii) *If $r \in \mathfrak{O}$ such that $(r, \mathfrak{A}) = 1$, then $L(x) = L(rx+y)$ for any $y \in I$ such that $B(y, x) \in \mathfrak{O}$;*

(iii) *If $(2, \mathfrak{A}) = 1$, then $L(x) = L(x')$ for any x' such that $Q(x') \in \mathfrak{O}$, $x - x' \in \overline{L}$;*

(iv) *Let $y \in \mathfrak{A}^{-1}L$, if there is an isometry $\sigma \in O(v)$ such that $\sigma x = y$ and $\sigma\{z \in L \mid B(x, z) \in \mathfrak{O}\} \subset L$, then $x \sim y$.*

(v) *If there is an element $r \in \mathfrak{O}$ and a lattice M on V such that $(r, \mathfrak{A}) = 1$, $M \cong L$, $rM \subseteq L(x)$ and $\mathfrak{A}\, M \subseteq L$, then $L(x) \cong I$.*

Proof (i)—(iv) are easy.

(v) We have $M = L(x)$, hence $L(x) \cong L$.

Definition 2. *Let $\mathfrak{A}$ be a non-zero ideal, V_n be a positive definite quadratic space over k. A set of finite lattices on $V_n\{L_i, i=1, 2, \cdots, r\}$ is called an $\mathfrak{A}$-chain if there are lattices $L_{i'} \in cls\ L_i$ such that $L_{i'}$ is $\mathfrak{A}$-adjacent to $L_{i'+1}$: $(1 \leqslant i \leqslant r-1)$.*

By Definition 2 we have an equivalence ralation "$\widetilde{\mathfrak{A}}$" such that $L \sim K$ if and only if L and K belong to an $\mathfrak{A}$-chain. (Usually, we write $L \widetilde{\mathfrak{A}} K$ as $L \sim K$ if we need not mention the ideal.)

Proposition 4. *Let $\mathfrak{P}$ be a prime ideal of k, and L, K be unimodular lattices on V_n. Then $L \sim K$ if and only if there is a lattice K' in class K such that the invariant factors of K' in L are $\mathfrak{P}^{r_i} (i=1, 2, \cdots, n)$ with $\sum_{i=1}^{n} r_i = 0$.*

Proof The necessity. If $L \sim K$, then $\exists \mathfrak{P}$-chain $L_1 = L, L_2, \cdots, L_{r-1}, L_r \in \text{cls } K$, such that L_i is $\mathfrak{P}$-adjacent to $L_{i+1} (1 \leqslant i \leqslant r-1)$. Let $K' = L_r$. Assume the invariant factors of K' in L are $\mathfrak{A}_i (i=1, 2, \cdots, n)$. Then there is a base of V_n such that

$$L = \mathscr{L}_1 x_1 + \cdots + \mathscr{L}_n x_n, \qquad (\mathscr{L}_i, \mathfrak{A}_i \text{ are fractional ideals of } k)$$
$$K = \mathscr{L}_1 \mathfrak{A}_1 x_1 + \cdots + \mathscr{L}_n \mathfrak{A}_n x_n.$$

We have $L_{\mathfrak{P}'} = L_{1\mathfrak{P}'} = \cdots = L_{r\mathfrak{P}'}$ $(\forall \mathfrak{P}' \neq \mathfrak{P})$ since L_i is $\mathfrak{P}$-adjacent to L_{i+1}. Hence $\mathfrak{A}_{1\mathfrak{P}'} = \mathfrak{A}_{2\mathfrak{P}'} = \cdots = \mathfrak{A}_{n\mathfrak{P}'} = \mathfrak{C}_{\mathfrak{P}'}$, and $\mathfrak{A}_i = \mathfrak{P}^{r_i}$. But L, K' are unimodular lattices, so $\mathfrak{O} = \mathfrak{d}L = \mathscr{L}_1^2 \cdots \mathscr{L}_n^2 \cdot d(x_1, \cdots, x_n)$, $\mathfrak{d} = \mathfrak{G}K' = \mathscr{L}_1^2, \cdots, \mathscr{L}_n^2 \mathfrak{P}^{2\sum_{i=1}^{n} r_i} d(x_1, \cdots, x_n)$. This implies $\sum_{i=1}^{n} r_i = 0$.

The sufficiency. We may assume $K' = K$. Let $m = \sum_{r_i \geqslant 0} r_i$. We proceed to prove by induction on m.

(1) $m = 0$. Then $\sum_{i=1}^{n} r_i = 0$ and $m = 0$ implies $r_i = 0$, $i = 1, 2, \cdots, n$. Hence $L = K$. So $L \underset{\mathfrak{P}}{\sim} K$.

(2) $m \geqslant 1$. Then the invariant factors of K in L are $\mathfrak{P}^{r_1}, \mathfrak{P}^{r_2}, \cdots, \mathfrak{P}^{r_n} (r_1 < 0)$ and

there is a base $\{x_i\}$ of V_n such that

$$L=\mathfrak{A}_1x_1+\cdots+\mathfrak{A}_nx_n,$$
$$K=\mathfrak{P}^{r_1}\mathfrak{A}_1x_1+\cdots+\mathfrak{A}^{r_n}\mathfrak{A}_nx_n. \quad (\mathfrak{A}_i \text{ are fractional ideals of } k)$$

Without loss of generality we may assume (i) $(\mathfrak{A}_i, \mathfrak{P})=1$; (ii) $B(\mathfrak{A}_1x_1, \mathfrak{A}_ix_i)\equiv O(\bmod \mathfrak{P})$, $i=1, 2, \cdots, n-1$; (iii) $B(\mathfrak{A}_1x_1, \mathfrak{A}_nx_n)\not\equiv O$ $(\bmod \mathfrak{P})$.

By ([4] 22: 5) there is an element $\alpha\in k$ such that $\alpha\mathfrak{O}+\mathfrak{A}_1=\mathfrak{A}_1\mathfrak{P}^{-1}$. Then $\alpha x_1\in \mathfrak{A}_1\mathfrak{P}^{-1}x_1\subseteq\mathfrak{P}^{-1}L$ and $Q(\alpha x_1)\in B(\mathfrak{A}_1\mathfrak{P}^{-1}x_1, \mathfrak{A}_1\mathfrak{P}^{-1}x_1)\subseteq B(\mathfrak{A}_1\mathfrak{P}^{r_1}x_1, \mathfrak{A}_1\mathfrak{P}^{r_1}x_1)\subseteq\mathfrak{O}$. It is easy to check that

$$L(\alpha x_1)\supseteq\mathfrak{O}(\alpha x_1)+\mathfrak{A}_1x_1+\mathfrak{A}_2x_2+\cdots+\mathfrak{A}_{n-1}x_{n-1}+\mathfrak{A}_n\mathfrak{P}x_n$$
$$=\mathfrak{A}_1\mathfrak{P}^{-1}x_1+\mathfrak{A}_2x_2+\cdots+\mathfrak{A}_{n-1}x_{n-1}+\mathfrak{A}_n\mathfrak{P}x_n=L'.$$

But $L(\alpha x_1)$ and L' are both unimodular lattices. Hence $L(\alpha x_1)=L'$. It is clear that the invariant factors K in L' are $\mathfrak{P}^{r_1+1}, \mathfrak{P}^{r_2}, \cdots, \mathfrak{P}^{r_{n-1}}, \mathfrak{P}^{r_n-1}$ and the volume m of K with respect to L' is $m-1$. Hence $L'\sim K$ by induction hypothesis. Thus $L\sim K$ since L' is $\mathfrak{P}$-adjacent to L.

Remark. It is easy to know by the proof of the proposition that if L is $\mathfrak{P}$-adjacent to L', then L' is adjacent to L.

Let L, K be unimodular lattices on V_n, S the set of all finite spots of k, and $\mathfrak{P}$ a non-dyadic spot. Denote $\mathfrak{O}_{(\mathfrak{P})}=\mathfrak{O}(S-\mathfrak{P})$, $L_{(\mathfrak{P})}=\mathfrak{O}_{(\mathfrak{P})}L$, $K_{(\mathfrak{P})}=\mathfrak{O}_{(\mathfrak{P})}L$, $\mathfrak{A}_{(\mathfrak{P})}=\mathfrak{A}\mathfrak{O}_{(\mathfrak{P})}$ ($\mathfrak{A}$ is a fractional ideal of k).

Proposition 5. *When* $\dim V_n\geqslant3$,

$$L\underset{\mathfrak{P}}{\sim}K \text{ if and only if } \operatorname{spn} L_{(\mathfrak{P})}=\operatorname{spn} K_{(\mathfrak{P})}$$

Proof. The necessity. Since $L\underset{\mathfrak{P}}{\sim}K$, there is a lattice $K'\in\operatorname{cls} K$ such that the invariant factors of K' in L are $\mathfrak{P}^{r_i}$ $(i=1, 2, \cdots, n)$ and a base of V_n $\{x_i\}$ such that

$$L=\mathfrak{A}_1x_1+\cdots+\mathfrak{A}_nx_n,$$
$$K'=\mathfrak{A}_1\mathfrak{P}^{r_1}x_1+\cdots+\mathfrak{A}_n\mathfrak{P}^{r_n}x_n.$$

Then

$$L_{(\mathfrak{P})}=\mathfrak{A}_{1(\mathfrak{P})}x_1+\cdots+\mathfrak{A}_{n(\mathfrak{P})}x_n,$$
$$K'_{(\mathfrak{P})}=\mathfrak{A}_{1(\mathfrak{P})}\mathfrak{P}^{r_1}_{(\mathfrak{P})}x_1+\cdots+\mathfrak{A}_{n(\mathfrak{P})}\mathfrak{P}^{r_n}_{(\mathfrak{P})}x_n.$$

Since $\mathfrak{P}_{(\mathfrak{P})}=\mathfrak{O}_{(\mathfrak{P})}$, we have $L_{(\mathfrak{P})}=K'_{(\mathfrak{P})}$. Hence $\operatorname{cls} K_{(\mathfrak{P})}=\operatorname{cls} L_{(\mathfrak{P})}$.

On the other hand, L, K are unimodular lattices with $\dim V_n\geqslant3$ and $\mathfrak{P}$ is a non-dyadic spot of k. Then V_n is isotropic at $\mathfrak{P}$. Hence $\operatorname{cls} L_{(\mathfrak{P})}=\operatorname{spn} L_{(\mathfrak{P})}$, $\operatorname{cls} K_{(\mathfrak{P})}=\operatorname{spn} K_{(\mathfrak{P})}$ by ([4]104: 25). Thus $\operatorname{spn} L_{(\mathfrak{P})}=\operatorname{spn} K_{(\mathfrak{P})}$.

The sufficiency. We have $\operatorname{cls} L_{(\mathfrak{P})}=\operatorname{cls} K_{(\mathfrak{P})}$ by ([4] 104:25). Hence there is an isometry $\sigma\in O(V_n)$ such that $\sigma K_{(\mathfrak{P})}=L_{(\mathfrak{P})}$. Let $K'=\sigma K$. Then $K'_{(\mathfrak{P})}=L_{(\mathfrak{P})}$. Assume the invariant factors of K' in L are $\mathscr{L}_1, \mathscr{L}_2, \cdots, \mathscr{L}_n$. Then there is a base of $V_n\{x_i\}$ such that

$$L=\mathfrak{A}_1x_1+\cdots+\mathfrak{A}_nx_n,$$
$$K'=\mathfrak{A}_1\mathscr{L}_1x_1+\cdots+\mathfrak{A}_n\mathscr{L}_nx_n.$$

Since $K'_{(\mathfrak{P})}=L_{(\mathfrak{P})}$, we have $\mathscr{L}_{i(\mathfrak{P})}=\mathfrak{O}_{(\mathfrak{P})}$. This implies $\mathscr{L}_i=\mathfrak{P}^{\gamma_i}$ $(i=1, 2, \cdots, n)$.

But K', L are unimodular lattices on V_n, so $\sum_{i=1}^{n} r_i = 0$. Hence $K \underset{\mathfrak{P}}{\approx} L$ by Proposition 4.

It is easy to know that if $\mathfrak{P}$ is a non-dyadic prime ideal, L is a unimodular lattice on V_n and K is $\mathfrak{P}$-adjacent to L, then $K \in \text{gen } L$[11]. Thus we can partition gen L into equivalence classes by the equivalence "$\underset{\mathfrak{P}}{\approx}$". We use $M_{\mathfrak{P}}L$ to denote the class number of $\mathfrak{P}$-chain equivalence classes in gen L.

Proposition 6. *Let L be a unimodular lattice on V_n with $\dim V_n \geqslant 3$ and $\mathfrak{P}$ be a non-dyadic prime ideal. Then*

$$M_{\mathfrak{P}}L \leqslant (J: P_D J^{\mathfrak{P}}).$$

Proof We have $\theta(O^+(V_n)) = D$ by ([4]101:8) and $J^{\mathfrak{P}} \subseteq J_k^{L(\mathfrak{P})}$ by ([4]102:10). Then

$$M_{\mathfrak{P}}L \leqslant g(L_{(\mathfrak{P})}) \leqslant g^+(L_{(\mathfrak{P})}) = (J: P_D J_k^{L(\mathfrak{P})}) \leqslant (J: P_D J^{\mathfrak{P}}),$$

Immediately from Proposition 1 and Proposition 6 we have the following theorem.

Theorem 1. *Let $k = Q(\sqrt{d})$ be a real quadratic field over Q with a square-free rational integer d, and the ideal class number of k be 1. And let L be a positive definite unimodular lattice of rank $n \geqslant 3$ and $\mathfrak{P} = (\pi)$ be a non-dyadic prime ideal of k. Then*

(i) $M_{\mathfrak{P}}L \leqslant 2$,

(ii) $M_{\mathfrak{P}}L = 1$ if $N_{k/Q}\varepsilon < 0$ or $N_{k/Q}\pi < 0$.

Here ε is the fundamental unit of k.

§3. The Mixed-Type Lattices of Rank $n \geqslant 3$ over $Z[\sqrt{3}]$
$\varepsilon = 2 + \sqrt{3} \gg 0$, $\pi = \sqrt{3}$, $\mathfrak{P} = (\pi)$

1. The classes in gen $(E_1 \perp I_3)$

A. The adjacent lattices of $E_1 \perp I_3$.

Let $L = E_1 \perp I_3 = \perp_{i=1}^{4} \mathfrak{O}_{e_i}$, $Q(e_1) = \varepsilon$, $Q(e_i) = 1$ $(i = 2, 3, 4)$. It is clear that τ_{e_i} $(i = 1, 2, 3, 4)$, $\tau_{e_i - e_j}$ $(2 \leqslant i < j \leqslant 4) \in O(L)$.

Considering $x = \frac{1}{\sqrt{3}} \sum_{i=1}^{4} r_i e_i$, $r_i = a_i + b_i\sqrt{3}$, $a_i, b_i \in Z$ with $Q\{x\} \in \mathfrak{O}$, we have only following two cases to consider by Proposition 3 (ii), (iii), (iv):

(1) $x_1 = \frac{1}{\sqrt{3}} \sum_{i=2}^{4} e_i$.

It is easy to check

$$L(x_1) = \sum_{i=1}^{4} \mathfrak{O}k_i \cong \langle\varepsilon\rangle \perp \langle 1 \rangle \perp \begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}.$$

Here $k_1 = e_1$, $k_2 = x_1$, $k_3 = e_2 - e_3$, $k_4 = \frac{1}{\sqrt{3}}(2e_2 - e_3 - e_4)$. Note $\begin{pmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 2 \end{pmatrix}$ is indecomposable since it is an even unimodular lattice of rank 2. Hence $I(x_1)\not\cong L$ and $L(x_1)\not\cong E_3\perp L_1$.

(2) $x_2=\frac{1}{\sqrt{3}}(e_1+e_2)-e_1$.

We have

$$L(x_2)=\sum_{i=1}^{4}\mathfrak{O}k_i'\cong\langle\varepsilon\rangle\perp\langle1\rangle\perp\langle1\rangle\perp\langle1\rangle\cong L.$$

Here $k_1'=\frac{1}{\sqrt{3}}(e_1+e_2)+e_2$, $k_2'=\frac{1}{\sqrt{3}}(e_1+e_2)-e_1$, $k_3'=e_3$, $k_4'=e_4$.

B. The adjacent lattices of $K=L(x_1)$.

Clearly $\tau_{k_i}(1\leqslant i\leqslant4)$; $\tau_{k_3-k_4}$ and σ: $k_3\to-k_3$, $k_4\to-k_4\in O(K)$.

Considering $x=\frac{1}{\sqrt{3}}\sum_{i=1}^{4}r_ik_i$, $r_i=a_i+b_i\sqrt{3}$, a_i, $b_i\in Z$ with $Q(x)\in\mathfrak{O}$, we have only following four cases to consider by Proposition 3 (ii), (iii), (iv):

(1) $x_1=\frac{1}{\sqrt{3}}(k_2+k_4)$.

We have $K(x_1)=L$ since $x_1=e_2\in L-K$.

(2) $x_2=\frac{1}{\sqrt{3}}(k_1+k_3+k_4)$.

We have $K(x_2)=\sum_{i=1}^{4}\mathfrak{O}l_i\cong\langle1\rangle\perp\langle\varepsilon\rangle\perp\langle\varepsilon\rangle\perp\langle\varepsilon\rangle\cong E_3\perp I_1$. Here $l_1=k_2$, $l_2=x_2$, $l_3=\frac{1}{\sqrt{3}}(-k_1+(2+\sqrt{3})k_3-(1+\sqrt{3})k_4)$, $l_4=\frac{1}{\sqrt{3}}(k_1+(1+\sqrt{3})k_3-(2+\sqrt{3})k_4)$. Clearly $E_3\perp I_1\not\cong E_1\perp I_3$ by Proposition 2 (ii) and ([4] 105:1).

(3) $x_3=\frac{1}{\sqrt{3}}(k_1+k_3-k_4)-k_4$.

We have $\tau_{k_4}\in O(K)$ and $\tau_{k_4}x_2=x_3$. Hence $K(x_3)\cong K(x_2)$ by Proposition 3 (iv).

(4) $x_4=\frac{1}{\sqrt{3}}(k_1+k_2)-k_1$.

We have $K(x_4)=\sum_{i=1}^{4}\mathfrak{O}l_i'=\langle1\rangle\perp\langle\varepsilon\rangle\perp\begin{pmatrix}2 & \sqrt{3}\\ \sqrt{3} & 2\end{pmatrix}\cong K$. Here $l_1'=\frac{1}{\sqrt{3}}\cdot(k_1+k_2)-k_1$, $l_2'=\frac{1}{\sqrt{3}}(k_1+k_2)+k_2$, $l_3'=k_3$, $l_4'=k_4$.

We know by A there is only one adjacent lattice of $E_1\perp I_3$ which does not isometry to $E_1\perp I_3$. But $E_3\perp I_1=(E_1\perp I_3)^\varepsilon$. Hence there is only one adjacent lattice of $E_3\perp I_1$ which does not isometry to $E_3\perp I_1$, and the unique lattice is $K^\varepsilon\cong K$ by B. Thus, by Theorem 1 we have the following theorem.

Theorem 2 (1). *There are exactly three classes in* gen $E_1\perp I_3$, *and they are*

(i) $E_1\perp I_3$, (ii) $E_3\perp I_1$, (iii) $K\cong\langle1\rangle\perp\langle\varepsilon\rangle\perp\begin{pmatrix}2 & \sqrt{3}\\ \sqrt{3} & 2\end{pmatrix}$.

II. The classes in gen $(E_2\perp I_2)$

A. The adjacent lattices of $E_2\perp I_2$.

Let $L=E_2\perp I_2=\perp_{i=1}^{4}\mathfrak{O}e_i$, $Q(e_1)=Q(e_2)=\varepsilon$, $Q(e_3)=Q(e_4)=1$. Clearly, $\tau_{e_i}(1\leqslant i\leqslant$

4), $\tau_{e_1-e_2}, \tau_{e_3-e_4}\in O(L)$.

Considering $x=\frac{1}{\sqrt{3}}\sum_{i=1}^{4}r_ie_i,\ r_i=a_i+b_i\sqrt{3},\ a_i,\ b_i\in Z$ with $Q(x)\in\mathfrak{O}$, we have only following two cases to consider by Proposition 3(ii), (iii), (iv).

(1) $x_1=\frac{1}{\sqrt{3}}((1-\sqrt{3})e_1+e_3)$.

We have $L(x_1)=\sum_{i=1}^{4}\mathfrak{O}k_i'=\langle 1\rangle\perp\langle\varepsilon\rangle\perp\langle\varepsilon\rangle\perp\langle 1\rangle\cong L$. Here $k_1'=x_1,\ k_2'=\frac{1}{\sqrt{3}}(e_1+(1+\sqrt{3})e_3),\ k_3'=e_2,\ k_4'=e_4$.

(2) $x_2=\frac{1}{\sqrt{3}}(e_1+e_2+e_3+e_4)+e_1$.

It is easy to check that $L(x_2)$ has base $k_1=\frac{1}{\sqrt{3}}(e_1+e_2+e_3+e_4)+e_1,\ k_2=\frac{1}{\sqrt{3}}(e_1+e_2+e_3+e_4)+e_2,\ k_3=\frac{1}{\sqrt{3}}(e_1+e_2+e_3+e_4)-e_3,\ k_4=\frac{1}{\sqrt{3}}(e_1+e_2+e_3+e_4)-e_4$. Hence

$$L(x_2)=\sum_{i=1}^{4}\mathfrak{O}k_i\cong\begin{bmatrix}3\varepsilon & 2\varepsilon & 3+\sqrt{3} & 3+\sqrt{3}\\ 2\varepsilon & 3\varepsilon & 3+\sqrt{3} & 3+\sqrt{3}\\ 3+\sqrt{3} & 3+\sqrt{3} & 3 & 2\\ 3+\sqrt{3} & 3+\sqrt{3} & 2 & 3\end{bmatrix}$$

Now we are going to show $1,\ \varepsilon\notin Q(L(x_2))$ which will imply $L(x_2)\ncong L$ since $1\in Q(L)$. If $1\in Q(L(x_2))$, then $\exists x\in L(x_2)$ such that $Q(x)=1$. Hence $3\in Q(L)$ since $\sqrt{3}x\in L$. But this would imply $\sqrt{3}x=\pm(1-\sqrt{3})e_i\pm e_j(i=1,2,\ j=3,4)$ or $\sqrt{3}x=\pm\sqrt{3}e_i\ (i=3,4)$. This is a contradiction since $x=\pm\frac{1}{\sqrt{3}}[(1-\sqrt{3})e_i\pm e_j)]\ (i=1, 2,\ j=3,\ 4)$ and $x=\pm e_i\ (i=3,\ 4)\notin L(x_1)$. Similarly we can verify $\varepsilon\notin Q(L(x_2))$.

Remark. Using the fact $1,\varepsilon\notin Q(L(x_2))$ we can prove $L(x_2)$ is indecomposable.

B. The adjacent lattices of $K=L(x_2)$

Let $y_1=\frac{1}{\sqrt{3}}(e_1+e_2+e_3+e_4)+(1-\sqrt{3})e_1,\ y_2=\frac{1}{\sqrt{3}}(e_1+e_2+e_3+e_4)+(1-\sqrt{3})e_2,\ y_3=\frac{1}{\sqrt{3}}(e_1+e_2+e_3+e_4)-(1+\sqrt{3})e_3,\ y_4=\frac{1}{\sqrt{3}}(e_1+e_2+e_3+e_4)-(1+\sqrt{3})e_3$ and $Y=\sum_{i=1}^{4}\mathfrak{O}y_i$. Clearly $y_i\perp y_j(i\neq j)$. $Q(y_1)=Q(y_2)=2,\ Q(y_3)=Q(y_4)=2\varepsilon,\ Y\subseteq K$ and $k_1=\frac{1}{2}y_1+\frac{\varepsilon}{2}y_2+\frac{1}{2}y_3+\frac{1}{2}y_4,\ k_2=\frac{\varepsilon}{2}y_1+\frac{1}{2}y_2+\frac{1}{2}y_3+\frac{1}{2}y_4,\ k_3=\frac{1}{2}y_1+\frac{1}{2}y_2+\frac{1}{2}y_3+\frac{\bar{\varepsilon}}{2}y_4,\ k_4=\frac{1}{2}y_1+\frac{1}{2}y_2+\frac{\bar{\varepsilon}}{2}y_3+\frac{1}{2}y_4(\bar{\varepsilon}=2-\sqrt{3}),\ 2K\subseteq Y$.

If $x\in\frac{1}{\sqrt{3}}K$ such that $Q(x)\in\mathfrak{O}$, then $K(x)=K(2x)$ by Proposition 3 (iii) since $(2,\ \sqrt{3})=1$. Note $2x\in\frac{1}{\sqrt{3}}2K\subseteq\frac{1}{\sqrt{3}}Y$. Hence we may assume $x\in\frac{1}{\sqrt{3}}Y$. Note that $O(Y)$ contains the isometries $\tau_{y_i}(1\leqslant i\leqslant 4),\ \tau_{y_1-y_2},\ \tau_{y_3-y_4}$.

Considering $y'=\frac{1}{\sqrt{3}}\sum_{i=1}^{4}r_iy_i,\ r_i=a_i+b_i\sqrt{3}.\ a_i,\ b_i\in Z$ with $Q(y')\in\mathfrak{O}$, we have only following two cases to consider by proposition 3 (ii), (iii), (iv).

(1) $y_1' = \frac{1}{\sqrt{3}}(y_1+y_2-y_3-y_4)-y_1$.

We have $K(y_1')=L$ since $y_1'=-e_1-e_2+e_3+e_4-(1-\sqrt{3})e_1 \in L-K$.

(2) $y_2' = \frac{1}{\sqrt{3}}(y_1+y_3)+y_1$.

Let $w_1=\frac{1}{\sqrt{3}}(y_1+y_3)-y_3$, $w_2=y_2$, $w_3=\frac{1}{\sqrt{3}}(y_1+y_3)+y_1$, $w_4=y_4$ and $k_1'=\frac{1}{2}w_1+\frac{\varepsilon}{2}w_2+\frac{1}{2}w_3+\frac{1}{2}w_4$, $k_2'=\frac{\varepsilon}{2}w_1+\frac{1}{2}w_2+\frac{1}{2}w_3+\frac{1}{2}w_4$, $k_3'=\frac{1}{2}w_1+\frac{1}{2}w_2+\frac{1}{2}w_3+\frac{\bar{\varepsilon}}{2}w_4$, $k_4'=\frac{1}{2}w_1+\frac{1}{2}w_2+\frac{\bar{\varepsilon}}{2}w_3+\frac{1}{2}w_4$, $M=\sum_{i=1}^{4}\mathfrak{O}k_i'$. It is easy to check that $M\cong K$ (by σ: $w_i\to y_i$ $i=1, 2, 3, 4$), $\sqrt{3}M\subseteq K$, $2M\subseteq K(y_2')$ and $(2, \sqrt{3})=1$. Hence $K(y_2')\cong K$ by Proposition 3 (v).

Thus, by Theorem 1 we have the following theorem.

Theorem 2 (2). *There are exactly two classes in* gen $E_2\perp I_2$ *and they are*

(i) $E_2\perp I_2$;

(ii) $K\cong\begin{bmatrix} 3\varepsilon & 2\varepsilon & 3+\sqrt{3} & 3+\sqrt{3} \\ 2\varepsilon & 3\varepsilon & 3+\sqrt{3} & 3+\sqrt{3} \\ 3+\sqrt{3} & 3+\sqrt{3} & 3 & 2 \\ 3+\sqrt{3} & 3+\sqrt{3} & 2 & 3 \end{bmatrix}$.

By Theorem 2 (2) we have the following theorem.

Theorem 2 (3). *There is only one class in* gen $(E_1\perp I_2)$ *and in* gen$(E_2\perp I_1)$ *respectively.*

Proof Let N be a class in gen $(E_1\perp I_2)$. Then $N\perp E_1$ is a class in gen $(E_2\perp I_2)$ which represents ε. But $\varepsilon\notin Q(K)$, *hance* $N\perp E_1\cong E_2\perp I_3$. Thus $N\cong E_1\perp I_2$. The proof for $E_2\perp I_1$ is similar.

§4. The Mixed-Type Lattices of Rank $n\geqslant 3$ over $Z[\sqrt{6}]$ $\varepsilon=5+2\sqrt{6}$. $\pi=3+\sqrt{6}$. $\mathfrak{P}=(\pi)$

I. The classes in gen $(E_1\perp I_3)$

A. The adjacent lattices of $E_1\perp I_3$.

Let $L=E_1\perp I_3=\perp_{i=1}^{4}\mathfrak{O}e_i$. $Q(e_1)=\varepsilon$, $Q(e_i)=1$ $(i=2, 3, 4,)$. Clearly $\tau_{e_i}(1\leqslant i\leqslant 4)$, $\tau_{e_i-e_j}$ $(2\leqslant i<j\leqslant 4)\in O(L)$.

Considering $x=\frac{1}{\pi}\sum_{i=1}^{4}r_ie_i$, $r_i=a_i+b_i\sqrt{6}$, $a_i, b_i\in Z$ with $Q(x)\in\mathfrak{O}$ we have only following two cases to consider by Proposition 3 (ii), (iii), (iv):

(1) $x_1=\frac{1}{\pi}(e_1+e_2)-e_2$.

We have $L(x_1)\cong\sum_{i=1}^{4}\mathfrak{O}k_i'\cong\langle 1\rangle\perp\langle\varepsilon\rangle\perp\langle 1\rangle\perp\langle 1\rangle\cong L$. Here $k_1'=x_1$, $k_2'=\frac{\varepsilon}{\pi}(e_1+e_2)-e_1$, $k_3'=e_3$, $k_4'=e_4$.

(2) $x_2=\frac{\varepsilon}{\pi}(e_2+e_3+e_4)$.

We have $L(x_2)=\sum_{i=1}^{4}\mathfrak{O}k_i=\langle\varepsilon\rangle\perp\langle\varepsilon\rangle\perp\begin{pmatrix}2 & \pi\\ \pi & 2\varepsilon\end{pmatrix}$. Here $k_1=e_1$, $k_2=x_2$, $k_3=e_2-e_3$, $k_4=\frac{\varepsilon}{\pi}(2e_2-e_3-e_4)$. Note $\begin{pmatrix}2 & \pi\\ \pi & 2\varepsilon\end{pmatrix}$ is indecomposable since it is an even unimodular lattice of rank 2. Hence $L(x_2)\ncong L$, $L(x_2)\ncong E_3\perp I_1$.

B. The adjacent lattices of $K=L(x_2)$.

Let $z_1=k_1$, $z_2=k_2$, $z_3=k_3-\bar{\pi}k_4$, $z_4=-\pi k_3+k_4$ $(\bar{\pi}=3-\sqrt{6})$, $Z=\sum_{i=1}^{4}\mathfrak{O}z_i$. Clearly $2K\subseteq Z\subseteq K$, τ_{z_i}, $\tau_{z_1-z_2}\in O(K)$. We may assume the generators of adjacent lattices of K are in $\frac{1}{\pi}Z$ by Proposition 3 (ii) since $(2,\pi)=1$.

Considering $y=\frac{1}{\pi}\sum_{i=1}^{4}r_iz_i$, $r_i=a_i+b_i\sqrt{6}$, a_i, $b_i\in Z$ with $Q(y)\in\mathfrak{O}$, we have only following three cases to consider by Proposition 3 (ii), (iii), (iv):

(1) $y_1=\frac{1}{\pi}(z_2+z_4)$.

We have $K(y_1)\cong L$ since $y_1=e_3\in L-K$.

(2) $y_2=\frac{1}{\pi}(z_3+z_4)-z_3$.

We have $K(y_2)=\sum_{i=1}^{4}\mathfrak{O}l_i'=\langle\varepsilon\rangle\perp\langle\varepsilon\rangle\perp\begin{pmatrix}2 & \pi\\ \pi & 2\varepsilon\end{pmatrix}\cong K$. Here $l_1'=k_1$, $l_2'=k_2$, $l_3'=\frac{1}{\pi}(k_3-2k_4)$, $l_4'=\frac{\varepsilon}{\pi}(2k_3-k_4)$.

(3) $y_3=\frac{1}{\pi}(z_1+z_2+z_3)-z_3$

$$K(y_3)\cong\sum_{j=1}^{4}\mathfrak{O}l_i\cong\begin{bmatrix}(1-6)^2\varepsilon^2 & \varepsilon & -\pi\varepsilon & 0\\ \varepsilon & 2 & -(\sqrt{6}+2) & -(\sqrt{6}+2)\\ -\pi\varepsilon & -(\sqrt{6}+2) & 2\varepsilon & \varepsilon\\ 0 & -(\sqrt{6}+2) & \varepsilon & 2\varepsilon\end{bmatrix}.$$

Here $l_1=z_1+\varepsilon k_3$, $l_2=y_3$, $l_3=-k_4$, $l_4=-k_4-z_4$.

On the analogy of 3. II. A. (2) we can prove $\varepsilon\notin Q(K(y_3))$. Hence $K(y_3)\ncong L$, $K(y_3)\ncong K$ and $K(y_3)\ncong E_3\perp I_1$.

C. The adjacent lattices of $J=K(y_3)$.

Let
$$f_1=\frac{1}{\pi}(z_1+z_2-(\sqrt{6}+2)z_3),$$
$$f_2=\frac{1}{\pi}\left(z_1+z_2+\frac{\sqrt{6}+2}{2}z_3-\frac{\sqrt{6}}{2}z_4\right),$$

$$f_3=\frac{1}{\pi}\left(z_1+z_2+\frac{\sqrt{6}+2}{2}z_3+\frac{\sqrt{6}}{2}z_4\right),$$

$$f_4=z_1-z_2,\ Y=\sum_{j=1}^{4}\mathfrak{O}f_i.$$

Clearly $$l_1=\frac{\varepsilon}{2}f_1+\frac{\varepsilon}{2}f_2+\frac{(1-\sqrt{6})\varepsilon}{2}f_3+\frac{1}{2}f_4,\ l_2=f_1,$$

$$l_3=-\frac{\sqrt{6}+2}{2}f_1+\frac{\sqrt{6}+2}{2}f_3,\ l_4=-\frac{\sqrt{6}+2}{2}f_1+\frac{\sqrt{6}+2}{2}f_2,$$

$2J\subseteq Y\subseteq J$ and τ_{f_i} $(1\leqslant i\leqslant 4)$. $\tau_{f_1-f_2}$, $\tau_{f_2-f_3}\in O(J)$. And we may assume that the generators of the adjacent lattices of J are in $\frac{1}{\pi}Y$ by Proposition 3 (ii).

Considering $x=\frac{1}{\pi}\sum_{j=1}^{4}r_if_i$, $r_i=a_i+b_i\sqrt{6}$, a_i, $b_i\in Z$ with $Q(x)\in\mathfrak{O}$, we have only following two cases to consider by Proposition 3 (ii), (iii), (iv):

(1) $x_2=\frac{1}{\pi}(f_1+f_2+f_3)$.

We have $J(x_2)\cong K$ since $x_2=\bar{\varepsilon}(z_1+z_2)\in K-J$.

(2) $x_1=\frac{1}{\pi}(f_1+f_4)-f_1$.

Let $$w_1=\frac{1}{\pi}(f_1+f_4)-f_1,\quad w_2=f_2,\quad w_3=f_3,\quad w_4=\frac{\varepsilon}{\pi}(f_1+f_4)-f_4$$

and $$l_1'=\frac{\varepsilon}{2}w_1+\frac{\varepsilon}{2}w_2+\frac{(1-\sqrt{6})\varepsilon}{2}w_3+\frac{1}{2}w_4,\quad l_2'=w_1,$$

$$l_3'=-\frac{\sqrt{6}+2}{2}w_1+\frac{\sqrt{6}+2}{2}w_3,$$

$$l_4'=-\frac{\sqrt{6}+2}{2}w_1+\frac{\sqrt{6}+2}{2}w_2,\qquad M=\sum_{j=1}^{4}\mathfrak{O}l_1'.$$

Then $J(x_1)\cong J$ by Proposition 3(v).

We know there are exactly three classes in the $\mathfrak{P}$-adjacent equivalence class of $(E_1\perp I_3)$. But $E_3\perp I_1\in\text{gen}\ (E_1\perp I_3)$ and $E_3\perp I_1\not\cong E_1\perp I_3$ (by Proposition 2), $E_3\perp I_1\not\cong J$ and $E_3\perp I_1\not\cong K$. Hence $M_\pi(E_1\perp I_3)=2$ by Theorem 1. On the other hand we have exactly three classes in the adjacent equivalence class of $(E_3\perp I_1)$ since $E_3\perp I_1=(E_1\perp I_3)^\varepsilon$. Then we have the following theorem.

Theorem 3 (1). *There are exactly six classes in* gen$(E_1\perp I_3)$ *and they are*

(i) $E_1\perp I_3$,

(ii) $$\langle\varepsilon\rangle\perp\langle\varepsilon\rangle\perp\begin{pmatrix}2 & \pi\\ \pi & 2\varepsilon\end{pmatrix},$$

(iii) $$J\cong\begin{bmatrix}(1-\sqrt{6})^2\varepsilon^2 & \varepsilon & -\pi\varepsilon & 0\\ \varepsilon & 2 & -(\sqrt{6}+2) & -(\sqrt{6}+2)\\ -\pi\varepsilon & -(\sqrt{6}+2) & 2\varepsilon & \varepsilon\\ 0 & -(\sqrt{6}+2) & \varepsilon & 2\varepsilon\end{bmatrix},$$

(iv) $E_3\perp I_1$,

(v) $\langle 1\rangle \perp \langle 1\rangle \perp \begin{pmatrix} 2 & \pi \\ \pi & 2\varepsilon \end{pmatrix}$,

(vi) J^{ε}.

II. The classes in gen $(E_2 \perp I_2)$

A. The adjacent lattices of $E_2 \perp I_2$.

Let $L = E_2 \perp I_2 = \perp_{i=1}^{4} \mathfrak{O}e_i$, $Q(e_1) = Q(e_3) = \varepsilon$, $Q(e_2) = Q(e_4) = 1$. Clearly τ_{e_i} $(1 \leqslant i \leqslant 4)$, $\tau_{e_1-e_3}$, $\tau_{e_2-e_4} \in O(L)$.

Condidering $x = \frac{1}{\pi}\sum_{j=1}^{4} r_i e_i$, $r_i = a_i + b_i\sqrt{6}$, $a_i, b_i \in Z$ with $Q(x) \in \mathfrak{O}$, we have only following two cases to consider by Proposition 3 (ii), (iii), (iv):

(1) $x_1 = \frac{1}{\pi}(e_1 + e_2) - e_2$.

We have $L(x_1) = \sum_{i=1}^{4} \mathfrak{O}k_i' = \langle 1\rangle \perp \langle \varepsilon\rangle \perp \langle \varepsilon\rangle \perp \langle 1\rangle = L$. Here $k_1' = x_1$, $k_2' = \frac{\varepsilon}{\pi}(e_1 + e_2) - e_2$, $k_3' = e_3$, $k_4' = e_4$.

(2) $x_2 = \frac{1}{\pi}(e_1 + e_2 + e_3 + e_4) + e_2$.

It is easy to check that $L(x_2)$ has the base

$$k_1 = \frac{1}{\pi}(e_1 + e_2 + e_3 + e_4) - e_1,$$

$$k_2 = \frac{1}{\pi}(e_1 + e_2 + e_3 + e_4) + e_2,$$

$$k_3 = \frac{1}{\pi}(e_1 + e_2 + e_3 + e_4) - e_3$$

$$k_4 = \frac{1}{\pi}(e_1 + e_2 + e_3 + e_4) + e_4.$$

Hence

$$L(x_2) = \sum_{j=1}^{4} \mathfrak{O}k_i \cong \begin{bmatrix} 7 & 2(1-\sqrt{6}) & 2(2-\sqrt{6}) & 2(2-\sqrt{6}) \\ 2(1-\sqrt{6}) & 7 & 2(2-\sqrt{6}) & 2(2-\sqrt{6}) \\ 2(2-\sqrt{6}) & 2(2-\sqrt{6}) & 7-2\sqrt{6} & 2(3-\sqrt{6}) \\ 2(2-\sqrt{6}) & 2(2-\sqrt{6}) & 2(3-\sqrt{6}) & 7-2\sqrt{6} \end{bmatrix}.$$

Analogous to 3, II, A, (2) we can show $1, \varepsilon \notin Q(L(x_2))$. Hence $L(x_2) \not\cong L$, $L(x_2) \not\cong I_4$ and $L(x_2) \not\cong E_4$.

Remark. Using the fact $1, \varepsilon \notin Q(L(x_2))$ we can prove $L(x_2)$ is indecomposable.

B. The adjacent lattices of $S = L(x_2)$.

Let

$$y_1 = \frac{\varepsilon}{\pi}(e_1 + e_2 + e_3 + e_4) - e_1 - e_3,$$

$$y_2 = \frac{1}{\pi}(e_1 + e_2 + e_3 + e_5) - e_2 - e_4,$$

$$y_3=e_1-e_3, \quad y_4=e_2-e_4, \quad Y=\sum_{i=1}^{4}\mathfrak{O}y_i.$$

Clearly

$$k_1=\frac{\bar{\pi}-1}{2}y_1+\frac{1-\sqrt{6}}{2}y_2-\frac{1}{2}y_3,$$

$$k_2=\frac{\bar{\pi}}{2}y_1+\frac{\bar{\pi}-1}{2}y_2+\frac{1}{2}y_4,$$

$$k_3=\frac{\bar{\pi}-1}{2}y_1+\frac{1-\sqrt{6}}{2}y_2+\frac{1}{2}y_3,$$

$$k_4=\frac{\bar{\pi}}{2}y_1+\frac{\bar{\pi}-1}{2}y_2,$$

$2S\subseteq Y\subseteq S$ and $\tau_{y_i}\in O(S)$, σ: $y_1\to y_3$, $y_3\to y_1$, $y_2\to y_4$. $y_4\to y_2\in O(S)$. By proposition 3(ii) we may assume the generators of adjacent lattices of S are in $\frac{1}{\pi}Y$.

Considering $x=\frac{1}{\pi}\sum_{i=1}^{4}r_iy_i$, $r_i=a_i+b_i\sqrt{6}$, a_i, $b_i\in Z$ with $Q(x)\in\mathfrak{O}$, we have only following three cases to consider by Proposition 3 (ii), (iii), (iv):

(1) $x_1=\frac{1}{\pi}(y_1+y_2)-y_2$.

We have $S(x_1)\cong L$ since $x_1=e_2+e_4\in L-S$.

(2) $x_2=\frac{1}{\pi}(y_2+y_4)-y_4$.

Let

$$y_1'=\frac{\varepsilon}{\pi}(y_1+y_4)-y_1, \quad y_2'=y_2, \quad y_3'=y_3,$$

$$y_4'=\frac{1}{\pi}(y_1+y_4)-y_4,$$

$$k_1'=\frac{\bar{\pi}-1}{2}y_1'+\frac{1-\sqrt{6}}{2}y_2'-\frac{1}{2}y_3',$$

$$k_2'=\frac{\bar{\pi}}{2}y_1'+\frac{\bar{\pi}-1}{2}y_2'+\frac{1}{2}y_4',$$

$$k_3'=\frac{\bar{\pi}-1}{2}y_1'+\frac{1-\sqrt{6}}{2}y_2'+\frac{1}{2}y_3',$$

$$k_4'=\frac{\bar{\pi}}{2}y_1'+\frac{\bar{\pi}-1}{2}y_2'-\frac{1}{2}y_4', \quad M=\sum_{i=1}^{4}\mathfrak{O}k_i'.$$

Then $S(x_2)\cong S$ by Proposition 3(v).

(3) $x_3=\frac{1}{\pi}(y_1+y_2+y_3+y_4)+y_3-y_4$.

We have $x_3\sim x_3'=\frac{1}{\pi}(e_1+e_2)-e_2+e_4$ by Proposition 3 (ii).

Let

$$y_1'=\frac{\varepsilon}{\pi}(e_1+e_2)-e_1+e_3, \quad y_2'=\frac{1}{\pi}(e_1+e_2)-e_2+e_4,$$

$$y_3'=\frac{\varepsilon}{\pi}(e_3+e_4)-e_1-e_3, \quad y_4'=\frac{1}{\pi}(e_3+e_4)-e_2-e_4,$$

$$k_1'=\frac{\bar{\pi}-1}{2}y_1'+\frac{1-\sqrt{6}}{2}y_2'-\frac{1}{2}y_3',$$

$$k_2'=\frac{\bar{\pi}}{2}y_1'+\frac{\bar{\pi}-1}{2}y_2'+\frac{1}{2}y_4',$$

$$k_3'=\frac{\bar{\pi}-1}{2}y_1'+\frac{1-\sqrt{6}}{2}y_2'+\frac{1}{2}y_3'.$$

$$k_4'=\frac{\bar{\pi}}{2}y_1'+\frac{\bar{\pi}-1}{2}y_2'-\frac{1}{2}y_4',\quad M=\sum_{i=1}^{4}\mathfrak{O}k_i'.$$

Then $S(x_3)\cong S(x_3')\cong S$ by Proposition 3 (v).

C. The adjacent lattices of I_4.

Let $I_4=\perp_{i=1}^{4}\mathfrak{O}e_i$, $Q(e_i)=1$, $(1\leqslant i\leqslant 4)$. Clearly τ_{e_i}, $\tau_{e_i-e_j}$ $(i\neq j)\in O(L)$.

Considering $x=\frac{1}{\pi}\sum_{i=1}^{4}r_ie_i$, $r_i=a_i+b_i\sqrt{6}$, a_i, $b_i\in Z$ with $Q(x)\in\mathfrak{O}$, we may choose $x=\frac{\varepsilon}{\pi}\sum_{i=1}^{4}e_i$. We have $I_4(x)=\sum_{i=1}^{4}\mathfrak{O}l_i\cong\langle 1\rangle\perp\langle\varepsilon\rangle\perp\begin{pmatrix}2&\pi\\ \pi&2\varepsilon\end{pmatrix}$, where $l_1=e_1$, $l_2=x_1$, $l_3=e_2-e_3$, $l_4=\frac{\varepsilon}{\pi}(2e_2-e_3-e_4)$. Clearly $I_4(x)\not\cong I_4$, E_4.

D. The adjacent lattices of $T=I_4(x)$.

Let $z_1=l_1$, $z_2=l_2$, $z_3=e_3-e_4$, $z_4=l_4$. $Z=\sum_{i=1}^{4}\mathfrak{O}z_i$. Clearly $2T\subseteq Z\subseteq T$.

Hence we may assume the generators of adjacent lattices of T are in $\frac{1}{\pi}Z$. Note $\tau_{z_i}\in O(T)$.

Considering $y=\frac{1}{\pi}\sum_{i=1}^{4}r_iz_i$, $r_i=a_i+b_i\sqrt{6}$, a_i, $b_i\in Z$ with $Q(y)\in\mathfrak{O}$, we have only following five cases to consider by Proposition 3 (ii), (iii), (iv):

(1) $y_1=\frac{1}{\pi}(z_1+z_2)-z_2$.

We have

$$T(y_1)=\sum_{i=1}^{4}\mathfrak{O}k_i\cong\langle 1\rangle\perp\langle\varepsilon\rangle\perp\begin{pmatrix}2&\pi\\ \pi&2\varepsilon\end{pmatrix}\cong T.$$

Here

$$k_1=y_1,\quad k_2=\frac{\varepsilon}{\pi}(z_1+z_2)-z_2,\quad k_3=l_3,\quad k_4=l_4.$$

(2) $y_2=\frac{1}{\pi}(z_1+z_3)$.

We have

$$T(y_2)=\sum_{i=1}^{4}\mathfrak{O}k_i=\langle\varepsilon\rangle\perp\langle\varepsilon\rangle\perp\langle\varepsilon\rangle\perp\langle\varepsilon\rangle\cong E_4.$$

Here

$$k_1=y_2,\ k_2=\frac{1}{\pi}(e_2+e_3+e_4),$$

$$k_3=\frac{1}{\pi}(e_1-e_2+e_4),\quad k_4=\frac{1}{\pi}(e_1+e_2-e_3).$$

We know that I_4 has unique adjacent lattice which does not isometry to I_4. But $E_4=I_4^\varepsilon$. Hence E_4 has unique adjacent lattice which does not isometry to E_4 and the unique adjacent lattice is $T^\varepsilon\cong T$.

(3) $y_3=\frac{1}{\pi}(z_2+z_4)$.

We have $T(y_3)\cong I_4$ since $y_3=e_2\in I_4-T$.

(4) $y_4=\frac{1}{\pi}(z_3+z_4)-z_3$.

It is easy to know $T(y_4)=\langle 1\rangle\perp\langle\varepsilon\rangle\perp\left[\text{an adjacent lattice of }\begin{pmatrix}2&\pi\\\pi&2\varepsilon\end{pmatrix}\right]$. But the adjacent lattices of $\begin{pmatrix}2&\pi\\\pi&2\varepsilon\end{pmatrix}$ are equivalent to $\begin{pmatrix}2&\pi\\\pi&2\varepsilon\end{pmatrix}$ by I, B, (2). Hence $T(y_4)\cong T$.

(5) $y_5=\frac{1}{\pi}(z_1+z_2+z_3+z_4)$.

$$T(y_5)=\sum_{i=1}^{4}\mathfrak{O}h_i\cong\begin{bmatrix}2\bar\pi&\sqrt{6}-2&-1&-1\\\sqrt{6}-2&2\bar\pi&-(\sqrt{6}+2)&-(\sqrt{6}+1)\\-1&-(\sqrt{6}+2)&3\varepsilon&0\\-1&-(\sqrt{6}+1)&0&3\varepsilon\end{bmatrix}.$$

Here $h_1=e_1-\frac{1}{\pi}(e_2+e_3+e_4)$, $h_2=\frac{1}{\pi}(e_1-(2+\sqrt{6})e_3-(1+\sqrt{6})e_4)$, $h_3=\pi e_3$, $h_4=\pi e_4$. We can show $1, \varepsilon\notin Q(T(y_5))$. Hence $T(y_5)\ncong I_4, E_4, T$.

E. The adjacent lattices of $H=T(y_5)$.

We can check that σ_1: (e_2, e_3, e_4), σ_2: $(e_1, -e_2, e_4)$, $\sigma_3=(e_1, e_2, -e_3)$, σ_4: $(e_1, e_3, -e_4)$ and σ_5: $\begin{pmatrix}e_1&e_2&e_3&e_4\\e_2&-e_1&e_4&-e_3\end{pmatrix}$ are in $O(H)$.

Let $h_1'=y_5$, $h_2'=\sigma_4h_1$, $h_3'=\pi e_2$, $h_4'=h_3$, $H'=\sum_{i=1}^{4}\mathfrak{O}h_i'$. Then $2H\subseteq H'\subseteq H$. Thus we may assume the generators of the adjacent lattices of H are in $\frac{1}{\pi}H'$.

Considering $w=\frac{1}{\pi}\sum_{i=1}^{4}r_ih_i'$, $r_i=a_i+b_i\sqrt{6}$, a_i, $b_i\in Z$ with $Q(w)\in\mathfrak{O}$, we have following four cases to consider by Proposition 3 (ii), (iii), (iv):

(1) $w_1=e_3\sim-e_1$ (by σ_3) and $e_1\in T-H$, hence $H(w_1)\cong T$.

(2) $$w_2=\frac{1}{\pi}z_3-\pi e_2=\frac{1}{\pi}z_3-\frac{1}{2\pi}(\pi z_2-z_1-z_3+z_4)$$
$$=\frac{1}{2\pi}(z_1-\pi z_2+(11-4\sqrt{6})z_3+z_4)$$
$$\sim\frac{1}{\pi}(z_1-\pi z_2+(11-4\sqrt{6})z_3-z_4)\quad\text{(by Proposition 3(ii))}$$

$$\sim\frac{1}{\pi}(z_1-z_3-z_4) \quad \text{(by Proposition 3 (iii))}$$

$$\sim\frac{1}{\pi}(z_2+z_3-z_4) \quad (\text{by } \sigma_5)$$

$$\sim\frac{1}{\pi}(z_3+z_4-z_2)=w_2' \quad (\text{by } \sigma_1).$$

Here $z_1=h_1$, $z_2=y_5$, $z_3=\sigma_2 z_1$, $z_4=\sigma_1 z_3$.

We have $H(w_5')=\mathfrak{O}\left(\frac{1}{\pi}(-z_2+z_3+z_4)\right)+N$, where

$$N=\left\{\sum_{i=1}^{4}\alpha_i h_i,\ \alpha_i\in\mathfrak{O}\,|\,B\left(\sum_{i=1}^{n}\alpha_i h_i,\ w_5'\right)\in\mathfrak{O}\right\}$$
$$=\left\{\sum_{i=1}^{4}\alpha_i h_i,\ \alpha_i\in\mathfrak{O}\,|\,\alpha_3-\alpha_4\equiv 0 \pmod{\pi}\right\}$$
$$=\{\alpha_1 h_1+\alpha_2 h_2+\alpha_3(h_3+h_4)+\pi\alpha_4 h_4\,|\,\alpha_i\in\mathfrak{O}\}.$$

Note $\tau_{z_2}\in O(kI_4)$ and

$$\tau_{z_2}w_2'=\frac{1}{\pi}(z_2+z_3+z_4)\in H$$

and $\tau_{z_2}N\subset H$. Hence

$$H(w_2)=H(w_2')=H\left(\frac{1}{\pi}(z_2+z_3+z_4)\right)=H.$$

(3) $w_3=\frac{1}{\pi}(z_3+\pi e_3)=\frac{1}{\pi}z_3+e_3\sim-\left(\frac{1}{\pi}z_1+e_1\right)$ (by σ_3)

$$\sim\frac{1}{\pi}z_1+e_1=w_3'.$$

We have

$$H(w_3)=H(w_3')=\sum_{i=1}^{4}\mathfrak{O}t_i\cong\langle 1\rangle\perp\langle\varepsilon\rangle\perp\begin{pmatrix}2 & \pi\\ \pi & 2\varepsilon\end{pmatrix}=T.$$

Here
$$t_1=\frac{1}{3}(\sqrt{6}e_1-e_2-e_3-e_4),$$
$$t_2=\frac{1}{3}(\pi e_1+(\pi-1)e_2+(\pi-1)e_3+(\pi-1)e_4),$$
$$t_3=\frac{1}{3}((1+\sqrt{6})e_2+(1-\sqrt{6})e_3-2e_4),$$
$$t_4=\frac{1}{3}(2(\sqrt{6}+2)e_2+e_3-\varepsilon e_4).$$

(4) $w_4=\frac{1}{\pi}(z_3-\pi e_3)+z_2=\frac{1}{6}(2\bar{\pi}z_3-\pi z_3+z_1-z_2+z_4)+z_2$

$$\sim\frac{1}{3}(z_1-z_2+z_4)+2z_2 \quad \text{(by Proposition 3 (ii) (iii))}$$

$$\sim\frac{1}{3}(z_1-z_2+z_4)-z_2 \quad \text{(by Proposition 3 (iii))}$$

$$\sim\frac{1}{3}(z_2+z_3+z_4)+z_3=w_4' \quad (\text{by } \sigma_3).$$

Let $\sigma=\tau_{z_2+z_3+z_4}$. Then $\sigma w_4'=-(z_2+z_4)\in H$. On the other hand, it is easy to know that for any $y\in H$, $B(y,\ w_4')\in\mathfrak{O}\Leftrightarrow B\left(\frac{1}{3}(z_2+z_3+z_4),\ y\right)\in\mathfrak{O}$. Hence if $y\in H$ such that $B(y,\ w_4')\in\mathfrak{O}$, then

$$\begin{aligned}\sigma y&=y-\frac{2B(z_2+z_3+z_4,y)}{Q(z_2+z_3+z_4)}(z_2+z_3+z_4)\\&=y-B\left(\frac{1}{3}(z_2+z_3+z_4,\ y\right)\cdot\frac{1}{\pi}(z_2+z_3+z_4)\in H.\end{aligned}$$

Hence $H(w_4)\cong H(w_4')\cong H$ by Proposition 3 (iv).

Thus by Theorem 1 we have the following theorem.

Theorem 3 (2). *There are exactly six classes in* gen $(E_2\perp I_2)$ *and they are:*

(i) $E_2\perp I_2$, (ii) I_4, (iii) E_4, (iv) $\langle 1\rangle\perp\langle\varepsilon\rangle\perp\begin{pmatrix}2&\pi\\\pi&2\varepsilon\end{pmatrix}$,

$$\text{(v)}\ \begin{bmatrix}7&2(1-\sqrt{6})&2(2-\sqrt{6})&2(2-\sqrt{6})\\2(1-\sqrt{6})&7&2(2-\sqrt{6})&2(2-\sqrt{6})\\2(2-\sqrt{6})&2(2-\sqrt{6})&7-2\sqrt{6}&2(3-\sqrt{6})\\2(2-\sqrt{6})&2(2-\sqrt{6})&2(3-\sqrt{6})&7-2\sqrt{6}\end{bmatrix}$$

$$\begin{bmatrix}2(3-\sqrt{6})&\sqrt{6}-2&-1&-1\\\sqrt{6}-2&2(3-\sqrt{6})&-(\sqrt{6}+2)&-(\sqrt{6}+1)\\-1&-(\sqrt{6}+2)&3\varepsilon&0\\-1&-(\sqrt{6}+1)&0&3\varepsilon\end{bmatrix}.$$

From the calculation in this part we can deduce following theorem.

Theorem 3 (3). *There are exactly three classes in* gen $(E_1\perp I_2)$ *and* gen $(E_2\perp I_1)$ *respectively and they are*

(i) $E_1\perp I_2$, (ii) E_3, (iii) $\langle 1\rangle\perp\begin{pmatrix}2&\pi\\\pi&2\varepsilon\end{pmatrix}$ *and*

(i) $E_2\perp I_1$, (ii) I_3, (iii) $\langle\varepsilon\rangle\perp\begin{pmatrix}2&\pi\\\pi&2\varepsilon\end{pmatrix}$ *respectively.*

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