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CLASSIFICATION OF GENERALIZED CARTAN MATRICES OF HYPERBOLIC TYPE

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Abstract

In this paper, the author gives all Dynkin diagrams of generalized Cartan matrices of hyperbolic type, the order of hyperbolic generalized Cartan matrix is less than or equal to 10. All indefinite generalized Cartan matrices of order 2 are hyperbolic. The number of Dynkin diagrams of hyperbolic generalized Cartan matrices whose order is larger than 2 is 238. Meanwhile, the author finds all Dynkin diagrams of strictly hyperbolic generalized Cartan matrices.

§1. Introduction

The theory of Kac–Moody Lie algebras has close connections to many areas of mathematics and mathematical physics[2-6]. The structure of a Kac–Moody Lie algebra depends on its generalized Cartan matrix.

Now we introduce some concepts as following:

An $n\times n$ matrix $A=(a_{ij})$ is called a generalized Cartan matrix if t satisfies the following conditions:

(C1) $a_{ii}=2$ for $i=1, \cdots, n$;

(C2) a_{ij} are non-positive integers for $i\neq j$;

(C3) $a_{ij}=0$ implies $a_{ji}=0$.

A matrix A is said to be decomposable if, after reordering the indices (i.e., a permutation of its rows and the same permutation of the columns), A has the decomposition $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. Otherwise, A is called indecomposable. As in the case of finite dimension, we associate to a generalized Cartan matrix A a graph $S(A)$, called the Dynkin diagram of A. The set of vertices of $S(A)$ is $\{1, 2, \cdots, n\}$, the vertices i and $j (i\neq j)$ are connected as follows. If $a_{ij}a_{ji}<4$, and $|a_{ij}|\geqslant|a_{ji}|$, the vertices i and j are connected by $|a_{ij}|$ lines, and these lines are equipped with an arrow pointing toward i if $|a_{ij}|>1$. If $a_{ij}a_{ji}>4$, the vertices i and j are connected by a boldfaced line equipped with an ordered pair of integers $|a_{ij}|$, $|a_{ij}|$. It is clear that A is

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indecomposable if and only if $S(A)$ is a connected graph.

If A is an indecomposable generalized Cartan matrix, then one and only one of the following three possibilities holds for A:

(Fin) $\exists\alpha>0$ such that $A\alpha>0$;

(Aff) $\exists\alpha>0$ such that $A\alpha=0$;

(Ind) $\exists\alpha>0$ such that $A\alpha<0$,

where α is a real column vector $\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_2\end{pmatrix}$, $\alpha>0$ means that all $\alpha_i>0$[1]. Refering to cases (Fin), (Aff) or (Ind), we will say A and its graph $S(A)$ are of finite, affine or indefinite type respectively. An indecomposable generalized Cartan matrix A is said to be of strictly hyperbolic type (resp. hyperbolic type) if it is of indefinite type and any connected proper subdiagram of $S(A)$ is of finite (resp. finite or affine) type.

The diagrams of finite and affine types are given in [1]. In this paper, we will give all Dynkin diagrams of generalized Cartan matrices of hyperbolic type.

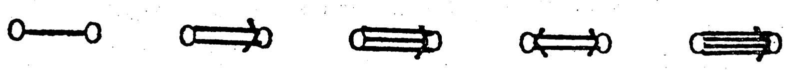
§ 2. Classification

Proposition 1. *If the order $n=2$, all generalized Cartan matrices of indefinite type are of strictly hyperbolic type.*

Lemma 1. *If A is a hyperbolic matrix of order n, S_1 is a proper connected affine subdiagram of $S(A)$, then S_1 has $n-1$ vertices.*

Proof The result follows from the fact: any diagram of finite or affine type has no proper connected affine subdiagram.

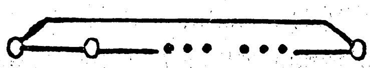
Lemma 2. *If $n\geqslant 3$ and $a_{ij}\neq 0$, the subdiagram consisting of the vertices i and j is one of the following forms:*



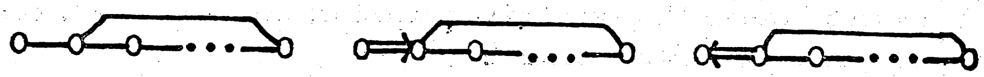
If $n\geqslant 4$, $a_{ij}\neq 0$, the subdiagram is one of the first three graphs. If $n\geqslant 5$, $a_{ij}\neq 0$, the subdiagram is one of the first two graphs.

Proposition 2. *The order of a hyperbolic matrix $n\leqslant 10$.*

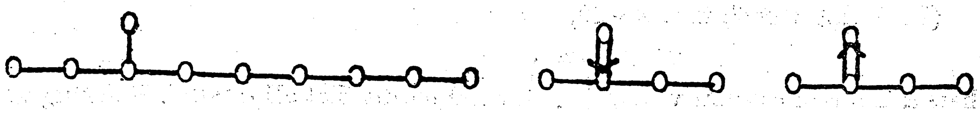
Proof Let $S(A)$ be the Dynkin diagram of a generalized Cartan matrix A of hyperbolic type and the order of A $n>10$. If there is a cycle in $S(A)$ and the number of vertices in the cycle is not n, the cycle must be of affine type. By Lemma 1, the cycle has $n-1$ vertices. It is of the form



By Lemma 2, $S(A)$ is one of the following forms:



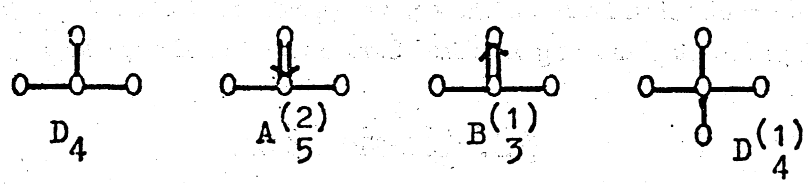
They have proper subdiagrams



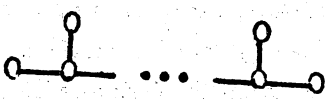
respectively. But these subdiagrams are neither of affine type nor of finite type. So $S(A)$ has no cycle $A_{n-2}^{(1)}$.

If the cycle has n vertices, then it has a subdiagram $0\Rightarrow 0$. In this case, $S(A)$ has a subdiagram $F_4^{(1)}$, $C_l^{(1)}$, $A_{2l}^{(2)}$ or $D_{l+1}^{(2)}$, $l<n-2$. By Lemma 1, this is impossible.

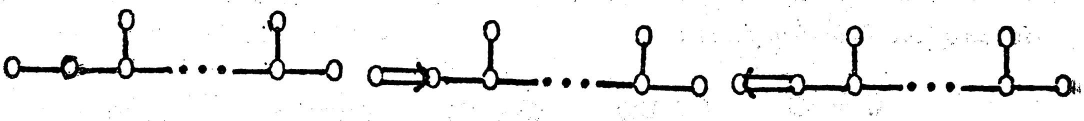
If $S(A)$ has branch vertices, any branch vertex has one of the forms



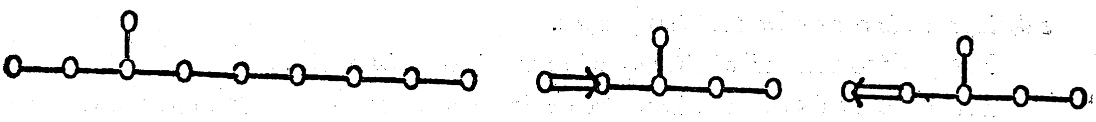
They are of affine type except first one. By Lemma 1, the branch vertex in $S(A)$ must be D_4. If $S(A)$ has two branch vertices, it has subdiagram



This is of affine type. Hence it has $n-1$ vertices. $S(A)$ has the form



They have subdiagrams



respectively. This is impossible. If $S(A)$ has only one branch vertex, $S(A)$ has a subdiagram D_{n-1}, $B_{n-2}^{(1)}$ or $A_{2n-5}^{(2)}$. But from these diagrams we cannot get hyperbolic diagram if $n>10$.

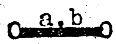
It remains to show for $S(A)$ a chain. If the chain has only one subdiagram $0\Rightarrow 0$, then $S(A)$ has proper subdiagram $F_4^{(1)}$ or $E_6^{(1)}$, this is impossible. If the chain has two subdiagrams of that form, then it has proper subdiagram $C_l^{(1)}$, $A_{2l}^{(2)}$ or $D_{l+1}^{(2)}$. By Lemma 1, $l=n-2$. $S(A)$ is obtained from one of these diagrams by adding one vertex in one line. It is not difficult to see that $S(A)$ has proper subdiagram $F_4^{(1)}$ or $E_6^{(2)}$. This is a contradiction.

Proposition 3. *If* $3 \leqslant n \leqslant 10$, *the Dynkin diagrams of hyperbolic matrices are listed in Section* 3. *In this case, we have* 238 *hyperbolic diagrams, in which* 35 *diagrams are strictly hyperbolic,* 142 *diagrams are symmetric or symmetrizable.*

The discussion is the same as that of Proposition 2, we omit the detail.

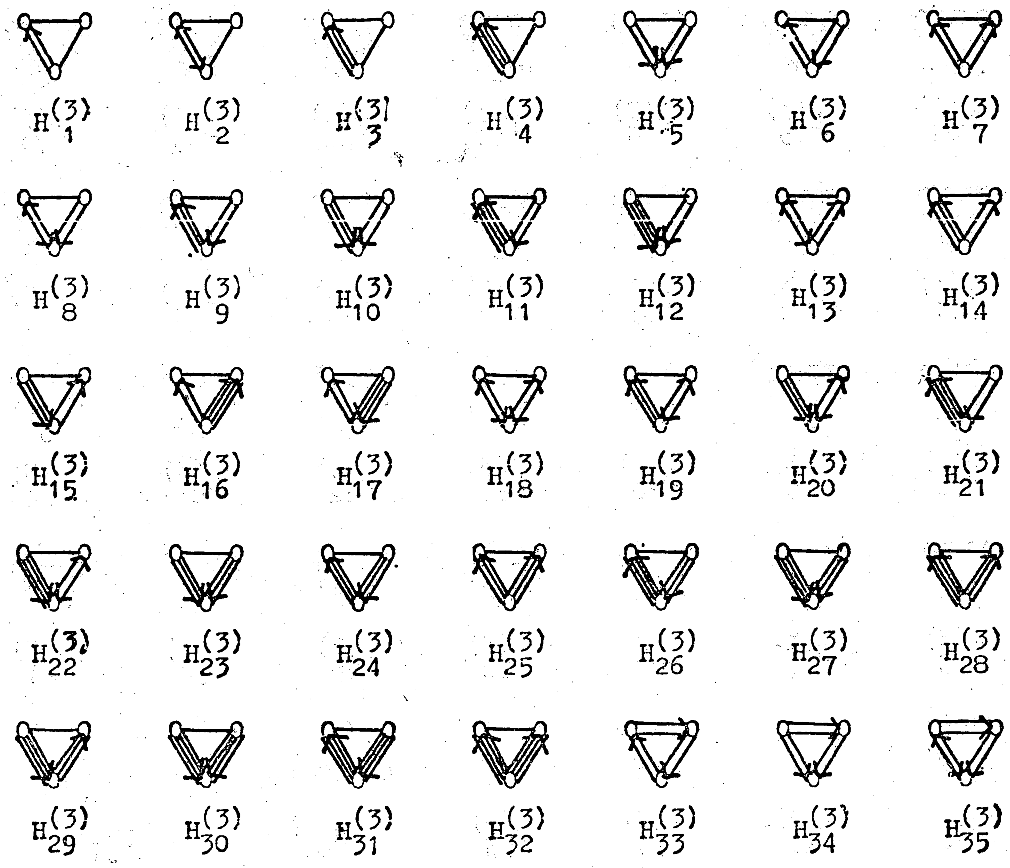
§3. Table of Dynkin Diagrams of Hyperbolic Matrices

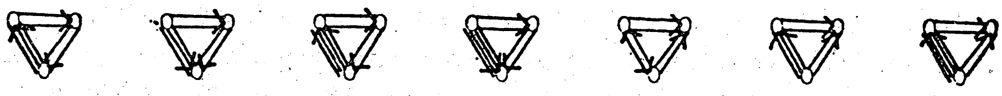
Hyp. 2



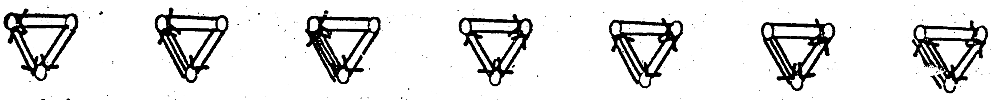
a,b ∈ N, ab > 4, a ≥ b

Hyp. 3

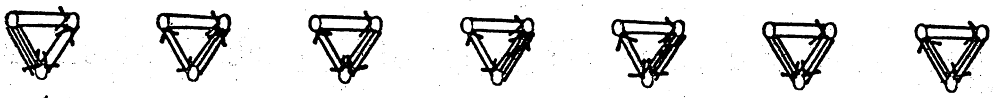




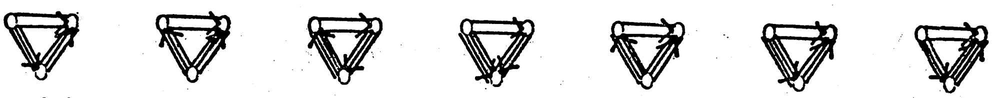
$H^{(3)}_{36}$ $H^{(3)}_{37}$ $H^{(3)}_{38}$ $H^{(3)}_{39}$ $H^{(3)}_{40}$ $H^{(3)}_{41}$ $H^{(3)}_{42}$



$H^{(3)}_{43}$ $H^{(3)}_{44}$ $H^{(3)}_{45}$ $H^{(3)}_{46}$ $H^{(3)}_{47}$ $H^{(3)}_{48}$ $H^{(3)}_{49}$

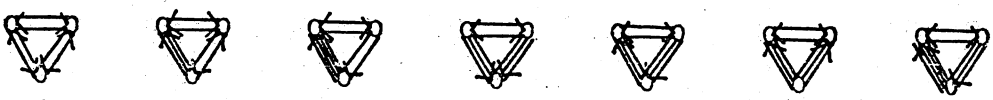


$H^{(3)}_{50}$ $H^{(3)}_{51}$ $H^{(3)}_{52}$ $H^{(3)}_{53}$ $H^{(3)}_{54}$ $H^{(3)}_{55}$ $H^{(3)}_{56}$

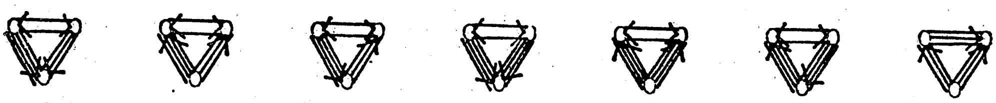


$H^{(3)}_{57}$ $H^{(3)}_{58}$ $H^{(3)}_{59}$ $H^{(3)}_{60}$ $H^{(3)}_{61}$ $H^{(3)}_{62}$ $H^{(3)}_{63}$

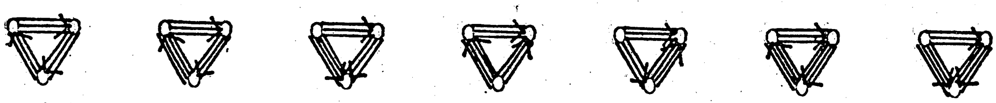
$H^{(3)}_{64}$ $H^{(3)}_{65}$ $H^{(3)}_{66}$ $H^{(3)}_{67}$ $H^{(3)}_{68}$ $H^{(3)}_{69}$ $H^{(3)}_{70}$



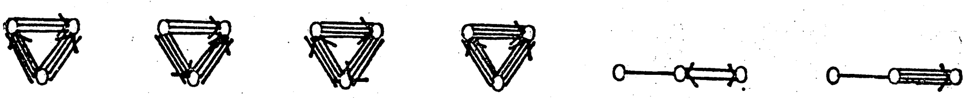
$H^{(3)}_{71}$ $H^{(3)}_{72}$ $H^{(3)}_{73}$ $H^{(3)}_{74}$ $H^{(3)}_{75}$ $H^{(3)}_{76}$ $H^{(3)}_{77}$



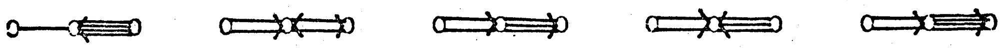
$H^{(3)}_{78}$ $H^{(3)}_{79}$ $H^{(3)}_{80}$ $H^{(3)}_{81}$ $H^{(3)}_{82}$ $H^{(3)}_{83}$ $H^{(3)}_{84}$



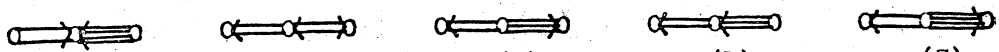
$H^{(3)}_{85}$ $H^{(3)}_{86}$ $H^{(3)}_{87}$ $H^{(3)}_{88}$ $H^{(3)}_{89}$ $H^{(3)}_{90}$ $H^{(3)}_{91}$



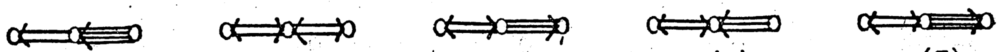
$H^{(3)}_{92}$ $H^{(3)}_{93}$ $H^{(3)}_{94}$ $H^{(3)}_{95}$ $H^{(3)}_{96}$ $H^{(3)}_{97}$



$H^{(3)}_{98}$ $H^{(3)}_{99}$ $H^{(3)}_{100}$ $H^{(3)}_{101}$ $H^{(3)}_{102}$

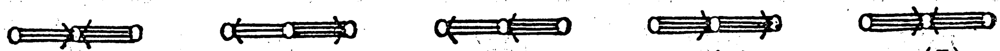


$H^{(3)}_{103}$ $H^{(3)}_{104}$ $H^{(3)}_{105}$ $H^{(3)}_{106}$ $H^{(3)}_{107}$

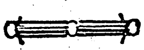


$H^{(3)}_{108}$ $H^{(3)}_{109}$ $H^{(3)}_{110}$ $H^{(3)}_{111}$ $H^{(3)}_{112}$

$H^{(3)}_{113}$ $H^{(3)}_{114}$ $H^{(3)}_{115}$ $H^{(3)}_{116}$ $H^{(3)}_{117}$

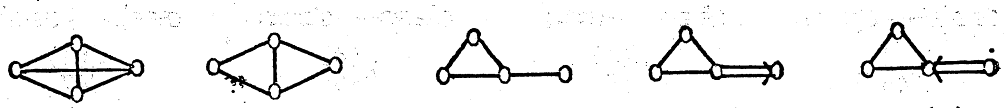


$H^{(3)}_{118}$ $H^{(3)}_{119}$ $H^{(3)}_{120}$ $H^{(3)}_{121}$ $H^{(3)}_{122}$

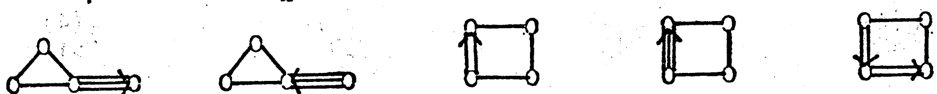


$H^{(3)}_{123}$

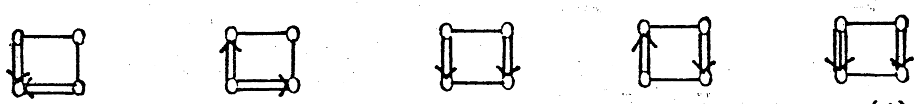
Hyp. 4



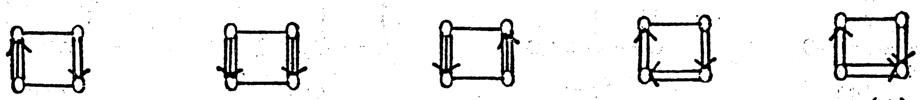
$H^{(4)}_{1}$ $H^{(4)}_{2}$ $H^{(4)}_{3}$ $H^{(4)}_{4}$ $H^{(4)}_{5}$



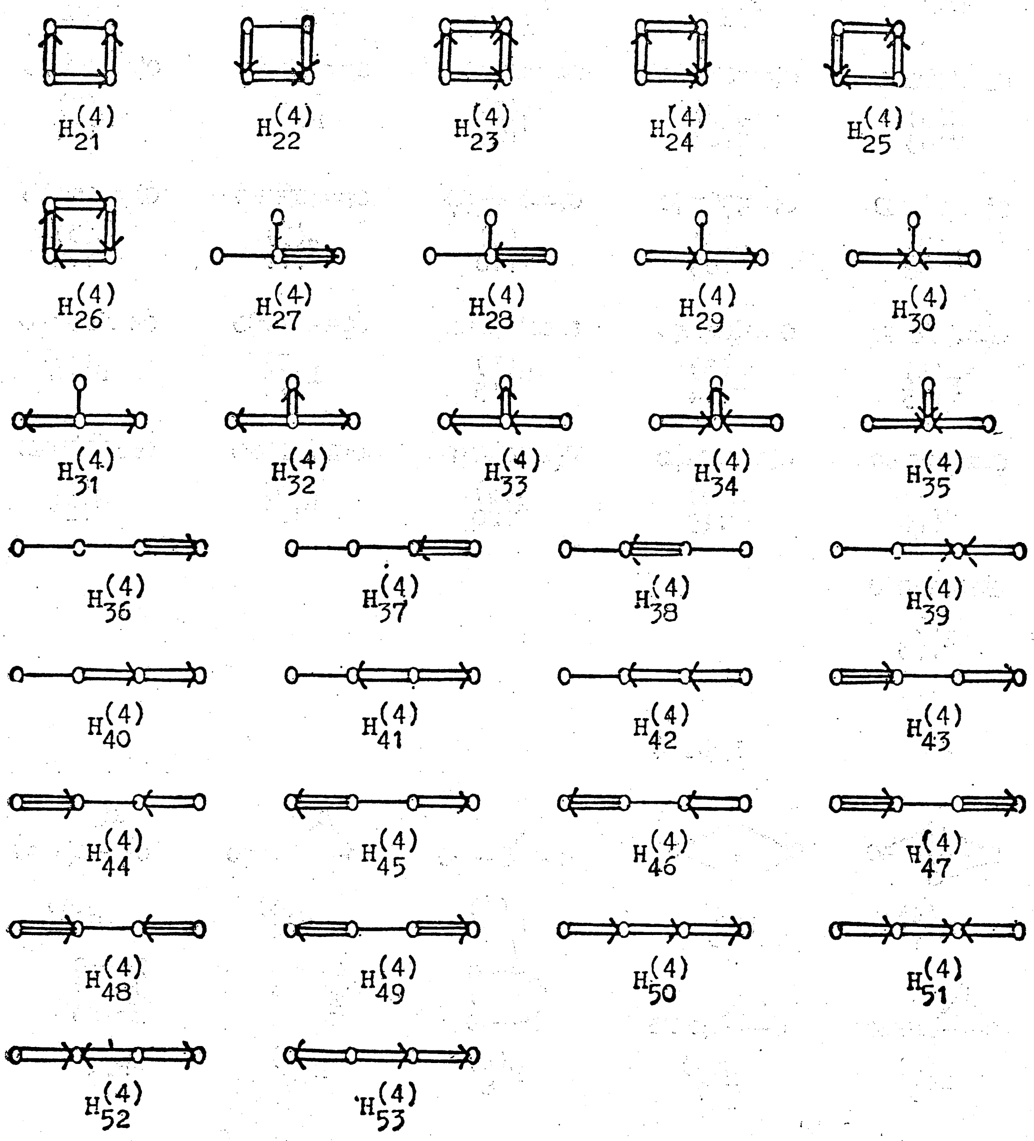
$H^{(4)}_{6}$ $H^{(4)}_{7}$ $H^{(4)}_{8}$ $H^{(4)}_{9}$ $H^{(4)}_{10}$



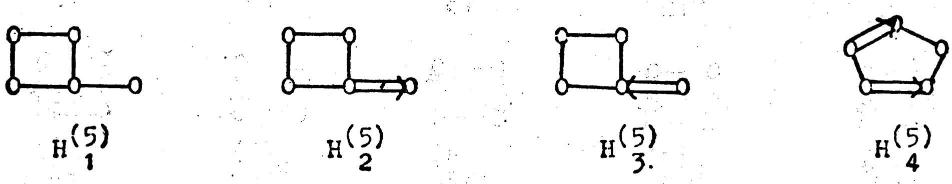
$H^{(4)}_{11}$ $H^{(4)}_{12}$ $H^{(4)}_{13}$ $H^{(4)}_{14}$ $H^{(4)}_{15}$

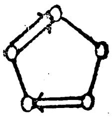


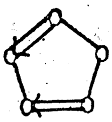
$H^{(4)}_{16}$ $H^{(4)}_{17}$ $H^{(4)}_{18}$ $H^{(4)}_{19}$ $H^{(4)}_{20}$


$H^{(4)}_{21}$ $H^{(4)}_{22}$ $H^{(4)}_{23}$ $H^{(4)}_{24}$ $H^{(4)}_{25}$
$H^{(4)}_{26}$ $H^{(4)}_{27}$ $H^{(4)}_{28}$ $H^{(4)}_{29}$ $H^{(4)}_{30}$
$H^{(4)}_{31}$ $H^{(4)}_{32}$ $H^{(4)}_{33}$ $H^{(4)}_{34}$ $H^{(4)}_{35}$
$H^{(4)}_{36}$ $H^{(4)}_{37}$ $H^{(4)}_{38}$ $H^{(4)}_{39}$
$H^{(4)}_{40}$ $H^{(4)}_{41}$ $H^{(4)}_{42}$ $H^{(4)}_{43}$
$H^{(4)}_{44}$ $H^{(4)}_{45}$ $H^{(4)}_{46}$ $H^{(4)}_{47}$
$H^{(4)}_{48}$ $H^{(4)}_{49}$ $H^{(4)}_{50}$ $H^{(4)}_{51}$
$H^{(4)}_{52}$ $H^{(4)}_{53}$

Hyp. 5


$H^{(5)}_{1}$ $H^{(5)}_{2}$ $H^{(5)}_{3}$ $H^{(5)}_{4}$


$H^{(5)}_{5}$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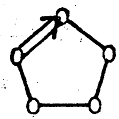
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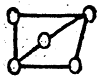
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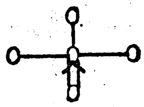
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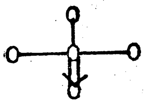
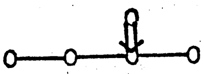

$H^{(5)}_{10}$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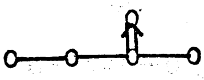
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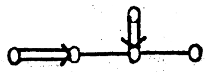
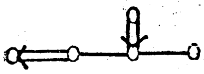

$H^{(5)}_{13}$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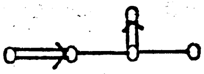
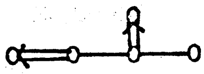

$H^{(5)}_{15}$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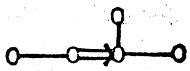
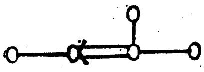
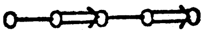
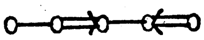
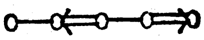
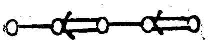

$H^{(5)}_{17}$


$H^{(5)}_{18}$

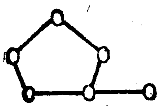

$H^{(5)}_{19}$

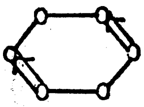

$H^{(5)}_{20}$

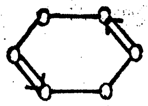

$H^{(5)}_{21}$

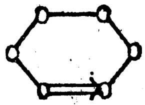

$H^{(5)}_{22}$

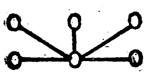
Hyp. 6


$H^{(6)}_{1}$


$H^{(6)}_{2}$


$H^{(6)}_{3}$


$H^{(6)}_{4}$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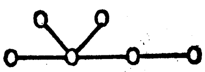
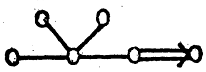
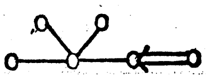
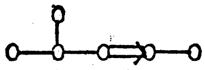
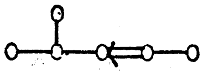

$H^{(6)}_{6}$


$H^{(6)}_{7}$


$H^{(6)}_{8}$

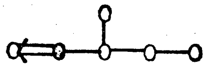


$H^{(6)}_{9}$

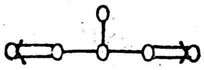


$H^{(6)}_{10}$

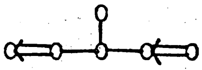
$H^{(6)}_{11}$



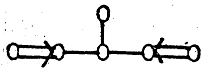
$H^{(7)}_{12}$



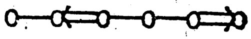
$H^{(6)}_{13}$



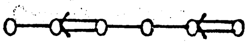
$H^{(6)}_{14}$



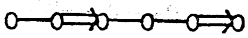
$H^{(6)}_{15}$



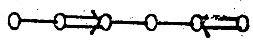
$H^{(6)}_{16}$



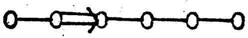
$H^{(6)}_{17}$



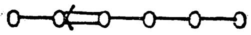
$H^{(6)}_{18}$



$H^{(6)}_{19}$



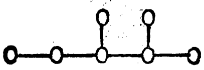
$H^{(6)}_{20}$



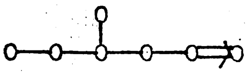
$H^{(6)}_{21}$

$H^{(6)}_{22}$

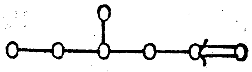
Hyp. 7



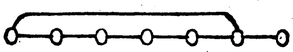
$H^{(7)}_{1}$



$H^{(7)}_{2}$

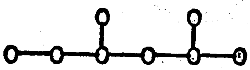


$H^{(7)}_{3}$

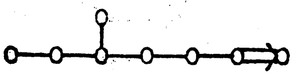


$H^{(7)}_{4}$

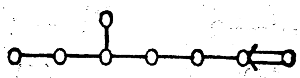
Hyp. 8



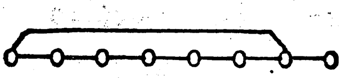
$H^{(8)}_{1}$



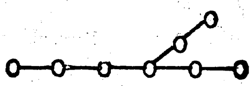
$H^{(8)}_{2}$



$H^{(8)}_{3}$

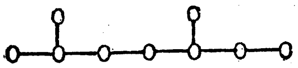


$H^{(8)}_{4}$

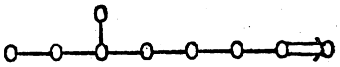


$H^{(8)}$

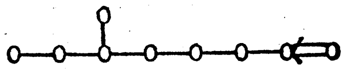
Hyp. 9



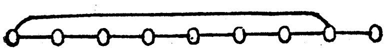
$H_1^{(9)}$



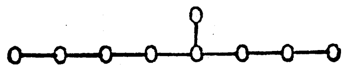
$H_2^{(9)}$



$H_3^{(9)}$

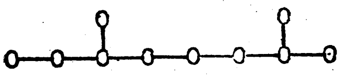


$H_4^{(9)}$

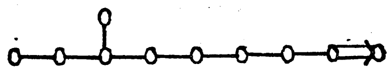


$H_5^{(9)}$

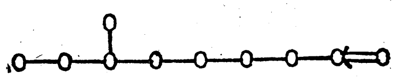
Hyp. 10



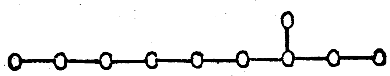
$H_1^{(10)}$



$H_2^{(10)}$



$H_3^{(10)}$



$H_4^{(10)}$

where $H_1^{(3)}$, $H_3^{(3)}$, $H_5^{(3)}$, $H_6^{(3)}$, $H_7^{(3)}$, $H_9^{(3)}$, $H_{10}^{(3)}$, $H_{14}^{(3)}$, $H_{15}^{(3)}$, $H_{23}^{(3)}$, $H_{24}^{(3)}$, $H_{25}^{(3)}$, $H_{33}^{(3)}$, $H_{34}^{(3)}$, $H_{36}^{(3)}$, $H_{37}^{(3)}$, $H_{41}^{(3)}$, $H_{44}^{(3)}$, $H_{55}^{(3)}$, $H_{56}^{(3)}$, $H_{57}^{(3)}$, $H_{58}^{(3)}$, $H_{84}^{(3)}$, $H_{85}^{(3)}$, $H_{100}^{(3)}$, $H_{101}^{(3)}$, $H_{105}^{(3)}$, $H_{106}^{(3)}$, $H_{114}^{(3)}$, $H_{115}^{(3)}$, $H_{116}^{(3)}$, $H_8^{(4)}$, $H_{13}^{(4)}$, $H_{14}^{(4)}$, $H_7^{(5)}$ are the diagrams of strictly hyperbolic matrices of the order $n \geqslant 3$.

I am very grateful to my teacher Professor Hao Bingxin for his advice and encouragement and to Mr. Lu Caihui for pointing out the diagram $H_8^{(5)}$.

Recently, I am told that Kobayashi and Morita[7] has listed all symmetric and symmetrizable hyperbolic diagrams. These diagrams are a part of that we get in this paper.

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[1] Kac, V. G., Infinite Dimensional Lie Algebras, Birkhäuser (1983).

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