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TIME-VARYING VECTOR FIELDS ON A COMPACT RIEMANNIAN MANIFOLD (I)

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Abstract

This paper studies critical points of a time-varying vector field $f:M\times R\to TM$ on a compact Riemannian manifold M. It is shown that if a critical point x_0 admits an exponential dichotomy, then there are two families of manifolds, stable manifold family and unstable manifold family of f through x_0 in some open neighborhood V of x_0, moreover, the critical point x_0 is isolated. Also it is shown that the solution curve family of the perturbed time-varying vector field yielded by a small change of f is qualitatively the same as that of f.

§ 1. Introduction

In study of ordinary differential equations, many works have been done for the autonomous systems, but few for the time-varying systems. On the other hand, since the theory of the exponential dichotomy was introduced by Lin Zhensheng and other mathematicians, there have been considerable development in study of stability of the time-varying systems in Euclidean space. In this paper we study the local property of a critical point x_0 of a time-varying system f on a manifold by using exponential dichotomy. We show that if the critical point x_0 admits an exponential dichotomy there exist unique stable manifold family $\{W_t^+(x_0)\,|\,t\in R\}$ and unstable manifold family $\{W_t^-(x_0)\,|\,t\in R\}$. We also show that such point is isolated. If all of the time-varying vector fields make up a Banach space in C^r norm, there is a unique solution curve of the perturbed time-varying field yielded by a small change of f in this norm, the vicinity of which is the same structure as that cf x_0. Thus, the point x_0 admiting an exponential dichotomy has similar properties with the elementary critical point of (autonomous) vector fields. The similar argument for the periodic solutions of the periodic systems can be given, we shall study them in [9].

This paper is organized as follows: Section 2 is concerned with some definitions and basic properties. In Section 3 we present the statement of the main theorems and corollaries. Section 4 is concerned with the proof of these therems. Finally in Section 5 we study the generic poblem.

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§ 2. Difinitions and Basic Properties

Difinition 2.1. *Let M be a C^∞ manifold, and f: $M\times R\to TM$ a $C^r(r\geqslant 1)$ map. We say f is a C^r time-varying vector field on M if $f(x, t)\in T_xM$ for all $t\in R$. (R is the real axis, TM is the tangent bundle of M.)*

Difinition 2.2. *Let f be a C^r time-varying vector field. A C^{r+1} curve c: $I\to R$ (I is an open interval of R) is called an integral curve of f through x_0 at time t_0 (or solution) if the equction*

$$c'(t)=f(c(t), t), \ c(t_0)=x_0, \quad t_0\in I$$

holds, where $c'(t)=Dc(t)*1$.

Proposition 2.1. *Let f be a C^r time-varying vector field. Then*

(i) For each $(x, \bar{t}_0)\in M\times R$, there exists an open interval I containing $\bar{t}_0$ and a C^{r+1} curve c: $I\to M$ such that c is an integral curve of f through x_0 at time $\bar{t}_0$;

(ii) There are an open neighborhood U of x_0, open intervals I and I_0 which both contain $\bar{t}_0$ and a C^r map ϕ: $I\times U\times I_0\to M$, $\phi(t_0, x, t_0)=x$, $t_0\in I_0$, $x\in U$, such that $\phi_{(x,t_0)}$: $I\to M$ is an integral curve of f and $\phi_{(t,t_0)}$: $U\to M$ a C^r imbedding from U to M, where $\phi_{(x,t_0)}(t)=\phi(t, x, t_0)$, $\phi_{(t,t_0)}(x)=\phi(t, x, t_0)$.

Proof See [6, p60]

Proposition 2.2. *Let f be a C^r time-varying vector field on M. If M is compact, for every non-continuable integral curve c:$I\to M$ of f we have $I=R$.*

Proof Let c: $I\to M$ be a non-continuable integral curve of f, $I=(t_-, t_+)$. If $t_+<+\infty$, we take t_n, $n=1, 2, \cdots$, $t_1\leqslant t_2\leqslant\cdots\leqslant t_n\leqslant\cdots$, $t_n\to t_+$. Since M is compact, without loss of generality there is a point $x_0\in M$ such that $c(t_n)\to x_0$. For (x_0, t_+), by Proposition 2.1 (ii) there exists a neighborhood U of x_0, an open interval I_+ containing t_+ and a C^r map ϕ: $I_+\times U\times I_+\to M$. Take n large enough, so that $t_n\in I_+$ and $c(t_n)\in U$, $\phi(t, x_n, t_n)$ is an integral curve of f on I_+. On the other hand, by uniqueness $c(t)=\phi(t, x_n, t_n)$ for $t\in I\cap I_+$. So c is continuable to $I\cup I_+$, but this contradicts the fact that I is non-continuable. Therefore, $t_+=+\infty$. The proof for t_- is similar.

Let M be a compact Riemannian manifold. $\mathscr{C}^r$ is the set that consists of all of C^r time-varying vector fields such that $f\in\mathscr{C}^r$ iff $\sup_{t\in R}\{\|f_t\|_r\}<+\infty$, where $f_t(x)=f(x, t)$, $\|\cdot\|_r$ is the C^r norm of C^r vector fields on M (cf. [1, p. 31]).

Let $\sup_{t\in R}\{\|f_t\|_r\}=\|f\|$. Then $\mathscr{C}^r$ is a Banach space in the norm $\|\cdot\|$.

Difinition 2.3. *Let $\psi(x, f, t)$ be the integral curve of f satisfying $\psi(x, f, 0)=x_0$. For $(x_0, f)\in M\times\mathscr{C}^r$, we denote $\Phi(x_0, f, t)=D_1\psi(x_0, f, t)$. If there exists a projection operator $P(x_0, f)$: $T_{x_0}M\to T_{x_0}M$, and positive numbers α, K such that*

$$\|\Phi(x_0, f, t)P(x_0, f)\Phi^{-1}(x_0, f, s)\|\leqslant Ke^{-\alpha(t-s)}, \quad t\geqslant s,$$

$$\|\Phi(x_0, f, t)(I-P(x_0, f))\Phi^{-1}(x_0, f, s)\| \leqslant Ke^{-\alpha(s-t)}, \quad s \geqslant t,$$

then we say (x_0, f) *admits an exponential dichotomy.*

Difinition 2.4. *Let* $f: M \times R \to TM$ *be a* C^r *time-varying vector field, and* $x_0 \in M$. *If for all* $t \in R$ *we have* $f(x_0, t) = O_{x_0}$, *where* O_{x_0} *is the null element of* $T_{x_0}M$, *then* x_0 *is called a critical point of* f.

It is easy to prove that if x_0 is an elementary critical point of C^r vector field $f: M \to TM$, (x_0, f) admits an exponential dichotomy.

§3. Main Theorems

In this section, we suppose that M is a connected n-dimensional compact Riemannian manifold without any loss of the generality.

Theorem 3.1. *Let* $f: M \times R \to TM$ *be* $C^{r+1}(r \geqslant 1)$ *time-varying vector field, and* x_0 *be a critical point of* f. *If* (x_0, f) *admits an exponential dichotomy, there exist two manifold families* $\{W_t^+(x_0) \mid t \in R\}$, $\{W_t^-(x_0) \mid t \in R\}$ *satisfying:*

(i) *For each* $x \in W_s^+(x_0)$, *let* $\phi(t, x, s)$ *be the integral curve of* f *through* x *at time* s, *then*

$$\lim d(\phi(t, x, s), x_0) = 0,$$

where d *is the metric on* M *generated by Riemannian metric;*

(ii) *There exists a neighborhood* U *of* x_0 *(indepedent of* t *and* s*) such that if* $\phi(t, x, s) \in U$, $t \geqslant s$, *then* $x \in W_s^+(x_0)$;

(iii) *There are positive numbers* λ, β *(independent of* t *and* s*) such that for each pair* $x, y \in W_s^+(x_0)$ *we have*

$$d(\phi(t, x, s), \phi(t, y, s)) \leqslant \lambda e^{-\beta(t-s)}, \quad t \geqslant s;$$

(iv) *For every* $s \in R$, $W_s^+(x_0)$ *is a* C^r *submanifold;*

(v) $\{W_t^+(x_0) \mid t \in R\}$ *is continuous in* t, *that is, for every* $s \in R$ *and any* ϵ*-neighborhood* V_s *of* $W_s^+(x_0)$, *there exists* $\delta > 0$ *such that* $W_t^+(x_0) \subset V_s$ *as* $t \in (s-\delta, s+\delta)$;

((i)—(v) also hold when *"$-$", "t", "s" and "$-\infty$" substitute for "$+$", "s", "t" and "$+\infty$", respectively.)*

(vi) *For every* $t \in R$,

$$W_t^+(x_0) \cap W_t^-(x_0) = x_0;$$

$$W_t^+(x_0) \pitchfork_{x_0} W_t^-(x_0);$$

$$\dim T_{x_0} W_t^+(x_0) = \operatorname{rank} P(x_0, f), \ \dim T_{x_0} W_t^-(x_0) = \operatorname{rank}(I - P(x_0, f)).$$

Corollary 3.1. *If* (x_0, f) *satisfies the conditions in Theorem 3.1, there exists a neighborhood* U *of* x_0, *in which no integral curves lie except* $\phi(t) = x_0$.

Proof It is true by the Theorem 3.1 (ii) and (vi).

Corollary 3.2. *If* (x_0, f) *satisfies the conditions in Theorem 3.1, then* x_0 *is an isolated critical point of* f.

Theorem 3.2. *If (x_0, f) satisfies the conditions in Theorem 3.1, then for any sufficient small $\epsilon>0$ there exists a neighborhood $\mathscr{U}$ of f in $\mathscr{C}^{r+1}$ such that for every $g\in\mathscr{U}$ there exists a unique integral curve $\phi_g(t)$ of g satisfying $d(\phi_g(t), x_0)<\epsilon$. Meanwhile, $(\phi_g(0), g)$ admits an exponential dichotomy.*

Corollary 3.3. *If (x_0, f) satisfies the conditions in Theorem 3.2, then there are two manifold families $\{W_t^+(\phi_g(t))\,|\,t\in R\}$ and $W_t^-(\phi_g(t))\,|\,t\in R\}$, where $\phi_g(t)$ is given in Theorem 3.2, such that*

(i) *For every $x\in W_s^+(\phi_g(s))$,*

$$\lim_{t\to+\infty} d(\phi_g(t, x, s), \phi_g(t))=0,$$

where $\phi_g(t, x, s)$ is an integral curve of g through x at time s;

(ii) *There is a positive $\delta>0$ such that if $d(\phi_g(t, x, s), \phi_g(t))<\delta$ for all $t\geqslant s$, then $x\in W_s^+(\phi_g(s))$;*

(iii) *There are positive numbers λ and β such that for every pair $x, y\in W_s^+(\phi_g(s))$,*

$$d(\phi_g(t, x, s), \phi_g(t, y, s))\leqslant\lambda e^{-\beta(t-s)}, \quad t\geqslant s;$$

(iv) *$W_s^+(\phi_g(s))$ is a C^r submanifold for all $s\in R$;*

(v) *$\{W_t^+(\phi_g(t))\,|\,t\in R\}$ is continuous in t;*

(The conclusions above are true for $\{W_t^-(\phi_g(t))\,|\,t\in R\}$, when "$-$", "$t$", "$s$" and "$-\infty$" substitute for "$+$", "$s$", "$t$" and "$+\infty$".)

(vi) *For all $t\in R$,*

$$W_t^+(\phi_g(t))\cap W_t^-(\phi_g(t))=\phi_g(t),$$
$$W_t^+(\phi_g(t))\pitchfork_{\phi_g(t)} W_t^-(\phi_g(t)),$$
$$\dim T_{\phi_g(t)}W_t^+(\phi_g(t))=\operatorname{rank}P(x_0, f),$$
$$\dim T_{\phi_g(t)}W_t^-(\phi_g(t))=\operatorname{rank}(I-P(x_0, f)).$$

§4. The Proof of Main Theorems

First, we prove some lemmas.

Let $f\in\mathscr{C}^{r+1}(r\geqslant1)$ and x_0 be a critical point of f. Under a local chart (U, α) around x_0, $\alpha(x_0)=0$, f is represented by the equation

$$x'=A(t)x+f_1(x, t),$$
$$f_1(0, t)=0, \quad D_1f_1(0, t)=0.$$

Lemma 4.1. *(x_0, f) admits an exponential dichotomy if and only if for $x'=A(t)x$, there are a projection operator $P\in L(R^n, R^n)$, positive K and α such that*

$$\|X(t)PX^{-1}(s)\|\leqslant Ke^{-\alpha(t-s)}, \quad t\geqslant s,$$
$$\|X(t)(I-P)X^{-1}(s)\|\leqslant Ke^{-\alpha(s-t)}, \quad s\geqslant t,$$

where $X(t)$ is a fundemental solution matrix of $x'=A(t)x$, satisfying $X(0)=I$, and $\operatorname{rank}P=\operatorname{rank}P(x_0, f)$.

We also say that $x'=A(t)x$ admits an exponential dichotomy.

Proof Put $X(t)=T\alpha\circ\Phi(x_0, f, t)\circ T\alpha^{-1}$, $P=T\alpha\circ P(x_0, f)\circ T\alpha^{-1}$. By Difinition 2.3, we prove easily the lemma.

Lemma 4.2. *For every n-dimensional Riemannian manifold, there exists a metric d on it. For given local chart (U, α) there is a positive λ such that $d(x, y)\leqslant \lambda\|\alpha(x)-\alpha(y)\|$ for each pair $x, y\in U$, where $\|\cdot\|$ is the Euclidean norm.*

Proof See [2, p.187].

Lemma 4.3. *Let $\phi(t)$ be a non-negative continuous function on $[s, +\infty)$, satisfying $\phi(t)\leqslant 2K$ and*

$$\phi(t)\leqslant Ke^{-\alpha(t-s)}+\frac{\alpha}{4}\int_s^{+\infty} e^{-\alpha|t-\tau|}\phi(\tau)d\tau, \quad t\geqslant s,$$

where K, α are positive. Then

$$\phi(t)\leqslant\frac{4}{2+\sqrt{2}}Ke^{-\frac{\alpha}{\sqrt{2}}(t-s)}.$$

Proof Consider the integral equation

$$\psi(t)=Ke^{-\alpha(t-s)}+\frac{\alpha}{4}\int_s^{+\infty} e^{-\alpha|t-\tau|}\psi(\tau)d\tau, \quad t\geqslant s.$$

If $\psi(t)$ is a bounded solution of this equation, then

$$\psi''(t)=\frac{1}{2}\alpha^2\psi(t).$$

So
$$\psi(t)=Ce^{-\frac{\alpha}{\sqrt{2}}(t-s)}, \quad \text{and} \quad C=\frac{4}{2+\sqrt{2}}K.$$

Thus this equation has a unique bounded solution

$$\psi(t)=\frac{4}{2+\sqrt{2}}Ke^{-\frac{\alpha}{\sqrt{2}}(t-s)}.$$

Taking $\psi_0(t)=2K$, and

$$\psi_n(t)=Ke^{-\alpha(t-s)}+\frac{\alpha}{4}\int_s^{+\infty} e^{-\alpha|t-\tau|}\psi_{n-1}(\tau)d\tau,$$

by induction it is easy to prove

$$\psi_n(t)\leqslant 2K,$$

$$\|\psi_n(t)-\psi_{n-1}(t)\|\leqslant\frac{1}{2}\|\psi_{n-1}(t)-\psi_{n-2}(t)\|.$$

So, $\psi_n(t)$ converges uniformly to $\psi(t)$.

Since $\phi(t)\leqslant\psi_0(t)$, by induction $\phi(t)\leqslant\psi_n(t)$ for every positive integer n. Thus $\phi(t)\leqslant\psi(t)$, that is

$$\phi(t)\leqslant\frac{4}{2+\sqrt{2}}Ke^{-\alpha(t-s)}.$$

Proof of Theorem 3.1 Choose a local chart (V, β) around x_0, $\beta(x_0)=0$, and f is represented under the chart by

$$x'=A(t)x+f_1(x, t), \ \|x\|<r_0, \ r_0>0, \quad (1)$$
$$f_1(0, t)=0, \ D_1f_1(0, t)=0.$$

Let $X(t)$, $X(0)=I$, be a fundemental solution matrix of

$$x'=A(t)x. \tag{2}$$

By Lemma 4.1 there are P, K, and α such that

$$\|X(t)PX^{-1}(s)\|\leqslant Ke^{-\alpha(t-s)},\quad t\geqslant s,$$
$$\|X(t)(I-P)X^{-1}(s)\|\leqslant Ke^{-\alpha(s-t)},\quad s\geqslant t.$$

Choose $0<a<\frac{r_0}{2}$ such that when $\|x\|\leqslant a$,

$$\|D_1f_1(x,t)\|<\frac{\alpha}{4K}.$$

Now, the proof is divided into four steps.

Step 1. Define $W_s^+(x_0)$ (In the proof below, s is always fixed). Define

$$E_s^+=\left\{x\in R^n \mid \|x\|<\frac{a}{2K},\ \exists\tilde{x}\in R^n \text{ such that } x=X(s)P\tilde{x}\right\}.$$

Then E_s^+ is continuous in s and for each $x\in E_s^+$,

$$\|\tilde{\phi}(t,x,s)\|\leqslant\frac{a}{2}e^{-\alpha(t-s)},\quad t\geqslant s,$$

where $\tilde{\phi}$ is a solution of (2).

Next, define the map

$$H_x^+:\ \mathscr{B}_s^+(a)\to\mathscr{B}_s^+(a),$$

where $\mathscr{B}_s^+(a)$ is the closed subset of the Banach space of all the continuous functions from $[s,+\infty)$ to R^n satisfying $\|y(t)\|\leqslant a$, by

$$H_x^+y(t)=\tilde{\phi}(t,x,s)+\int_s^t X(t)PX^{-1}(\tau)f_1(y(\tau),\tau)d\tau$$
$$-\int_t^{+\infty}X(t)(I-P)X^{-1}(\tau)f_1(y(\tau),\tau)d\tau.$$

It is easy to prove that $\|H_x^+y\|\leqslant a$,

$$\|H_x^+y_1-H_x^+y_2\|\leqslant\frac{1}{4}\|y_1-y_2\|.$$

So, there exists a unique fixed point, denoted as $y(t,x)\in\mathscr{B}_s^+(a)$. Hence the equation

$$y(t,x)=\tilde{\phi}(t,x,s)+\int_s^t X(t)PX^{-1}(\tau)f_1(y(\tau,x),\tau)d\tau$$
$$-\int_t^{+\infty}X(t)(I-P)X^{-1}(\tau)f_1(y(\tau,x),\tau)d\tau$$

holds. Now we define L_s^+: $E_s^+\to L_s^+(E_s^+)=\widetilde{W}_s^+(0)$ by

$$L_s^+(x)=x-\int_s^{+\infty}X(s)(I-P)X^{-1}(\tau)f_1(y(\tau,x),\tau)d\tau,$$

and put $W_s^+(x_0)=\beta^{-1}(\widetilde{W}_s^+(0))$. Thus the conclusion (i) is reached.

If $y(t)$ is a solution of (1) and $\|y\|<\frac{a}{4K}$, putting

$$\tilde{\phi}(t)=y(t)-\int_s^t X(t)PX^{-1}(\tau)f_1(y(\tau),\tau)d\tau$$
$$+\int_t^{+\infty} X(t)(I-P)X^{-1}(\tau)f_1(y(\tau),\tau)d\tau,$$

then $\tilde{\phi}$ is a solution of (2) and $\|\tilde{\phi}\|<\frac{a}{2K}$. So $\tilde{\phi}(s)\in E_s^+$ and (ii) also holds. By Lemma 4.2, (iii) is true.

Step 2. Prove (iv).

First, we prove L_s^+ is a homeomorphism. If $x_1, x_2\in E_s^+$, $x_1\neq x_2$, $L_s^+(x_1)=L_s^+(x_2)$, by the uniqueness of solution, $y(t, x_1)=y(t, x_2)$, hence $x_1=x_2$. This contradicts the fact that $x_1\neq x_2$. Thus L_s^+ is a bijection.

Since $\tilde{\phi}$ is linear in x, for $x_i\in E_s^+$, $i=1, 2$,

$$\|y(t, x_1)-y(t, x_2)\|\leqslant\|x_1-x_2\|Ke^{-\alpha(t-s)}$$
$$+\frac{\alpha}{4}\int_s^{+\infty} e^{-\alpha|t-\tau|}\|y(\tau, x_1)-y(\tau, x_2)\|d\tau.$$

By Lemma 4.3,

$$\|y(t, x_1)-y(t, x_2)\|\leqslant\frac{4K}{2+\sqrt{2}}\|x_1-x_2\|.$$

Hence, L_s^+ is continuous.

Next, if there are $y(t, x_0)$, $y(t, x_n)$, $n=1, 2, \cdots$, $y(s, x_n)\to y(s, x_0)$ as $n\to+\infty$, from the proof of Step 1 one has

$$\|y(t, x_0)\|\leqslant\frac{2a}{2+\sqrt{2}}e^{-\frac{\alpha}{\sqrt{2}}(t-s)},\quad t\geqslant s,$$
$$\|y(t, x_n)\|\leqslant\frac{2a}{2+\sqrt{2}}e^{-\frac{\alpha}{\sqrt{2}}(t-s)},\quad t\geqslant s.$$

Thus, for any given $\varepsilon>0$, there is a $T>0$ such that when $t>s+T$,

$$\|y(t, x_0)\|<\frac{\varepsilon}{2},\ \|y(t, x_n)\|<\frac{\varepsilon}{2},\ n=1, 2, \cdots.$$

On $[s, s+T]$, since the solutions are continuous in the initial valus, there is a positive integer N such that $n\geqslant N$ implies $\|y(t, x_n)-y(t, x_0)\|<\varepsilon$. Thus $y(t, x_n)$ converges uniformly to $y(t, x_0)$ on $[s, +\infty)$. Since

$$x_0=y(s, x_0)+\int_s^{+\infty} X(s)(I-P)X^{-1}(\tau)f_1(y(\tau, x_0), \tau)d\tau,$$
$$x_n=y(s, x_n)+\int_s^{+\infty} X(s)(I-P)X^{-1}(\tau)f_1(y(\tau, x_n), \tau)d\tau,$$

we have $x_n\to x_0$ as $n\to+\infty$. Therefore L_s^+ is a homeomorphism.

Now we prove $L_s^+\in C^r$. For given $x\in E_s^+$, we consider the integration equation:

$$\Psi(t)=X(t)PX^{-1}(s)+\int_s^t X(t)PX^{-1}(\tau)D_1f_1(y(\tau, x), \tau)\Psi(\tau)d\tau$$
$$-\int_t^{+\infty} X(t)(I-P)X^{-1}(\tau)D_1f_1(y(\tau, x), \tau)\Psi(\tau)d\tau.$$

In a way similar to the proof in Step 1, we can prove that it has a unique bounded solution, denoted as $\Psi^+(t, x)$. Let F_s^+ be the linear space generated by E_s^+. Define

$$\psi^+(t, x): F_s^+\to R^n$$

by the diagram

$$\begin{array}{ccc} R^n & \xrightarrow{\Psi^+(t,x)} & R^n \\ X(s)PX^{-1}(s)\downarrow & \nearrow \psi^+(t,x) & \\ F_s^+ & & \end{array}$$

It is easy to prove such map is unique. Give $\bar{x}\in E_s^+$, $\bar{x}\neq x$, and put

$$p(t)=\frac{\|y(t, \bar{x})-y(t, x)-\psi^+(t, x)(\bar{x}-x)\|}{\|\bar{x}-x\|}.$$

Then

$$\begin{aligned}
p(t)=&\frac{1}{\|\bar{x}-x\|}\Big\{\|X(t)X^{-1}(s)\bar{x}+\int_s^t X(t)PX^{-1}(\tau)f_1(y(\tau, \bar{x}), \tau)d\tau\\
&-\int_t^{+\infty} X(t)(I-P)X^{-1}(\tau)f_1(y(\tau, \bar{x}), \tau)d\tau-X(t)X^{-1}(s)x\\
&-\int_s^t X(t)PX^{-1}(\tau)f_1(y(\tau, x), \tau)d\tau\\
&+\int_t^{+\infty} X(t)(I-P)X^{-1}(\tau)f_1(y(\tau, x), \tau)d\tau-X(t)X^{-1}(s)(\bar{x}-x)\\
&-\int_s^t X(t)PX^{-1}(\tau)D_1f_1(y(\tau, x), \tau)\psi^+(\tau, x)(\bar{x}-x)d\tau\\
&+\int_t^{+\infty} X(t)(I-P)X^{-1}(\tau)D_1f_1(y(\tau, x), \tau)\psi^+(\tau, x)(\bar{x}-x)d\tau\|\Big\}\\
\leqslant&\frac{1}{\|\bar{x}-x\|}\Big\{\Big\|\int_s^t X(t)PX^{-1}(\tau)[f_1(y(\tau, \bar{x}), \tau)-f_1(y(\tau, x), \tau)\\
&-D_1f_1(y(\tau, x), \tau)\psi^+(\tau, x)(\bar{x}-x)]d\tau\Big\|\\
&+\int_t^{+\infty} X(t)(I-P)X^{-1}(\tau)[f_1(y(\tau, \bar{x}), \tau)-f_1(y(\tau, x), \tau)\\
&-D_1f_1(y(\tau, x), \tau)\psi^+(\tau, x)(\bar{x}-x)]d\tau\Big\|\Big\}\\
\leqslant&\frac{\alpha}{4}\int_s^{+\infty} e^{-\alpha|t-\tau|}p(\tau)d\tau+\Delta,
\end{aligned}$$

where

$$\begin{aligned}\Delta=K\int_s^{+\infty} e^{-\alpha|t-\tau|}\|&f_1(y(\tau, \bar{x}), \tau)-f_1(y(\tau, x), \tau)\\
&-D_1f_1(y(\tau, x), \tau)[y(\tau, \bar{x})-y(\tau, x)]\|/\|\bar{x}-x\|d\tau.\end{aligned}$$

Since for given $\varepsilon>0$ there is a $\delta>0$ such that when $\|\bar{x}-x\|<\delta$,

$$\|f_1(y(\tau, \bar{x},)\tau)-f_1(y(\tau, x), \tau)-D_1f_1(y(\tau, x), \tau)\cdot[y(\tau, \bar{x})-y(\tau, x)]\|$$
$$\leqslant\frac{\varepsilon\alpha}{4K}, \quad \Delta\leqslant\frac{\varepsilon}{2}.$$

Thus
$$p(t)\leqslant\frac{\alpha}{4}\int_s^{+\infty} e^{-\alpha|t-\tau|}p(\tau)d\tau+\frac{\varepsilon}{2}.$$

Putting $D=\sup_{t\geqslant s} p(t)$, one has $D\leqslant\frac{1}{2}D+\frac{\varepsilon}{2}$ and then $D\leqslant\varepsilon$. It follows that $y(t, x)$ is differentiable in x and $D_2y(t, x)=\psi^+(t, x)$. By induction we can easily prove $y(t,x)$ is C^r in x. Let $t=s$, one has $L_s^+\in C^r$. Since $L_s^+\in C^r$ and L_s^+ is a homeomorphism, L_s^+ is a diffeomorphism and $\widetilde{W}_s^+(0)$ is a C^r submanifold, so is $W_s^+(x_0)$.

It is similar for $W_s^-(x_0)$.

Step 3. Prove (vi).

First, we prove $W_s^+(x_0)\cap W_s^-(x_0)=\{x_0\}$. It suffices to prove $\widetilde{W}_s^+(0)\cap\widetilde{W}_s^-(0)=\{0\}$. For a solution $y(t, x)$ of (1), if $y(s, x)\in\widetilde{W}_s^+(0)\cap\widetilde{W}_s^-(0)$, then $\|y(t, x)\|\leqslant a$, $t\in R$. Define

$$\phi(t)=y(t, x)-\int_{-\infty}^{t} X(t)PX^{-1}(\tau)f_1(y(\tau, x), \tau)d\tau$$
$$+\int_{t}^{+\infty} X(t)(I-P)X^{-1}(\tau)f_1(y(\tau, x), \tau)d\tau.$$

Then $\phi(t)$ is a solution of (2), and $\|\phi(t)\|<r_0$. So $\phi(t)\equiv 0$. This means that

$$y(t, x)=\int_{-\infty}^{t} X(t)PX^{-1}(\tau)f_1(y(\tau, x), \tau)d\tau$$
$$-\int_{t}^{+\infty} X(t)(I-P)X^{-1}(\tau)f_1(y(\tau, x), \tau)d\tau.$$

It is easy to prove this integral equation has a unique bounded solution, that is the trivial solution. Hence $y(t, x)\equiv 0$, this implies $\widetilde{W}_s^+(0)\cap\widetilde{W}_s^-(0)=\{0\}$.

By the representations of $\psi^+(s, 0)$, $\psi^-(s, 0)$, one has $T_0L_s^+=id_{F_s^+}$, $T_0L_s^-=id_{F_s^-}$. As $F_s^+\oplus F_s^-=R^n$, $\widetilde{W}_s^+(0)$ is transversal to $\widetilde{W}_s^-(0)$ at 0. Therefore $W_s^+(x_0)\pitchfork_{x_0}W_s^-(x_0)$.

Finally, it follows from dim $F_s^+=\text{rank } P(x_0, f)$, dim $F_s^-=\text{rank}(I-P(x_0, f))$ that dim $T_{x_0}W_s^+(x_0)=\text{rank } P(x_0, f)$, dim $T_{x_0}W_s^-(x_0)=\text{rank}(I-P(x_0, f))$.

Step 4. Prove (v).

In order to prove (v), it suffices to prove that $\widetilde{W}_s^+(0)$ is continuous in s. If $\widetilde{W}_s^+(0)$ is not continuous in some point s_0, there are an ε-neighborhood V_ε of $\widetilde{W}_{s_0}^+(0)$ and $y(s_n, x_n)\in\widetilde{W}_{s_n}^+(0)$, $s_n\to s_0$, such that $y(s_n, x_n)\overline{\in}V_\varepsilon$. Since $\|y(s_n, x_n)\|\leqslant a$, $\|x_n\|\leqslant\frac{a}{2K}$, we can assume there are points x_0', y_0 such that $x_n\to x_0'$, $y(s_n, x_n)\to y_0$, $y_0\overline{\in}V_\varepsilon$. As in Step 2, one can prove $\tilde{\phi}(t, x_n, s_n)$, $y(t, x_n)$ converge uniformly to $\tilde{\phi}(t, x_0', s_0)$, $y(t)$, $y(s_0)=y_0$ respectely. But

$$y(t, x_n)=\tilde{\phi}(t, x_n, s_n)+\int_{s_n}^{t} X(t)PX^{-1}(\tau)f_1(y(\tau, x_n), \tau)d\tau$$
$$-\int_{t}^{+\infty} X(t)(I-P)X^{-1}(\tau)f_1(y(\tau, x_n), \tau)d\tau.$$

Let $n\to+\infty$, one has

$$y(t)=\tilde{\phi}(t, x_0', s_0)+\int_{s_0}^{t} X(t)PX^{-1}(\tau)f_1(y(\tau), \tau)d\tau$$
$$-\int_{t}^{+\infty} X(t)(I-P)X^{-1}(\tau)f_1(y(\tau), \tau)d\tau.$$

Since $x'_0 \in \text{closure}(E_s^+)$, $y_0 \in \text{closure}(W_{s_0}^+(0))$, this contradicts $y_0 \bar{\in} V_\varepsilon$. So we complete the proof.

Proof of Theorem 3.2 Choose a local chart (V, β) around x_0. Let $f, g \in \mathscr{C}^{r+1}$ and be represented under the chart by

$$x' = \tilde{f}(x, t), \ x' = \tilde{g}(x, t), \ \|x\| < r_0.$$

In a way similar to the proof of Lemma 4.1, one can prove that if $\phi_g(t)$ is a solution of g and $\phi_g(t) \in V$, $t \in R$, $(\phi_g(0), g)$ admits an exponential dichotomy iff $x' = B(t)x$ admits an exponential dichotomy, where $B(t) = D_1\tilde{g}(\beta \cdot \phi_g(t), t)$. So it suffices to prove that, for $\varepsilon > 0$ small enough, there is a $\delta > 0$ such that when

$$\|D_1^k[\tilde{g}(x, t) - \tilde{f}(x, t)]\| < \delta, \quad k = 0, 1, \cdots, r+1,$$

$x' = \tilde{g}(x, t)$ has a unique solution $\tilde{\phi}_g(t)$ satisfying $\|\tilde{\phi}_g\| < \varepsilon$ and $x' = B(t)x$ admits an exponential dichotomy.

Denoting $\tilde{f}(x, t) = A(t)x + f_1(x, t)$, we choose $0 < \varepsilon < \frac{r_0}{2}$ such that when $\|x\| < \varepsilon$,

$$\|f_1(x, t)\| < \frac{\alpha}{8K}, \ \|D_1 f_1(x, t)\| < \frac{\alpha}{8K}.$$

Putting $\delta = \frac{\alpha}{8K}$ and $\tilde{g}(x, t) = A(t)x + g_1(x, t)$ one has

$$\|g_1(x, t)\| < \frac{\alpha}{4K}, \ \|D_1 g_1(x, t)\| < \frac{\alpha}{4K}, \ \|x\| < \varepsilon.$$

Consider the integral equation

$$\tilde{\phi}(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\tau)g_1(\tilde{\phi}(\tau), \tau)d\tau$$
$$-\int_{t}^{+\infty} X(t)(I-P)X^{-1}(\tau)g_1(\tilde{\phi}(\tau), \tau)d\tau.$$

The same argument as in the proof of Theorem 3.1 proves this integral equation has a unique bounded solution $\phi_g(t)$ and $\|\phi_g\| < \varepsilon$. Moreover, this solution is also the unique solution of $x' = \tilde{g}(x, t)$ lying in the region $\{x \mid \|x\| < \varepsilon\}$.

Finally, since

$$D_1\tilde{g}(\phi_g(t), t) = A(t) + D_1 g_1(\phi_g(t), t),$$

by [8, Corollary 2.7], $x' = B(t)x$ admits an exponential dichotomy and rank $P(\phi_g(0), g) = \text{rank } P(x_0, f)$, provided that ε is small enough.

§5. Problem of Genericity

By Theorem 3.2, we have known that if a critical point x_0 of a time-varying vector field admits an exponential dichotomy, then there are a neighborhood V of x_0 and a neighborhood $\mathscr{U}$ of f such that for every $g \in \mathscr{U}$, g has a unique integral curve ϕ_g lying in V. However, is ϕ_g trivial (that is $\phi_g(t) = \phi_g(0)$)? And how many time-varying vector fields satisfying $\phi_g(t) = \phi_g(0)$ are there in $\mathscr{U}$? In this section, we shall show that for almost every $g\in\mathscr{U}$, g has no critical points on V. This case is different from that of the (autonomous) vector fields.

Let $f\in\mathscr{C}^{r+1}$ and x_0 be a critical point of f. If (x_0, f) admits an exponential dichotomy, by Theorem 3.2 one can choose a local chart (U, β) around x_0, an open neighborhood $V\subset\overline{V}\subset U$ of x_0 and an open neighborhood $\mathscr{U}$ of f such that for every $g\in\mathscr{U}$ there is a unique integral curve of g, which lies in $\overline{V}$. In this case, we assume that $\beta(U)=\{e\in R^n \mid \|e\|<1\}=B$, $\beta(\overline{V})=\{e\in R^n \mid \|e\|\leqslant r<1\}=D_r$. Denote the set $\mathscr{V}=\{g\in\mathscr{U} \mid g$ has no critical points on $\overline{V}\}$.

Lemma 5.1. *$\mathscr{V}$ is an open set of $\mathscr{U}$.*

Proof For each $g\in\mathscr{U}$, define

$$S_g(x)=\sup_{t\in R}\|g(x, t)\|_0,$$

where $\|\cdot\|_0$ is the norm on the tangent bundle of M defined by Riemannian metric.

Since $g\in\mathscr{C}^{r+1}$, S_g is continuous on M. Recall that g has no critical points on $\overline{V}$ and $\overline{V}$ is compact. Hence there is a $\delta>0$ such that $S_g(x)\geqslant\delta$, $x\in\overline{V}$. Thus for $h\in\mathscr{U}$, when $\|h-g\|<\frac{\delta}{2}$, $S_h(x)\geqslant S_g(x)-|S_h(x)-S_g(x)|\geqslant\delta-\|h-g\|>\frac{\delta}{2}$. So h has no critical points on $\overline{V}$.

Lemma 5.2. (Riesz's Lemma[12]) *Let E be a Banach spac e. If there is a closed ball in E, which is compact, then E is a finite dimension space.*

Proof See [12, p. 81].

Lemma 5.3. *Let $C^{r+1}(R, R^n)$ be the Banach space of all the bounded C^{r+1} maps from R to R^n in the C^{r+1} norm, then $C^{r+1}(R, R^n)$ is an infinite-dimension space.*

Proof For $n=1$, let $\tilde{\phi}_k: R\to R$ be the C^∞ map satisfying

$$\tilde{\phi}_k(k)=1,$$

$$\tilde{\phi}_k(t)=0,\ t\in(-\infty, k-1]\cup[k+1, +\infty).$$

Then $\{\tilde{\phi}_k\}_{k=1}^\infty$ is linearly independent.

Generally, define $\phi_k(t)=(\tilde{\phi}_k(t), 0, 0, \cdots, 0)$. $\{\phi_k\}_{k=1}^\infty$ is also linearly independent.

Lemma 5.4. *If C^{r+1} map f: $B\times R\to R^n$ has only one critical point (y_0 is a critical point of f iff $f(y_0, t)=0$), there are C^{r+1} maps $\tilde{g}_n$: $B\times R\to R^n$ such that $\tilde{g}_n\to\tilde{f}$ in C^{r+1} norm as $n\to+\infty$, and for every $y\in D_r$, $\tilde{g}_n(y, t)\not\equiv 0$. Moreover, if W is an open set and $D_r\subset W\subset\overline{W}\subset B$, $\tilde{g}_n$ can be chosen such that $\tilde{g}_n(y, t)=\tilde{f}(y, t)$, $y\in B-W$.*

Proof Without any loss of generality, we can assume that 0 is the only critical point of f. Define F: $D_r\to C^{r+1}(R, R^n)$ by

$$F(y)t=\tilde{f}(y, t),\quad y\in D_r.$$

Then F has only one zero point $y=0$, and $F\in C^{r+1}$. Now, we shall prove that there are $P_n\in C^{r+1}(R, R^n)$, $n=1, 2, \cdots$, such that $P_n\bar{\in}F(D_r)$ and $P_n\to 0$, $n\to+\infty$. If it is not true, then there is a closed ball C with center O and radius ε_0 in $C^{r+1}(R, R^n)$ such that $F(D_r)\supset C$. Since D_r is compact and F is continuous, $F(D_r)$ is compact, so is C.

By Riesz's Lemma, $C^{r+1}(R, R^n)$ is finite-dimensional, this contradicts Lemma 5.3. Hence there exist p_n, $n=1, 2, \cdots$, such that $p_n \to 0$, $p_n \in F(D_r)$.

Put $\tilde{h}_n(y, t) = \tilde{f}(y, t) - p_n(t)$. Then $\tilde{h}_n \in C^{r+1}$ and, for every $y \in D_r$, $\tilde{h}_n(y, t) = 0$. Thus $\tilde{h}_n$ has no critical points on D_r. Now, for the open set W, $D_r \subset W \subset \overline{W} \subset B$, we choose C^∞ function $\lambda: B \to [0, 1]$ such that

$$\lambda(y) = 1,\ y \in D_r,\ \lambda(y) = 0,\ y \in B - W,$$

and define

$$\tilde{g}_n(y, t) = \tilde{f}(y, t) - \lambda(y) p_n(t).$$

Then $\tilde{g}_n$ is the desired map.

Theorem 5.1. *$\mathscr{V}$ is an open dense subset of $\mathscr{U}$.*

Proof Let (U, β) be a local chart around x_0 and f is represented under the chart by $\tilde{f}: B \times R \to R^n$. By Lemma 5.4, there are $\tilde{g}_n: B \to R^n$ such that $\tilde{g}_n$ has no critical points on D_r and $\tilde{f} \equiv \tilde{g}_n$ on $B-W$. Define g_n: $M \times R \to TM$ by

$$g_n(x, t) = (T\beta)^{-1}(\beta(x), \tilde{g}_n(\beta(x_1), t)), \quad x \in U,$$
$$g_n(x, t) = f(x, t), \qquad x \in M - U.$$

Then $g_n \in C^{r+1}$, $g_n \to f$ and g_n has no critical points on $\overline{V}$.

For any $h \in \mathscr{U}$, if x_1 is a critical point of h on $\overline{V}$, then by Theorem 3.2 x_1 is the only critical point of h on $\overline{V}$. The same argument as above easily proves that there are $h_n \in \mathscr{U}$, $h_n \to h$ and h_n has no critical points on $\overline{V}$. Hence $\mathscr{V}$ is a dense subset of $\mathscr{U}$. By Lemma 5.1, $\mathscr{V}$ is an open dense subset of $\mathscr{U}$.

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