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COMPLETE DISTRIBUTIVITY OF AN *l*-GROUP

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Abstract

This paper gives twelve equivalent topological conditions to the complete distributivity of an $O(D)$-contractible l-group.

§1. Introduction

There is a variety of known ways in which an l-group G may be given a topology. For example, we may define a natural, interesting and important order topology on any l-group as follows. Let G be an l-group. A net $\{x_\alpha|\alpha\in A\}$ of elements in G is O_1-convergent to x, in symbols $O_1\text{-}\lim\limits_{\alpha\in A}^{(G)} x_\alpha=x$, if there exists a net $\{y_\alpha|\alpha\in A\}$ in G such that $|x_\alpha-x|\leqslant y_\alpha$ for all $\alpha\in A$ and $y_\alpha\downarrow 0$. O_1-convergence is also called order convergence in [4] and O-convergence in [5]. A convergence in a set is called σ-convergence, if each subnet of a convergent net is also convergent to the same limit. From a σ-convergence of nets in a set E we can induce a topology, denoted by τ_σ. That is, a subset $S\subseteq E$ is τ_σ-closed if, whenever a net of elements in S σ-converges, the limit is also in S. The topology induced by O_1-convergence in an l-group G is called the order topology, denoted by O.

We may also introduce Dedekind topolsgy D. Let P be a partially ordered set and S be a subset of P. S is Dedekind closed if for every subset K of S which is upper directed and has a least upper bound in P, $\vee K$ is in S, and dually (see[6]). The reader may verify that the class of all Dedekind closed subsets of P is closed with respect to arbitrary intersection and finite union. By the Dedekind topology of a partially ordered set P, in symbol D, we mean the topology defined by taking all Dedekind subsets as its closed sets (see[7]). In general, this is a different topology from the order topology. However, it is clear that $O\geqslant D$.

In [1] we showed the following results;

Lemma 1. *An l-group G is a topological l-group in its order topology O if and only if G is O-contractible.*

Lemma 2. *An l-group G is a topological l-group in its Dedekind topology D if*

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and only if G is D–contractible.

In this paper we mainly show some equivalent topological conditions to the complete distributivity of an $O(D)$–contractible l–group. For the standard definitions and notations the reader is referred to [1, 2, 3, 8, 9].

§ 2. Complete Distributivity

Now we turn to the complete distributivity of an l–group in view of a topological l–group. An l–group G is completely distributive if

$$\bigwedge_{i\in I}\bigvee_{j\in J} g_{ij}=\bigvee_{f\in J^I}\bigwedge_{i\in I} g_{if(i)},$$

where $g_{ij}\in G$ and the indicated joins and meets exist. Let G be an l–group with a topology τ. The lattice order in G is lower (resp. upper) semicontinuous provided, whenever $a\nleqslant b$ (resp. $b\nleqslant a$) in G, there exists $U\in\tau a$ such that if $x\in U$ then $x\nleqslant b$ (resp. $b\nleqslant x$), where τa is the neighborhoods filter of a. The lattice order in G is semicontinuous if it is both upper and lower semicontinuous. Clearly, the fact that the lattice order in an l–group G is semicontinuous with respect to a topology τ is equivalent to $i\geqslant\tau$, where i is the interval topology on G (see [10]).

Let (G, τ) be a topological l–group. If M is a convex l–subgroup of G, then there exists a unique natural way to partially order the right coset G/M of M so that G/M is a lattice and the natural map φ: $G\to G/M$ is a lattice homomorphism. If τ_φ is the quotient topology generated by φ in G/M, then the quotient space $(G, \tau)/M=(G/M, \tau_\varphi)$.

Lemma 3. *Let (G, τ) be a topological l–group. If M is a convex l–subgroup of G, then the quotient space is a topological lattice.*

Proof Suppose $M+x$, $M+y\in G/M$ with x, $y\in G$. Then $\varphi(x\vee y)=M+x\vee y=(M+x)\vee(M+y)$. Let $\tau_\varphi[(M+x)\vee(M+y)]$ be the neighborhoods filter of $(M+x)\vee(M+y)$ with respect to τ_φ. Assume $P\in\tau_\varphi[(M+x)\vee(M+y)]$. Then there exists a τ_φ–open set P' such that

$$P'\subseteq P \text{ and } P'\in\tau_\varphi[(M+x)\vee(M+y)].$$

Let $\tau(x\vee y)$ be the neighborhoods filter of $x\vee y$ with respect to τ. Then $\varphi^{-1}(P')$ is τ–open by the definition of quotient topology. Moreover, $x\vee y\in\varphi^{-1}(P')$. Hence $\varphi^{-1}(P')\in\tau(x\vee y)$. Since (G, τ) is a topological lattice, there exist $N_1\in\tau x$ and $N_2\in\tau y$ such that

$$N_1\vee N_2=\{n_1\vee n_2 \mid n_1\in N_1,\ n_2\in N_2\}\subseteq\varphi^{-1}(P'), \tag{1}$$

that is $\varphi(N_1\vee N_2)\subseteq P'$. By [9, 36.10] φ is an open map from (G, τ) onto $(G/M, \tau_\varphi)$. Hence $\varphi(N_1)\in\tau_\varphi(M+x)$ and $\varphi(N_2)\in\tau_\varphi(M+y)$ because $M+x\in\varphi(N_1)$ and $M+y\in\varphi(N_2)$, where $\tau_\varphi(M+x)$ and $\tau_\varphi(M+y)$ are the neighborhoods filters of $M+x$

and $M+y$, respectively. Thus, it follows from (1) that $\varphi(N_1)\vee\varphi(N_2)=\varphi(N_1\vee N_2)\subseteq P'\subseteq P$. Therefore the operation of join in G/M is continuous. Dually we can show that the operation of meet in G/M is also continuous. That is to say, $(G/M, \tau_\varphi)$ is a topological lattice.

Lemma 4. *Let (G, τ) be a topological lattice of T_1-type. Then the lattice order in G is semicontinuous with respect to τ.*

Proof Assume $a, b\in G$ and $a\nleqslant b$. Let τa be the neighborhoods filter of a. If for every $U_\alpha\in\tau a$ there exists $x_\alpha\in U_\alpha$ such that $x_\alpha\leqslant b$, then we may regard the index set A of all U_α as an upper directed set, i.e., $\alpha\geqslant\beta$ if and only if $U_\alpha\subseteq U_\beta$. So we get a net $\{x_\alpha\mid\alpha\in A\}$ such that $x_\alpha\leqslant b$ for all $\alpha\in A$. Obviously, $\tau\text{-}\lim\limits_{\alpha\in A}x_\alpha=a$. On the other hand, $\tau\text{-}\lim b=b$ because τ is of T_1-type. Thus

$$b=\tau-\lim b=\tau-\lim_{\alpha\in A}(x_\alpha\vee b)=(\tau-\lim_{\alpha\in A}x_\alpha)\vee(\tau-\lim b)=a\vee b.$$

Hence $a\leqslant b$, contrary to the assumption $a\nleqslant b$. Therefore there certainly exists $U\in\tau a$ such that $x\nleqslant b$ when $x\in U$. Dually we can show the upper semicontinuity.

We recall that a subset M of an l-group G is called order closed, if $\{g_\alpha\mid\alpha\in A\}\subseteq M$ and $g=\bigvee\limits_{\alpha\in A}g_\alpha$ imply $g\in M$. Now we discuss the question when a convex l-subgroup M of an l-group G is order closed.

Lemma 5. *Let M be a convex l-subgroup of an O-contractible (a D-contractible) l-group G. Then the following conditions are equivalent:*

(I) *M is order closed.*

(II) *M is closed with respect to the order topology O (the Dedekind topology D).*

(III) *The quotient topology O_φ (D_φ) of G/M is of T_1-type.*

(IV) *The lattice order of G/M is semicontinuous with respect to $O_\varphi(D_\varphi)$.*

(V) *$O_1\text{-}\lim\limits_{\alpha\in A}{}^{(G)}x_\alpha=x$ and $M+x\nleqslant M+b$ imply that $M+x_\alpha\nleqslant M+b$ holds eventually ($x_\alpha\uparrow x$ and $M+x\nleqslant M+b$ imply that $M+x_\alpha\nleqslant M+b$ holds eventually).*

(VI) *$x=\bigvee\limits_{\alpha\in A}{}^{(G)}x_\alpha$ implies $M+x=\bigvee\limits_{\alpha\in A}{}^{(G/M)}(M+x_\alpha)$.*

Proof (I)$\Leftrightarrow$(II): This follows immediately from the above Lemma 1 and the Proposition 4.6 in [4].

(II)$\Leftrightarrow$(III): By [9, 36.10], the quotient topology $O_\varphi(D_\varphi)$ of G/M is of T_1-type if and only if M is O-closed (D-closed).

(III)$\Leftrightarrow$(IV): Suppose that the quotient topology $O_\varphi(D_\varphi)$ of G/M is of T_1-type. It follows from Lemma 1 and Lemma 3 that the quotient space $(G/M, O_\varphi)$ ($(G/M, D_\varphi)$) is a topological lattice of T_1-type. And Lemma 4 implies that the lattice order of G/M is semicontinuous with respect to $O_\varphi(D_\varphi)$.

(IV)$\Rightarrow$(V): Suppose that the lattice order of G/M is semicontinuous with respect to O_φ. Assume $O_1\text{-}\lim\limits_{\alpha\in A}{}^{(G)}x_\alpha=x$ and $M+x\nleqslant M+b$. Since O_1-convergence is stronger than the convergence with respect to the order topology[11], $O\text{-}\lim_{\alpha\in A}{}^{(G)}x_\alpha = x$. This implies $O_\varphi\text{-}\lim_{\alpha\in A}^{(G/M)}(M+x_\alpha) = M+x$. By definition of semicontinuity there exists a neighborhood U of $M+x$ with respect to the topology O_φ such that $U \not\leqslant M+b$. Therefore there exists $\alpha_0 \in A$ such that $M+x_\alpha \not\leqslant M+b$ when $\alpha \geqslant \alpha_0$.

Now suppose that the lattice order of G/M is semicontinuous with respect to D_φ. Let $x_\alpha \uparrow x (\alpha \in A)$ and $M+x \not\leqslant M+b$. Let U be a D-open set containing x. If there exists a subnet $\{x_{\alpha'} | \alpha' \in A'\}$ of $\{x_\alpha | \alpha \in A\}$ such that $\{x_{\alpha'} | \alpha' \in A'\} \subseteq G \setminus U$, then $x \in G\setminus U$, since $G\setminus U$ is D-closed and $\{x_{\alpha'} | \alpha' \in A'\}$ is increasing with $x = \bigvee_{\alpha' \in A'}^{(G)} x_{\alpha'}$. This contradicts $x \in U$. This contradiction shows $D\text{-}\lim_{\alpha\in A} x_\alpha = x$. In addition, by [9, 36.10] φ is a continuous map from (G, D) onto $(G/M, D_\varphi)$. It follows from this that

$$D-\lim_{\alpha\in A}(M+x_\alpha) = M+x. \tag{2}$$

By semicontinuity there exists a neighborhood U of $M+x$ with respect to the topology D_φ such that $U \not\leqslant M+b$. The formula (2) implies that there exists $\alpha_0 \in A$ such that $M+x_\alpha \not\leqslant M+b$ when $\alpha \geqslant \alpha_0$.

(V)⇒(VI): Suppose that (V) holds. If $S = \{x_\alpha | \alpha \in A\} \subseteq G$ and $x = \bigvee_{\alpha\in A}^{(G)} x_\alpha$, put

$$S' = \{x_{\alpha_1} \vee \cdots \vee x_{\alpha_k} | x_{\alpha_i} \in S,\ i=1, \cdots, k\}.$$

Then S' is upper directed. Since $x = \bigvee_{\alpha\in A}^{(G)} x_\alpha$, $x' \leqslant x$ for any $x' \in S'$. On the other hand, if $\bar{x}$ is any upper bound of S', then $\bar{x}$ is also an upper bound of S, and so $x \leqslant \bar{x}$. Therefore $x = \bigvee^{(G)} S'$. If we regard S' as an increasing net, then

$$O-\lim S' = x.$$

Since φ is a lattice homomorphism, and so is an order-preserving map, we have $M+x' \leqslant M+x$ for any $x' \in S'$. Assume $M+x' \leqslant M+b$ with all $x' \in S'$ and some $b \in G$. By (V) we have $M+x \leqslant M+b$. Thus $M+x = \bigvee_{x'\in S'}^{(G/M)}(M+x')$. In addition, $\{M+x' | x' \in S'\}$ is consists of the finite joins of elements in $\{M+x | x \in S'\}$. If $M+b$ is an upper bound of $\{M+x | x \in S\}$, then $M+b$ is also an upper bound of $\{M+x' | x' \in S\}$. But $M+x$ is an upper bound of $\{M+x | x \in S\}$. Therefore

$$M+x = \bigvee_{\alpha\in A}^{(M/G)}(M+x_\alpha).$$

(VI)⇒(I): Suppose that (VI) holds, that is, the natural map φ from G onto G/M is complete. It follows from Lemma 4.4 in [12] that M is order closed.

Let $\{M_\delta | \delta \in \Delta\}$ be the collection of all minimal prime subgroups of an l-group G. Put

$$D(G) = \bigcap_{\delta\in\Delta} M_\delta^*,$$

where M_δ^* is the order closure of M_δ for every $\delta \in \Delta$, $D(G)$ is called the distributive radical of the l—group G. A family $\{M_\delta | \delta \in \Delta\}$ of subgroups of an l-group G is said to intersect to zero, if $\bigcap_{\delta\in\Delta} M_\delta = \{0\}$. The following lemma is known[10][12, Theorems3, 4, 310.].

Lemma 6. *$D(G)$ is the intersection of all order closed prime subgroups of G. Equivalently $D(G)$ is the intersectiom of all order closed regular subgroups of G. G is completely distributive if and only if $D(G)=\{0\}$.*

From Lemma 5 and Lemma 6 we obtain the following theorem.

Theorem. *Let G be an O-contractible (D-contractible) l-group. Then the following conditions are equivalent:*

(1) *G is completely distributive.*

(2) *There exists a family $\{M_\delta \mid \delta\in\Delta\}$ of order closed prime (or regular) subgroups that intersects to zero.*

(3) *There exists a family $\{M_\delta \mid \delta\in\Delta\}$ of O-closed (D-closed) prime (or regular) subgroups that intersects to zero.*

(4) *There exists a family $\{M_\delta \mid \delta\in\}$ of prime (or regular) subgroups that intersects to zero and the quotient topology $O_{\varphi_\delta}(D_{\varphi_\delta})$ of G/M_δ is of T_1-type for every $\delta\in\Delta$.*

(5) *There exists a family $\{M_\delta \mid \delta\in\Delta\}$ of prime (or regular) subgroups that intersects to zero and the lattice order of G/M_δ is semicontinuous with respect to $O_{\varphi_\delta}(D_{\varphi_\delta})$ for every $\delta\in\Delta$.*

(6) *There exists a family $\{M_\delta \mid \delta\in\Delta\}$ of prime (or regular) subgroups that intersects to zero, moreover $O\text{-}\lim\limits_{\alpha\in\Delta}{}^{(G)}x_\alpha=x$ and $M_\delta+x\nleqslant M_\delta+b$ imply that $M+x_\alpha\nleqslant M+b$ holds eventually for every $\delta\in\Delta$ ($x_\alpha\uparrow x$ and $M_\delta+x\nleqslant M_\delta+b$ imply that $M_\delta+x_\alpha\nleqslant M_\delta+b$ holds eventually for every $\delta\in\Delta$).*

(7) *There exists a family $\{M_\delta \mid \delta\in\Delta\}$ of prime (or regular) subgroups that intersects to zero and $x=\bigvee\limits_{\alpha\in\Delta}{}^{(G)}x_\alpha$ implies $M_\delta+x=\bigvee\limits_{\alpha\in\Delta}{}^{(G/M^\delta)}(M_\delta+x_\alpha)$ for every $\delta\in\Delta$.*

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