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ON THE CONSTRUCTION OF INDECOMPOSABLE POSITIVE DEFINITE QUADRATIC FORMS OVER $\mathbb{Z}$*

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Abstract

The aim of this paper is to construct indecomposable positive definite $\mathbb{Z}$-lattices with given rank $n(2\leqslant n\leqslant 9)$ and discriminant a, a being composite and square-free, but for a finite number of exceptions. And all exceptional cases for $3\leqslant n\leqslant 9$ are determined. These are unsettled cases of a paper of O'Meara in 1975.

§ 1. Introduction

Iet $\mathbb{Q}$ be the field of rational numbers, V a regular quadratic space over $\mathbb{Q}$ and let L be a $\mathbb{Z}$-lattice on V. If L can be expressed as the othogonal sum of two non-zero sublattices $L=P\perp R$, L is called decomposable or splitting. If there is no such expression we call L indecomposable. By an integral quadratic form we mean one whose associated symmetric matrix has integral entries or, in the language of lattices, one whose associated quadratic lattice has integral scale.

In 1975 and 1980 O. T. O'Meara[1,2] investigated on the existence of the indecomposable positive definite lattices over $\mathbb{Z}$ with given rank and given discriminant. In his 1975's paper he proved the following theorem.

Theorem M[1]. *Let n and a be natural numbers. Then*

(1) *for any n, if a is a prime or a is not square-free, or*

(2) *for any $a\geqslant 2$, if $n\geqslant 10$,*

there are n-ary indecomposable positive definite integral $\mathbb{Z}$-lattices L with discrimsnant $dL=a$, but for a finite number of exceptions.

The aim of this paper is to prove some results analogous to those of O'Meara, namely, for each n $(2\leqslant n\leqslant 9)$, there are n-ary indecomposable positive definite $\mathbb{Z}$-lattices with given discriminant $dL=a$, a being composite and square-free, but for a finite number of exceptions. And we determine all the exceptional cases for $3\leqslant n\leqslant 9$. The proof in general is based upon a method of O'Meara[1], but that of the binary

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case ($n=2$) is quite different from the other cases ($n>2$). When $n=2$, we apply some analytic method, and give a necessary and sufficient condition for the existence of the binary indecomposable positive definite $\mathbb{Z}$-lattice with given discriminant a.

For the notations in this paper we follow the book of O'Meara[3].

§ 2. Main Results

Theorem 1. *For any natural numbers n ($3\leqslant n\leqslant 9$) and a, a being composite and square-free, there are n-ary indecomposable positive definite $\mathbb{Z}$-lattices with*

$$\mathfrak{s}L\subseteq\mathbb{Z} \quad \text{and} \quad dL=a,$$

but for the following twelve exceptions:

$$n=3,\ a=6,\ 14,\ 15;\ n=4,\ a=6,\ 10,\ 14,\ 26;\ n=5,\ a=10;$$
$$n=6,\ a=6,\ 14;\ n=8,\ a=6,\ 10.$$

By Theorem 1 and the results of O'Meara[1] and Kneser[5] we have the following theorem.

Theorem 2. *For any natural numbers a and $n\geqslant 3$, there are n-ary indecomposable positive definite $\mathbb{Z}$-lattices with*

$$\mathfrak{s}L\subseteq\mathbb{Z} \quad \text{and} \quad dL=a,$$

but for 44 exceptions, which are listed in the following table:

Table of exceptions

n	dL								
3	1	2	3	5	6	9	11	14	15
4	1	2	3	6	7	10	14	26	
5	1	2	3	5	7	10	13		
6	1	2	5	6	14				
7	1	3	7						
8	2	6	10						
9	1	2	3	5					
10	1	3							
11	1								
12	2								
13	1								

Theorem 3. *For any natural number a, there are binary indecomposable positive definite $\mathbb{Z}$-lattices L with discriminant a, but for a finite number of exceptions.*

In order to find the exceptional discriminant $dL=a$, we require the following theorem.

Theorem 4. *Suppose a is square-free and $a>3$, there exists a binary indecompo-sable positive definite $\mathbb{Z}$-lattice of discriminant a if and only if there is an odd prime p, $p<\sqrt{4a/3}$ such that $-a$ is a quadratic residue* mod p.

§ 3. Proofs of Theorems 1–4

In [1] O'Meara proved the following proposition which was used in the proof of Theorem M (2).

Proposition #. *Let M be an n-ary positive definite lattice over $\mathbb{Z}$ with $\mathfrak{s}M \subseteq \mathbb{Z}$ and $n>1$, and let $M=M_1 \perp \cdots \perp M_t$ be its indecomposable splitting. Let $w=w_1+\cdots+w_t$ $(w_i \in M_i)$ be a vector in M with all w_i non-zero and with $Q(w)=p$ where p is a prime number and prime to dM. Then $L=M \cap (Qw)^*$ is an $(n-1)$-ary indecomposable positive definite lattice with $\mathfrak{s}L \subseteq \mathbb{Z}$ and $dL=p \cdot dM$.*

By this proposition O'Meara proved Theorem M (2) for $n \geqslant 10$ and natural number $a \geqslant 2$.

Because of the Proporsition #, in the following discussions we need only construct M and give an expression of the prime p, and when we say (M, p) satisfies # it means that M and p satisfy the conditions in Proposition #. In the following, the natural number a is composite but square-free, the lattices in discussions are positive definite integral lattices. If a and m are natural numbers, write $a=\square_m$ to signify that a is a sum of m non-zero squares, i.e. $a=n_1^2+\cdots+n_m^2$, $n_i \neq 0$, $n_i \in \mathbb{Z}$. We need the following facts about $\square_m$—— one may refer to [9].

If $a \geqslant 19$, a is odd, $a \neq 29, 41$, then $=\square_4$;
If $a \geqslant 19$ with $a \neq 33$, then $a=\square_5$;
If $a \geqslant 20$, then $a=\square_6$;
If $a \geqslant 21$, then $a=\square_7$;

and so on.

1. The cases for $6 \leqslant n \leqslant 9$

Proposition 1. *For any square-free composite number a, there are n-ary indecomposable positive definite $\mathbb{Z}$-lattices L with $dL=a$, but for the 4 exceptions: $n=8$, $a=6, 10$; $n=6$, $a=6, 14$. In the exceptional cases there are no lattices with the desired properties.*

Proof We prove the proposition for the case $n=6$ only, the arguments being similar for the other cases. First we consider that a is odd. Therefore we may assume that a has the form: $a=p_1 \cdots p_s$ with odd primes $p_1<\cdots<p_s$, where $s \geqslant 2$. Let

$$M_1=\begin{pmatrix} 2 & 1 \\ 1 & * \end{pmatrix} \perp I_5,$$

$$M_2=\begin{pmatrix} 2 & 1 \\ 1 & *1 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & *2 \end{pmatrix} \perp I_3.$$

(1) Suppose there is a prime p, $p \geqslant 19$ or $p=7, 13$, and $p \mid a$. We construct $M=$

M_1 with $dM=a/p$, and we have $p=2+\square_5$. So by proposition # we get an indecomposable lattice L, with $dL=dM\cdot p=a$.

In the following we assume that

$$a=p_1\cdots p_s,\ p_i\in\{3,5,11,17\},\quad s\geqslant 2.$$

(2) Suppose $p_s=17$. If $s=2$, construct $M=M_1$ with $dM=a/p_s$. Then the $*$ in M must be 2, 3 or 6; and $p_s=3+\square_5=6+\square_5$. Thus (M, p_s) satisfies proposition #. If $s\geqslant 3$, construct $M=M_2$ with $dM=a/p_s$ and $*1=3$ or 6. Then we get $P_s=*1+2+\square_3$, and so (M, P_s) satisfies Proposition #.

(3) Suppose $p_s=11$.

For $a=3\cdot 11$, construct $M=I_8$ in the base $\{e_1, \cdots, e_8\}$, and put $x=2e_1+\sum_{i=2}^{8} e_i$. Then $Q(x)=11$ and $M'=I_8\cap(\mathbb{Q}x)^*$ is a 7-ary indecomposable lattice by Proposition # with $dM'=11$. Put $y=e_1-e_2-e_3$. Then $y\in M'$, $Q(y)=3$ and $L=M'\cap(Qy)^*$ is a 6-ary indecomposable lattice with $dL=3\cdot 11$. It is clear that $2\in Q(L)$.

For $a=5\cdot 11$, construct $M=M_1$ with $dM=5$, $*=3$ and we have $p_s=11=3+\square_5$. So (M, p_s) satisfies Proposition #.

For $a=3\cdot 5\cdot 11$, construct $M=M_2$ with $*1=3$ and $dM=a/p_s$. Then (M, p_s) satisfies Proposition # for $p_s=3+2+\square_3$.

(4) Suppose $p_s=5$, i.e. $a=3\cdot 5$.

From the proof of ([1], 5.10) we know that there is a 6-ary indecomposable lattice M', $dM'=3$ and M' represents $(4)\perp(2)$. Construct $M=M'\perp(1)$. By Proposition # we get a 6-ary indecomposable lattice L with $dL=15$ and $2\in Q(L)$.

Now consider that a is even and has the form: $a=2\cdot p_1\cdots p_s$ with odd primes $p_1<\cdots<p_s$, where $s\geqslant 1$. Let

$$M_1=(2)\perp I_6,\ M_2=(2)\perp\begin{pmatrix}2&1\\1&*\end{pmatrix}\perp I_4,\ M_3=(2)\perp\begin{pmatrix}2&1\\1&*1\end{pmatrix}\perp\begin{pmatrix}2&1\\1&*2\end{pmatrix}\perp I_2.$$

(1) Suppose that there is a prime p, $p\geqslant 17$ or $p=11$ and $p\mid a$. If $s=1$, construct $M=M_1$ with $dM=a/p$, and (M, p) satisfies # (i.e. Proposition #). If $s\geqslant 2$, construct $M=M_2$ with $dM=a/p$, and (M, p) satisfies #.

In the following we can assume that

$$a=2\cdot p_1\cdots p_s,\ s\geqslant 1 \text{ and } p_i\in\{3, 5, 7\ 13\}.$$

(2) Suppose $p_s=13$.

For $s=1$, we know from the proof of ([1], 5.10) that there is a 7-ary indecomposable lattice M, $dM=13$ and M represents 2. So $(M, 2)$ satisfies #.

For $s=2$,

if $a=2\cdot 3\cdot 13$, let $M=(2)\perp(3)\perp I_5$;

if $a=2\cdot 7\cdot 13$, let $M=\begin{pmatrix}2&1&\\1&2&1\\&1&3\end{pmatrix}\perp(2)\perp I_3$;

if $a=2\cdot5\cdot13$, let $M=\begin{pmatrix}2&1&\\1&2&1\\&1&4\end{pmatrix}\perp I_4$ (see Lemma 4 in Section 2).

In these cases (M, p_s) satisfies $\#$.

For $s\geqslant3$,

if $7\mid a$ let $M=M_3$ with $*1=4$, $dM=a/p_s$;

if $7\nmid a$ let $M=(2)\perp(3)\perp I_3\perp\begin{pmatrix}2&1\\1&3\end{pmatrix}$.

In these cases (M, p_s) satisfies $\#$.

(3) Suppose $p_s=7$.

For $s=1$, i.e. $a=14$. It is an exception and will be treated later.

For $s=2$. If $a=2\cdot3\cdot7$, let $M''=\Phi_8\perp(1)$ in the base $\{e_1, \cdots, e_9\}$ and put $x=e_3+e_6+e_8+e_9$. Then $Q(x)=7$ and $M'=M''\cap(\mathbb{Q}x)^*$ is a 8-ary indecomposable lattice with $dM'=7$. Put $y=-e_4+e_9$. Then $y\in M'$ and $M=M'\cap(\mathbb{Q}y)^*$ is a 7-ary indecomposable lattice with $dM=21$. It is clear that $(M, 2)$ satisfies $\#$.

If $a=2\cdot5\cdot7$, we know from the proof in (4) later that there is a 6-ary indecomposable lattice M' with $dM'=10$ and $6\in Q(M')$. Construct $M=M'\perp(1)$ and $7=6+1$, so $(M, 7)$ satisfies $\#$.

For $a=2\cdot3\cdot5\cdot7$, let $M''=\Phi_8\perp(1)$ in the base $\{e_1, \cdots, e_9\}$ and put $x=e_2+e_9$. Then $Q(x)=5$ and $M'=M''\cap(\mathbb{Q}x)^*$ is a 8-ary indecomposable lattice with $dM'=5$. Put $y=e_1-e_9$. Then $y\in M'$ and $M=M'\cap(\mathbb{Q}y)^*$ is a 7-ary indecomposable lattice with $dM=15$. Put $z=e_4$. Then $z\in M$ and $N'=M\cap(\mathbb{Q}z)^*$ is a 6-ary indecomposable lattice with $dN'=30$. It is clear that $6\in Q(N')$. Construct $N=N'\perp(1)$, and $(N, 7)$ satisfies $\#$.

(4) Suppose $p_s=5$.

For a$=10$, the result holds from ([1], 5.4).

For $a=30$, there is a 7-ary indecomposable lattice M from ([1], 5.4) with $dM=15$ and $2\in Q(M)$, (so M, 2) satisfies $\#$.

(5) Finally, suppose $p_s=3$, i.e. $a=6$. It is an exception and treated below.

Now we consider the cases $n=6$, $a=6, 14$.

For $a=14$, we know from M. Kneser's paper[5] that every 6-ary lattice with discriminant a can be obtained as the orthogonal complement of the ternary sublattice with the same discriminant in a 9-ary unimodular lattice. But there are only four ternary lattices with discriminant 14 in view of isometry, namely

$$\begin{pmatrix}1&&\\&1&\\&&14\end{pmatrix},\ \begin{pmatrix}1&&\\&2&\\&&7\end{pmatrix},\ \begin{pmatrix}1&&\\&3&1\\&1&5\end{pmatrix},\ \text{and}\ \begin{pmatrix}2&&\\&2&1\\&1&4\end{pmatrix}.$$

And there are only two 9-ary unimodular lattices, I_9 and $\Phi_8\perp(1)$. One can see easily that the orthogonal complement of the ternary sublattice with discriminant a in I_9 or

$\Phi_8 \perp (1)$ is decomposable. So $n=6$, $a=14$ is an exceptional case. One may refer to [11]. Similarly we can prove that $n=6$, $a=6$ is also an exceptional case. This completes the proof of Proposition 1.

2. The cases for $3 \leqslant n \leqslant 5$

Proposition 2. *For any square-free composite number a, there are 5-ary indecomposable positive definite $\mathbb{Z}$-lattices L with $dL=a$, but for the only exception $a=10$. In the exceptional case there are no lattices with the desired properties.*

Proof First consider that a is odd. Assume that $a=p_1\cdots p_s$ with odd primes $p_1 < \cdots < p_s$, where $s \geqslant 2$.

If there is a prime p, $p \geqslant 23$ or $p=17$ and $p \mid a$, then construct $M=\begin{pmatrix} 2 & 1 \\ 1 & * \end{pmatrix} \perp I_4$ with $dM=a/p$, and (M, p) satisfies #.

If $5 \mid a$, construct

$$M_1=I_2 \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \perp I_2 \text{ when } s=2;$$

$$M_2=I_2 \perp \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & * \end{pmatrix} \text{ when } s \geqslant 3; \ dM_i=a/p_s.$$

Then (M_i, p_s) satisfies # when $p_s=7$, 11, 13, or 19.

For $a=3\cdot 5$, let

$$L \cong \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & -1 & 0 & 3 \end{pmatrix} \text{ in the base } \{e_1, \cdots, e_5\}.$$

Suppose f is the quadratic form associated with L. Then

$$\begin{aligned} f &= 2x_1^2-2x_1x_2-2x_1x_5+2x_2^2+2x_3^2-2x_3x_4-2x_3x_5+2x_4^2+3x_5^2 \\ &= 2\left(x_1-\frac{1}{2}x_2-\frac{1}{2}x_5\right)^2+\frac{3}{2}\left(x_2-\frac{1}{3}x_5\right)^2+2\left(x_3-\frac{1}{2}x_4-\frac{1}{2}x_5\right)^2 \\ &\quad +\frac{3}{2}\left(x_4-\frac{1}{3}x_5\right)^2+\frac{5}{3}x_5^2 \end{aligned}$$

If f represents 1, then we deduce that $\frac{5}{3}x_5^2 \leqslant 1$. But $x_5 \in \mathbb{Z}$, so $x_5=0$. On the other hand the lattice associated with $f'=f|_{x_5=0}$ is an even lattice, so f does not represent 1. Therefore $e_1, \cdots, e_5$ are irreducible vectors, and then L is indecomposable.

If $11 \mid a$, construct

$$M_1=I_2 \perp \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} \perp I_2 \text{ when } s=2,$$

$$M_2=I_2 \perp \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix} \perp \begin{pmatrix} 2 & 1 \\ 1 & * \end{pmatrix} \text{ when } s \geqslant 3, \ dM_i=a/p_s.$$

Then (M_i, p_s) satisfies # when $p_s=7, 13, 19$.

For $a=3\cdot 11$, let $M=(3)\perp I_5$ and $11=3+\square_5$.

If $19|a$, for $s=2$, construct $M=\begin{pmatrix}2 & 1\\ 1 & *\end{pmatrix}\perp I_4$, $dM=a/19$.

From the above discussion, we may assume that the * in M is 2, 4 or 7, and then $(M, 19)$ satisfies #. For $s\geqslant 3$, construct $M=\begin{pmatrix}2 & 1\\ 1 & *1\end{pmatrix}\perp\begin{pmatrix}2 & 1\\ 1 & *2\end{pmatrix}\perp I_2$, $dM=a/19$. We can choose $*1=4$ or 7. Then $(M, 19)$ satisfies#.

In the following we may assume that $a=p_1\cdots p_s$, $p_i\in\{3, 7, 13\}$.

For $a=3\cdot 13$, let $M=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\perp I_4$;

for $a=3\cdot 7\cdot 13$, let $M=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\perp\begin{pmatrix}2 & 1\\ 1 & 4\end{pmatrix}\perp I_2$;

for $a=7\cdot 13$, let $M=\begin{pmatrix}2 & 1 & \\ 1 & 2 & 1\\ & 1 & 3\end{pmatrix}\perp I_3$.

In any case $(M, 13)$ satisfies #.

For $a=3\cdot 7$, let

$$L\cong\begin{pmatrix}2 & -1 & -1 & 0 & 0\\ -1 & 2 & 0 & 0 & 0\\ -1 & 0 & 2 & -1 & 0\\ 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & -1 & 5\end{pmatrix}.$$

Similarly as in the case $a=15$, we can prove that L is an indecomposable lattice with $dL=21$.

Now consider that a is even. Assume $a=2p_1\cdots p_s$, with odd primes $p_1<\cdots<p_s$ and $s\geqslant 1$.

(i) $s=1$. Construct $M=(2)\perp I_5$, Then (M, p_1) satisfies # when $p_1\geqslant 19$ or $p_1=7$, 13. For $a=2\cdot 3$, $2\cdot 11$, $2\cdot 17$, there is a 6-ary indecomposable lattice M from ([1], 5.10), $dM=a/2$ and M represents 2. For $a=10$, it is an exceptional case by [7].

(ii) $s\geqslant 2$. First, we require the following lemma which will often be used later.

Lemma 1[4]. *Let $f(a, b)=x^2+ay^2+bz^2$. For given natural number m, m cannot be represented by $f(a, b)$ in integers for the a, b bolow if and only if*

$$a=1,\ b=2,\ m=4^v\,(16u+14);$$
$$a=1,\ b=3,\ m=9^v(9u+16);$$
$$a=2,\ b=3,\ m=4^v(16u+10),$$

where $v>0$ and are integers..

Let

$$M=(2)\perp\begin{pmatrix}2 & 1\\ 1 & *\end{pmatrix}\perp I_3,\ dM=a/p_s.$$

By Lemma 1, the equations:

$$p_s=2+16+2x^2+y^2+z^2,\ \text{when } p_s\equiv 1 \bmod 8 \text{ and } p_s>18;$$
$$p_s=8+4+2x^2+y^2+z^2,\ \text{when } p_s\equiv 3 \bmod 8 \text{ and } p_s>12;$$
$$p_s=2+4+2x^2+y^2+z^2,\ \text{when } p_s\equiv 5 \bmod 8 \text{ and } p_s>6;$$
$$p_s=8+16+2x^2+y^2+z^2,\ \text{when } p_s\equiv 7 \bmod 8 \text{ and } p_s>24,$$

where $xyz\neq 0$, are solvable in $\mathbb{Z}$. Because $(M, 23)$ satisfies Proposition #, in the following we can assume that

$$a=2p_1\cdots p_s,\ p_i\in\{3, 5, 7, 11, 17\},\ s\geqslant 2.$$

For $7|a$, construct $M=\begin{pmatrix}2 & 1\\ 1 & *\end{pmatrix}\perp I_5,\ dM=a/14.$

Then $(M, 7)$ satisfies #. So we get a sublattice M' of M with $dM'=a/2$, and M' represents 2.

For $5|a$, construct

$$M_1=(2)\perp\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}\perp I_3,\ \text{when } s=2;$$
$$M_2=(2)\perp\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}\perp\begin{pmatrix}2 & 1\\ 1 & *\end{pmatrix}\perp I_1,\ \text{when } s\geqslant 3;\ dM_i=a/p_s.$$

Then (M_i, p_s) satisfies # when $p_s=11, 17$.

If $a=2\cdot 3\cdot 5$, there is a 6–ary indecomposable lattice M with $dM=15$ and M represents 2.

For $11a|$, construct $M_1=(2)\perp\begin{pmatrix}3 & 1\\ 1 & 4\end{pmatrix}\perp I_3$, when $s=2$;

$$M_2=(2)\perp\begin{pmatrix}3 & 1\\ 1 & 4\end{pmatrix}\perp\begin{pmatrix}2 & 1\\ 1 & *\end{pmatrix}\perp I_1,\ \text{when } s\geqslant 3;\ dM_i=a/p_s.$$

Then (M_i, p_s) satisfies # when $p_s=17$.

Finally for $a=2\cdot 3\cdot 11$, or $a=2\cdot 3\cdot 17$, construct

$$M=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\perp I_5 \text{ in the base } \{e_1, \cdots, e_7\}.$$

Put $x=e_1+\cdots+e_7$ and $y=e_1+e_2+2e_3+2e_4+e_5+e_6+e_7$. Then $Q(x)=11$, $Q(y)=17$ and $M_1=M\cap(\mathbb{Q}x)^*$, $M_2=M\cap(\mathbb{Q}y)^*$ are 6–ary indecomposable lattices with $dM_1=33$ and $dM_2=51$ respectively. Both represent 2. This completes the proof.

Now consider the existence of quaternary indecomposable lattice. First we have the following lemma.

Lemma 2. *The lattice* $L\cong\begin{pmatrix}2 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 2 & 1\\ & & 1 & c\end{pmatrix}$ *is indecomposable and positive definite* *with* $dL=4c-3$, *where* $c\geqslant 2$.

Proof If L is decomposable, then $L=L_1\perp L_2$. First we assume that dim $(L_1)=1$, i.e. $L_1=\mathbb{Z}t$, $t=x_1e_1+\cdots+x_4e_4$, where $\{e_1, \cdots, e_4\}$ is the base of L. So $B(L, L_1)\subseteq Q(t)\mathbb{Z}$. Consider $B(t, e_1)=2x_1+x_2$,

$$Q(t)=2x_1^2+2x_1x_2+2x_2^2+2x_2x_3+2x_3^2+2x_3x_4+cx_4^2$$
$$=(x_1+x_2)^2+x_1^2+(x_2^2+2x_2x_3+2x_3^2+2x_3x_4+cx_4^2)\geqslant|2x_1+x_2|.$$

The equality holds if and only if $x_2=x_3=x_4=0$, $x_1=\pm 1$. Then $t=\pm e_1$, but $B(t, e_2)=\pm 1\notin 2\mathbb{Z}$, which is a contradiction. So we have $B(t, e_1)=0$. Similarly we obtain $B(t, e_i)=0$, $\forall i$. Thus $t=0$, which contradicts dim $(L_1)=1$. Therefore in the orthogonal splitting of L there is no one-dimensional component, hence there is no three dimensional component. Since $1\notin Q(L)$, the vectors e_1, e_2, e_3 are irreducible, they are in the same component of the splitting L. So L is indecomposable.

By Lemma 2, there is always a quaternary indecomposable lattice of discriminant a when $a\geqslant 5$ and $a\equiv 1$ mod 4.

Lemma 3. *The equation* $p=2x^2+2y^2+z^2+u^2$, $xyzu\neq 0$, *is solvable in* $\mathbb{Z}$ *for every prime* p, *if* $p\neq 2, 3, 5, 7, 11, 13$ *and* 19.

Proof For $p\equiv 1$ mod 8, the equation $p=2+2x^2+y^2+z^2$, with $xyz\neq 0$, is solvable in $\mathbb{Z}$. (See Lemma 1.)

For $p\equiv 7$ mod 8, $p\neq 7$, the equation $p=8+2x^2+y^2+z^2$ with $xyz\neq 0$ is solvable. (See Lemma 1.)

For $p\equiv 3$ mod 8, $p\geqslant 72$, let us consider the equation

$$p=2\cdot 6^2+2x^2+y^2+z^2.$$

By Lemma 1, it is solvable. If $xyz\neq 0$, there is nothing to prove. If $xyz=0$, considering the congruence modulus 8, we can assume $xy\neq 0$, $z=0$. So $p=2\cdot 2^2+2x^2+y^2+8^2$.

For $p\equiv 3$ mod 8, $p<72$. By computing, it is easily seen that $p=3, 11, 19$ are the only exceptions.

For $p\equiv 5$ mod 8, $p>200$, let us consider the equation

$$p=2\cdot 10^2+2x^2+y^2+z^2.$$

By Lemma 1, the equation is solvable. If $xyz\neq 0$, there is nothing to prove. If $xyz=0$, considering the congruence modulus 8, we can assume $x=0$, $yz\neq 0$. In this way $p=2\cdot 6^2+2\cdot 8^2+y^2+z^2$.

For $p\equiv 5$ mod 8, $p<200$. It is easily seen that $p=5, 13$ are the only exceptions. The proof is completed.

Proposition 3. *For any square-free composite number* a, *there is a quaternary indecomposable positive definite* $\mathbb{Z}$*-lattice of discriminant* a, *but for the* 4 *exceptions*: $a=6, 10, 14$ *and* 26. *In the exceptional cases there are no lattices with the desired properties.*

Proof (A) First we prove the proposition for $a=p_1, \cdots, p_s$, with odd primes p_1,

$\cdots, p_s$, and $s\geqslant 2$.

If there is a prime factor p of a which is congruent to 3, 5, or 7 mod 8, construct

$$M=\begin{pmatrix}2 & 1\\ 1 & *\end{pmatrix}\perp I_3,\ dM=a/p.$$

Then $p=8+\square_3$ when $p\equiv 3$ mod 8, $p\neq 3$; and $p=2+\square_3$ when $p\equiv 5$ mod 8. For $p\equiv 7$ mod 8, $p\neq 7$, by Lemma 1 the equation

$$p=16+2x^2+y^2+z^2,\ xyz\neq 0$$

is solvable. So in these cases there are quaternary indecomposable lattices L with $dL=a$.

Now we assume that a has one of the forms:

$$a=p_1\cdots p_s,\ a=3p_1\cdots p_s,\ a=7p_1\cdots p_s,\ a=3\cdot 7p_1\cdots p_s,$$

where $p_i\equiv 1$ mod 8.

Since the lattice in Lemma 2 represents 2 and 6, in the first three cases there are quaternary indecomposable lattices L with $dL=a$.

For $a=3\cdot 7p_1\cdots p_s$, $p_i\equiv 1$ mod 8. If $s\geqslant 1$. construct

$$M=\begin{pmatrix}2 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 2 & 1\\ & & 1 & c\end{pmatrix}\perp(1)\text{ in the base }\{e_1,\cdots e_5\}\text{ with }dM=a/21.$$

Put $x=e_1+e_5$. Then $Q(x)=3$ and $M'=M\cap(\mathbb{Q}x)^*$ in a quaternary indecomposable lattice of discriminant $a/7$. Put $y=e_1-2e_5$. Then $y\in M'$, $Q(y)=6$. Construct $N=M'\perp(1)$, and $(N, 7)$ satisfies #.

If $a=3\cdot 7$, construct $M=(3)\perp I_4$, and $(M, 7)$ satisfies #. This completes the proof if a is odd.

(B) Assume $a=2p_1\cdots p_s$, with odd primes p_i, where $s\geqslant 1$.

(i) $s=1$. Construct $M=(2)\perp I_4$.

For $p_1\geqslant 23$ or $p_1=17$, (M, p_1) satisfies #.

For $p_1=11$ or $p_1=19$, by ([1], 5.10) there is a quaternary lattice of discriminant a with desired properties.

For $a=6$, 10, 14, or 26, there is no lattice with the desired properties from [7] and [10].

(ii) $s\geqslant 2$. Construct $M=\begin{pmatrix}2 & 1\\ 1 & *\end{pmatrix}\perp(2)\perp I_2$, $dM=a/p_s$. Then (M, p_s) satisfies # by Lemma 3 when $p_s\neq 3, 5, 11, 13, 19, 7$. In the following $a=2p_1\cdots p_s$, $s\geqslant 2$ and $p_i\in\{3, 5, 7, 11, 13, 19\}$.

For $5\,|\,a$, construct

$$M_1=(2)\perp\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}\perp I_2,\text{ when }s=2;$$

$$M_2=(2)\perp\begin{pmatrix}2&1\\1&3\end{pmatrix}\perp\begin{pmatrix}2&1\\1&*\end{pmatrix}, \text{ when } s\geqslant 3;\ dM_i=a/p_s.$$

Then (M_i, p_s) satisfies # when $p_s=7$, 11, 13 or 19. For $a=2\cdot3\cdot5$, $n=4$, from the proof of Proposition 2 there is a 5-ary indecomposable positive definite $\mathbb{Z}$-lattice of discriminant 15, which represents 2.

For $11\,|\,a$, construct

$$M_1=(2)\perp\begin{pmatrix}3&1\\1&4\end{pmatrix}\perp I_2, \text{ when } s=2;$$

$$M_2=(2)\perp\begin{pmatrix}3&1\\1&4\end{pmatrix}\perp\begin{pmatrix}2&1\\1&*\end{pmatrix}, \text{ when } s\geqslant 3;\ dM_i=a/p_s.$$

Then (M_i, p_s) satisfies # when $p_s=7$, 13 or 19. For $a=2\cdot3\cdot11$, from the proof of Proposition 2, there is a 5-ary indecomposable lattice of discriminant 33 which represents 2. In the following

$$a=2p_1\cdots p_s,\ s\geqslant 2 \text{ and } p_i\in\{3, 7, 13, 19\}.$$

For $s=2$, from the proof of Porposition 2 there is a 5-ary indecomposable lattice of discriminant $a/2$, which represents 2.

For $s\geqslant 3$, if $7\,|\,a$, construct

$$M=\begin{pmatrix}3&1\\1&5\end{pmatrix}\perp\begin{pmatrix}2&1\\1&*\end{pmatrix}\perp(1),\ dM=a/p_s,$$

and (M, p_s) satisfies # when $p_s=13$ or 19.

Finally, for $a=2\cdot3\cdot13\cdot19$, construct

$$M=\begin{pmatrix}3&1\\1&9\end{pmatrix}\perp\begin{pmatrix}2&1\\1&2\end{pmatrix}\perp(1),$$

and $(M, 19)$ satisfies #. This completes the proof of Proposition 3.

Lemma 4. *The lattices*

$$L_1=\begin{pmatrix}2&1&\\1&2&1\\&1&c\end{pmatrix},\ L_2=\begin{pmatrix}2&1&\\1&3&1\\&1&c\end{pmatrix},\ L_3=\begin{pmatrix}3&1&\\1&2&1\\&1&c\end{pmatrix}$$

are indecomposable with $dL_1=3c-2$, $dL_2=5c-2$, $dL_3=5c-3$, *where* $c\geqslant 2$.

The proof is similar to that of Lemma 2 and is omitted.

Lemma 5[4]. *If the congruence* $ak^2+b\equiv 0 \bmod p$ *is solvable (where* a, b *are given natural numbers and* p *is a prime), then there are* $x, y\in\mathbb{Z}$ *such that* $ax^2+by^2=mp$, *where* $m\leqslant\sqrt{4ab}$ *is a positive integer.*

Lemma 6[9]. *The equation* $x^2+2y^2=p$ *is solvable when* $p\equiv 1 \bmod 8$ *or* $p\equiv 3 \bmod 8$.

Lemma 7. *The equation* $3x^2+2y^2=p$ *is solvable when* $p\equiv 2 \bmod 3$ and $p\equiv 3$ *or* $5 \bmod 8$.

Proof Since $\left(\frac{-6}{p}\right)=1$ $\left(\text{where } \left(\frac{-6}{p}\right) \text{ denotes the Legendre's symbol}\right)$, we can prove the lemma easily by using Lemma 5.

Lemma 8. (1) *The equation* $2x^2+y^2+z^2=p$, $xyz\neq 0$, *is solvable in* $\mathbb{Z}$ *when* $p\equiv 1$ mod 3.

(2) *The equation* $6x^2+y^2=p$ *is solvable in* $\mathbb{Z}$ *when* $p\equiv 1$ mod 3 *and* $p\equiv 7$ mod 8.

Proof (1) For $p\equiv 1$ or 5 mod 8, the equation $p=x^2+y^2$ is solvable, since $p\equiv 1$ mod 3, we may assume that $x=3z$, and so $p=2(2z)^2+z^2+y^2$, $yz\neq 0$.

For $p\equiv 3$ mod 8, the equation $p=x^2+2y^2$ is solvable by Lemma 6, since $p\equiv 1$ mod 3, then $y=3z$ and $p=(4z)^2+2z^2+x^2$, $xz\neq 0$.

For $p\equiv 7$ mod 8 (the condition $p\equiv 1$ mod 3 is not necessary), the equation $p=2x^2+y^2+z^2$, $xyz\neq 0$, is solvable by Lemma 1. This proves Lemma 8(1).

The proof for (2) is similar to that of Lemma 7 and is omitted.

Proposition 4. *For any square-free compositive number a, there is a ternary indecomposable positive definite $\mathbb{Z}$-lattice of discriminant a, but for the 3 exceptions: $a=6$, 14, and 15. In the exceptional cases there is no lattice with the desired properties.*

Proof (A) First we prove the Proposition for odd a. By Lemma 8 (1), we construct $M=\begin{pmatrix} 2 & 1 \\ 1 & * \end{pmatrix}\perp I_2$. We can assume that a has no prime factor which is congruent to 1 mod 3 or congruent to 7 mod 8. Because the lattice L_1 in Lemma 4 represents 2, we need only consider the following cases: (i) $a=p_1\cdots p_s$; (ii) $a=3p_1\cdots p_s$; where p_i is a prime and $p_2\cdots p_s\equiv 2$ mod 3, $p_i\equiv 2$ mod 3, $p_i\not\equiv 7$ mod 8.

The proof of (i): Since $a\equiv 2$ mod 3 and $p_i\equiv 2$ mod 3, the s is odd and $s\geqslant 3$.

If $p_s\equiv 1$ or 3 mod 8, construct (see Lemma 4)

$$M=\begin{pmatrix} 2 & 1 & \\ 1 & 2 & 1 \\ & 1 & c \end{pmatrix}\perp(1),\ dM=a/p_s.$$

Then (M, p_s) satisfies # by Lemma 6.

If $p_s\equiv 5$ mod 8, construct

$$M=\begin{pmatrix} 2 & 1 \\ 1 & *1 \end{pmatrix}\perp\begin{pmatrix} 3 & 1 \\ 1 & *2 \end{pmatrix},\ dM=a/p_s.$$

Then (M, p_s) satisfies # by Lemma 7.

The proof of (ii): $a=3p_2\cdots p_s$, $a/3\equiv 2$ mod 3, and $p_i\equiv 2$ mod 3, $p_i\not\equiv 7$ mod 8. Since $a/3\equiv 2$ mod 3 and $p_i\equiv 2$ mod 3, s is odd.

If $s=1$, for $p_1\equiv 2$ or 3 mod 5, construct $M=L_i\perp(1)$, where $L_i(i=2, 3)$ are given by Lemma 4 with $dM=p_1$. Then $(M, 3)$ satisfies #. For $p_1\equiv 1$ or 4 mod 5, then $a\equiv 2$ or 3 mod 5 and by Lemma 4 the proposition is true for these cases. For $a=15$, it is an exceptional case by [8] p. 181.

If $s\geqslant 3$, for $5|a$, construct

$$M=\begin{pmatrix}2 & 1\\ 1 & *1\end{pmatrix}\perp\begin{pmatrix}3 & 1\\ 1 & *2\end{pmatrix},\ dM=a/5.$$

Then $(M, 5)$ satisfies #. For $5\nmid a$, the proof is similar to that of $s=1$, $p_1\neq 5$ as above. This completes the proof for (A).

(B) Now we prove the proposition for even a. Assume $a=2p_1\cdots p_s$ with odd primes p_i, where $s\geqslant 1$.

(i) $s=1$. If $p_1\equiv 1$ mod 8, by Lemma 2 the proposition is true for these cases.

Let $M=(2)\perp I_3$. $p_1=8+\Box_3$ when $p_1\equiv 3$ mod 8, $p_1\neq 3$; $p_1=2+\Box_3$ when $p_1\equiv 5$ mod 8. The equation $p_1=16+2x^2+y^2+z^2$, $xyz\neq 0$, is solvable by Lemma 5 when $p_1\equiv 7$ mod 8, $p_1\neq 7$. So in these cases the proposition is also true. $a=6$ and $a=14$ are exceptional cases by [8] p. 181.

(ii) $s\geqslant 2$. Let $M=(2)\perp\begin{pmatrix}2 & 1\\ 1 & *\end{pmatrix}\perp(1)$, $dM=a/p_s$. (M, p_s) satisfies # when $p_s\equiv 1$ or 5 mod 8. And because of the L_1 in Lemma 4, we can assume that (1) $a=2\cdot 3p_1\cdots p_s$; (2) $a=2p_1\cdots p_s$; where $p_i\equiv 3$ or 7 mod 8 and $p_1\cdots p_s\equiv 1$ mod 3.

The proof of (1): Since $5\nmid a$, if $a/3\equiv 3$ or 2 mod 5, construct $M=L_i\perp(1)$ with $dM=a/3$, where L_i is the lattice in Lemma 4 ($i=2, 3$). So $(M, 3)$ satisfies #. If $a/3\equiv 1$ or 4 mod 5, then $a\equiv 2$ or 3 mod 5 and Lemma 4 says that there is a ternary indecomposable lattice of discriminant a.

The proof of (2): $a=2p_1\cdots p_s$, $p_i\equiv 3$ or 7 mod 8, $a/2\equiv 1$ mod 3. Because of Lemma 2, we can also assume that $a/2\equiv 3$ mod 4, so s is an odd number and $s\geqslant 3$.

If a has a prime factor, say p_s, $p_s\equiv 3$ mod 8, construct

$$M=\begin{pmatrix}2 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 2 & 1\\ & & 1 & c\end{pmatrix}\perp(1),\ dM=a/2p_s.$$

By Lemma 6 there is a quaternary indecomposable sublattice M' of M with discriminant $a/2$ and M' represents 2. So in the following we can assume that a has no such prime factor, namely $a=2p_1\cdots p_s$ $p_i\equiv 7$ mod 8, $a/2\equiv 1$ mod 3, $a/2\equiv 3$ mod 4. If a has a prime factor, say p_s, which is congruent to 2 mod 3, construct

$$M=(1)\perp(2)\perp\begin{pmatrix}3 & 1\\ 1 & *\end{pmatrix},\ dM=a/p_s.$$

Then (M, p_s) satisfies # by Lemma 1. In the following we assume that each prime factor of a is congruent to 1 mod 3. Since $p_s\equiv 3$ mod 4, we have $p_1\cdots p_{s-1}\equiv 1$ mod 4. Construct (see Lemma 2)

$$M=\begin{pmatrix} 2 & 1 & & \\ 1 & 2 & 1 & \\ & 1 & 2 & 1 \\ & & 1 & c \end{pmatrix},\ dM=a/2p_s \text{ in the base } \{e_1, \cdots, e_4\}.$$

$M'=M\cap(\mathbb{Q}e_1)^*$ is a ternary indecomposable lattice of discriminant a/p_s. Put $x=e_1-2e_2$. Then $x\in M'$, $Q(x)=6$, and so 6 is represented by M'. Let $M_1=M'\perp(1)$, $dM_1=a/p_s$. Since $p_s\equiv1 \bmod 3$, $p_s\equiv7 \bmod 8$, by Lemma 8, (M_1, p_s) satisfies #. This completes the proof of Proposition 4.

Theorem 1 follows from Propositions 1—4.

3. The case for $n=2$

To prove Theorem 3 we need only prove the following theorem.

Theorem 3′. *For sufficiently large values of $a>0$, there exists a binary indecomposable positive definite quadratic form over $\mathbb{Z}$ with given discriminant a.*

Proof In view of Theorem $M(1)$ we need only consider the binary properly primitive positive definite quadratic form of the classic type

$$f(x, y)=\alpha x^2+2\beta xy+\gamma y^2$$

over $\mathbb{Z}$ with discriminant $a=\alpha\gamma-\beta^2>0$. The class number of these forms will be denoted by $h(a)$. Then by Siegel's Theorem[12] we have

$$\lim\frac{\log h(a)}{\log a}=\frac{1}{2},$$

or for every $\varepsilon>0$,

$$h(a)>a^{\frac{1}{2}-\varepsilon}.$$

On other hand, if the lattice L associated with the form $f(x, y)$ is decomposable, then

$$L=P\perp R$$

with non-zero sublattices P and R, and rank $P=1=$rank R. Hence we have the discriminant relation

$$dL=dP\cdot dR,$$

from which we deduce that $0<dP\mid dL$. So the number N_a of decomposable lattices is less than the number of divisors of $dL=a$. In fact, since we observe that both k and a/k are simultaneous divisors of a, they lead to the same decomposable lattice and so we have

$$N_a<\frac{1}{2}\,d(a),$$

where $d(a)$ is the divisor function of a. It is well-known that[13]

$$d(a)=O(a^{\varepsilon}),$$

i.e. $d(a)\leqslant Ka^{\varepsilon}$, where K is a constant, say $\frac{4}{\log 2}$, which is independent of a. Therefore we have

$$N_a/h(a)<\frac{1}{2}\,Ka^{2\varepsilon-\frac{1}{2}}\to 0$$

as a tends to $+\infty$. This means that the number of binary decomposable forms with $dL=a$ is finite, when a is sufficiently large. Theorem 3′ is proved.

We note that the proof of Theorem 3′ holds also for binary indecomposable positive definite quadratic forms of non-classic type.

It is easily seen that Theorem 3′ implies Theorem 3 or Theorem 3″.

Teeorem 3″. *There are binary indecomposable positive definite quadratic forms over $\mathbb{Z}$ with discriminant $a>0$, but for a finite number of exceptions.*

Finally we give some information on the exceptional cases, i.e. for those values of the discriminant a there does not exist binary indecomposable positive definite $\mathbb{Z}$-lattices.

It was proved by O'Meara[1] that for every odd number $a>1$ there is a binary indecomposable positive definite $\mathbb{Z}$-lattice

$$L\cong\begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2}(a+1) \end{pmatrix}$$

with scale $\mathscr{S}L\subseteq\mathbb{Z}$ and discriminant $dL=a$ and for every integer $b>1$ and $b\equiv1$ mod 3 there is a binary indecomposable positive definite $\mathbb{Z}$-lattice

$$L\cong\begin{pmatrix} 3 & 2 \\ 2 & \frac{2}{3}(b+2) \end{pmatrix}$$

with scale $\mathfrak{s}L\subseteq\mathbb{Z}$ and discriminant $dL=2b$. It follows that the exceptional case is possible only if $dL=2b$ is even and $b=1$ or $1<b\not\equiv1$ mod 3. From the proof of Theorem 3 we see that the number of values of such $b>0$ is finite. It remains to determine all the exceptional discriminants.

In order to find the exceptional discriminants we give a criterion–Theorem 4.

It is well-known that[5] there does not exist binary indecomposable positive definite $\mathbb{Z}$-lattices L with discriminants 1 and 2, but with discriminant 3 the binary $\mathbb{Z}$-lattice

$$L\cong\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

is indecomposable. In general we have Theorem 4.

To prove Theorem 4 we require the following lemma, which can be proved by Eisenstein reduction.

Lemma 9. *The necessary and sufficient condition for the existence of a binary indecomposable positive definite $\mathbb{Z}$-lattice with discriminant a is that the equation $xz-y^2=a$ $(0\leqslant2y\leqslant x\leqslant z)$ is solvable in $\mathbb{Z}$.*

Proof of Theorem 4. If there exists a binary indecomposable positive definite $\mathbb{Z}$-lattice with square-free discriminant a, by Lemma 9, the equation $xz-y^2=a$ $(0<2y\leqslant x\leqslant z,\ x,\ y,\ z\in\mathbb{Z})$ is solvable in $\mathbb{Z}$. We claim that there is a prime factor of x

which does not divide a. For otherwise, a is not square-free if x has a square factor, and $y\geqslant x$ if x is square-free. Hence in any case we get a contradiction. Thus there is a prime $p|x$ and $p\nmid a$, and so $y^2\equiv -a \bmod p$, i.e. $-a$ is a quadratic residue mod p. Since $a=xz-y^2\geqslant x^2-\frac{1}{4}x^2=\frac{3}{4}x^2$, $p\leqslant x\leqslant\sqrt{\frac{4}{3}a}$.

Conversely, if there is a prime $p_1\leqslant\sqrt{\frac{4}{3}a}$ such that $-a$ is a quadratic residue mod p_1, then $y_1^2\equiv -a \bmod p_1$ is solvable in y_1 with $1\leqslant y_1\leqslant\frac{p_1-1}{2}$ and so $y_1^2+a=p_1z$ $(z\in\mathbb{Z})$. If $z\geqslant p_1$, the equation $xz-y^2=a$ is solvable with $x=p_1$, $y=y_1$, $z=z_1$ and hence the sufficiency of Theorem 4 is proved. If $z_1<p_1$, then there is a prime p_2 such that $p_2|z_1$ and $p_2\nmid a$. For otherwise, a is not square-free if z_1 has a square factor, and if z_1 is square-free, we would have $y_1\geqslant z_1$ and then

$$a=p_1z_1-y_1^2\leqslant p_1y_1-y_1^2\leqslant\frac{1}{2}(p_1-1)^2$$

and so $p_1\geqslant\sqrt{2a}+1$, contrary to $p_1\leqslant\sqrt{\frac{4}{3}a}$. Thus $p_2\leqslant z_1<p_1$. Proceeding in this way, we can prove the sufficiency of Theorem 4 by induction.

By Theorem 4 and the information before Lemma 9 we obtain, by using computer, in the binary case $(n=2)$ with $dL\leqslant 4\cdot 14^4$ only the following 18 exceptions:

$dL=1, 2, 4, 6, 10, 18, 22, 30, 42, 58, 70, 78, 102, 130, 190, 210, 330, 462.$

It is probable that there are no other exceptions.

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