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ON ROTATED VECTOR FIELDS IN HIGHER-DIMENSIONAL SPACES

CHEN YIYUAN (陈一元)*

Abstract

In this paper, the author introduces the concept of rotated vector fields in higher-dimensional spaces and proves some theorems which are similar to those in theory of rotated vector fields in plane.

Besides, some applications are given.

§1. Definition and Basic Properties

The theory of rotated vector fields in plane has many important applications in the planar qualitative theory[1]. We now give a definition of rotated vector field in higher-dimensional spaces.

Let vector field $X(x, \mu)=\sum_{i=1}^{n} X_i(x, \mu)\frac{\partial}{\partial x_i}$ be continuous on $R^n\times[0, T]$, where $x=(x_1, x_2, \cdots, x_n)\in R^n$, μ is a parameter. $X(x, \mu)$ is uniformly Lipschitz continuous with respect to x in any bounded region $D\subset R^n$, and has continuous partial derivative $\frac{\partial}{\partial\mu}X(x, \mu)$, the singular points of $X(x, \mu)$ are isolated.

Definition 1. *If the singular points of $X(x, \mu)$ are fixed for $\mu\in[0, T]$, there exists a continuous differential 1-form $\omega(X)$ dependent on X such that*

(1) $\omega(-X)=-\omega(X)$;

(2) $\omega(X(x, \mu))X(x, \mu)=0$;

(3) *For the regular point x of $X(x, \mu)$ and $\mu\in[0, T]$,*

$$\omega(X(x, \mu))\Big(\frac{\partial}{\partial\mu}X(x, \mu)\Big)>0;$$

(4) *For the regular point x of $X(x, \mu)$ and $\mu\in(0, T]$*

$$\omega(X(x, 0))X(x, \mu)\neq 0;$$

(5) *Let $K(x, \mu)$ be a field of hyperplanes defined by $\omega(X(x, \mu))=0$ for the regular point x of $X(x, \mu)$, $A(x)=\bigcap_{\mu\in[0,T]}K(x, \mu)$ is a vector space of dimension $(n-2)$, and $X(x, \mu)\notin A(x)$.*

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* Department of Mathematics, Nanjing University, Nanjing, China.

The vector field $X(x, \mu)$ is called a rotated vector field in R^n with respect to parameter $\mu \in [0, T]$.

We can consider a region $D \subset R^n$ and give a definition of rotated vector field on D.

For $n=2$, the above definition coincides with the definition of rotated vector field in plane. Let $X(x, \mu) = (P(x, \mu), Q(x, \mu))$ be a vector field in plane and $\omega(X) = -Q(x, \mu)dx_1 + P(x, \mu)dx_2$. Conditions (1), (2), (5) of Definition 1 are satisfied, and conditions (3), (4) mean that

$$\begin{vmatrix} P & Q \\ \dfrac{\partial P}{\partial \mu} & \dfrac{\partial Q}{\partial \mu} \end{vmatrix} > 0$$

and $(P(x, 0), Q(x, 0)), (P(x, \mu), Q(x, \mu))$ are not on the same line for $\mu \in (0, T]$. Hence $X(x, \mu)$ is a rotated field in plane if it satisfies the conditions of Definition 1.

In view of $X(x, \mu) \in K(x, \mu)$, we have the following theorem.

Theorem 1. *Let $X(x, \mu)$ be a rotated vector field in R^n. Then the $(n-1)$-dimensional integral surfaces of the field of hyperplanes $K(x, \mu)$ are the invariant manifolds of $X(x, \mu)$.*

Theorem 2. *Let $X(x, \mu)$ be a rotated vector field in R^n. $(n-1)$-dimensinal integral surface C of $K(x, \mu_0)$ does not meet another $(n-1)$-dimensional integral surface of $K(x, \mu_0)$ and splits R^n into two parts. Then a $P(P^+, P^-)$ stable trajectory of $X(x, \mu)$, $\mu \neq \mu_0$, different from a singular point, can not meet C.*

Proof Let integral surface C satisfying the conditions of Theorem 2 split R^n into parts I and II, $x \in C$ the regular point of $X(x, \mu)$. $K(x, \mu_0)$ splits $T_x R^n$ into two parts I′ and II′, the vectors in I′(II′) point to I(II), $\omega(X(x, \mu_0))$ takes different signs, say, positive (negative) in I′(II′). From conditions (3), (4) of Definition 1, we know that $X(x, \mu)$ points to I(II) for $\mu > \mu_0 (\mu < \mu_0)$. It is true for any regular point $x \in C$, because $K(x, \mu_0)$ is continuous and C does not meet another $(n-1)$-dimensional integral surface of $K(x, \mu_0)$. Hence any trajectory of $X(x, \mu)$, $\mu \neq \mu_0$, can not intersect C twice, and in view of the conditions (3), (4) of Definition 1, $X(x, \mu) \notin T_x C$ for $\mu \neq \mu_0$. This means that a $P(P^+, P^-)$ stable trajectory of $X(x, \mu)$, $\mu \neq \mu_0$, can not meet C.

Corollary. *Let $X(x, \mu)$ be a rotated vector field, and the field of hyperplanes $K(x, \mu_0)$ completely integrable. Let all $(n-1)$-dimensional integral surfaces of $K(x, \mu_0)$ not meet each other in D and each $(n-1)$-dimensional integral surface splits D into two parts. Then $X(x, \mu)$, $\mu \neq \mu_0$, has no $P(P^+, P^-)$ stable trajectory which is not a singular point in D.*

§ 2. Examples

We now give two examples about the rotated vector fields in R^3 and apply the conclusion of § 1 to prove these fields have no $P(P^+, P^-)$ stable trajectory different from singular point.

In order to prove a given vector field $X(x, \mu)$ to be a rotated vector field, we must find a suitable continuous differential 1-form $\omega(X)$. We give several useful differential 1-forms in R^3, which can be generalized to higher-dimensional spaces.

Let $X(x, \mu)=\sum_{i=1}^{3} X_i(x, \mu)\frac{\partial}{\partial x_i}$ satisfy the assumption before Definition 1.

1. $\omega(X)=(a_2X_2-a_1X_3)dx_1+(a_3X_3-a_2X_1)dx_2+(a_1X_1-a_3X_2)dx_3$, where a_1, a_2, a_3 are C^1 functions of $x=(x_1, x_2, x_3)$. This form satisfies the conditions (1), (2) of Definitisn 1. Obviously, vector $(a_3(x), a_1(x), a_2(x))\in A(x)$. If $(a_3(x), a_1(x), a_2(x))\neq(0, 0, 0)$, and $(a_3(x), a_1(x), a_2(x))$ is not parallel to $X(x, \mu)$ at the regular points of $X(x, \mu)$, then condition (5) is satisfied. We can give an analogous 1-form in other odd-dimensional spaces R^n, $n>3$.

Example 1. Consider system

$$\begin{aligned}\frac{dx_1}{dt}&=u(x)+\mu\phi(x),\\ \frac{dx_2}{dt}&=\mu\psi(x),\\ \frac{dx_3}{dt}&=\mu(\psi(x)+u(x)).\end{aligned}\tag{1}$$

where u, ϕ, ψ are C^1 functions and $u>0$, $\mu\in(-\infty, +\infty)$,

In the above form, we take $a_1=a_2=a_3=1$,

$$\omega(X)=-\mu u dx_1+(\mu\psi+\mu u-u-\mu\phi)dx_2+(-\mu\psi+u+\mu\phi)dx_3.$$

Obviously, conditions (1), (2), (5) of Definition 1 are satisfied. On the other hand, we have

$$\omega(X(x, \mu))\left(\frac{\partial}{\partial\mu}X(x, \mu)\right)=u^2>0,$$

$$\begin{aligned}\omega(X(x, 0))(X(x, \mu))&=(-u\,dx_2+u\,dx_3)X(x, \mu)\\ &=\mu u^2\neq 0, \text{ if } \mu\neq 0.\end{aligned}$$

Hence system (1) is a rotated vector field.

We have $\omega(X(x, 0))=-u dx_2+u dx_3$, the field of hyperplanes $K(x, 0)$ is completely integrable, and integral surfaces are $x_2-x_3=$const. which satisfy the conditions of the corollary to Theorem 2, so system (1) has no $P(P^+, P^-)$ stable trajectory.

2. $\omega(X)=(-a_1X_2-a_2X_3)dx_1+a_1X_1dx_2+a_2X_1dX_3$, where a_1, a_2 are C^1 functions of $x=(x_1, x_2, x_3)$. This form is a special case of the form in 1, but it can be generalized to arbitrary higher-dimensional spaces.

Example 2. Consider system

$$\frac{dx_1}{dt} = -(x_2+x_3)x_1+\mu x_1,$$
$$\frac{dx_2}{dt} = (x_2+x_3)^3x_1+(x_2+x_3)x_1^3+\Phi(x), \tag{2}$$
$$\frac{dx_3}{dt} = \mu x_1-\Phi(x),$$

where $\Phi(x)$ is C^1 function, $\mu\in(-\infty, +\infty)$.

Taking $\omega(X) = -(X_2+X_3)dx_1+X_1dx_2+X_1dx_3$, we have

$$\omega(X(x, \mu))\left(\frac{\partial}{\partial\mu}X(x, \mu)\right) = -[(x_2+x_3)^2+x_1^2+1](x_2+x_3)x_1^2,$$
$$\omega(X(x, 0))X(x, \mu) = -\mu[(x_2+x_3)^2+x_1^2+1](x_2+x_3)x_1^2.$$

Planes $x_1=0$ and $x_2+x_3=0$ divide R^n into four regions where $(0, 1, -1)\in A(x)$ is not parallel to $X(x, \mu)$, so system (2) is a rotated vector field in the four regions.

The field of hyperplanes $K(x, 0)$ defined by

$$\omega(X(x, 0)) = -x_1(x_2+x_3)[(x_2+x_3)^2+x_1^2]dx_1-x_1(x_2+x_3)(dx_2+dx_3)=0$$

is completely integrable (in the four regions), but we can not obtain its first integral from

$$\frac{\partial x_3}{\partial x_1} = -(x_2+x_3)^2-x_1^2,$$
$$\frac{\partial x_3}{\partial x_2} = -1. \tag{3}$$

Let $x_2+x_3\to y$, $x_1\to x_1$, $x_2\to x_2$. (3) becomes

$$\frac{\partial y}{\partial x_1} = -y^2-x_1^2,$$
$$\frac{\partial y}{\partial x_2} = 0. \tag{3'}$$

Analysing the intersections of the integral surfaces of (3′) with x_1y-plane and x_2y-plane, we know that the integral surfaces in the four regions satisfy the conditions of the corollary to Theorem 2. Hence system (2) has no $P(P^+, P^-)$ stable trajectory in the four regions for $\mu\neq0$.

Plane $x_1=0$ is an invariant manifold of system (2) and there is no $P(P^+, P^-)$ stable trajectory different from singular point in the plane. A trajectory of system (2) intersects half plane $x_2+x_3=0$, $x_1>0$ (or $x_2+x_3=0$, $x_1<0$) at one point at most. Therefore system (2) has no $P(P^+, P^-)$ stable trajectory different from singular point for $\mu\neq0$. The conclution is also true for $\mu=0$, but we do not prove it.

§ 3. Limit Surfaces

Definition 2. *If B is an $(n-1)$-dimensional closed invariant manifold of vector field $X(x)$, which does not meet other $(n-1)$-dimensional invariant manifold in its exterior (interior) neighbourhood and the singular points on which are isolated, and for* *each point $x_0 \in B$, there is a trajectory $\phi(p, t)$ in the exterior (interior) neighbourhood of B such that $x_0 \in \Omega_p \subset B$ and $\phi(p, t)$ is uniformly asymptotic to Ω_p (cf. [2], Chapter 5, § 9, Definition 2) then B is called an exterior (interior) uniformly stable limit surface of $X(x)$.*

Similarly, we can give a definition of exterior (interior) uniformly unstable limit surface.

For abbreviation, we call the surface given by the above definition limit surface.

Theorem 3. *If $X(x, \mu)$ is a rotated vector field and closed integral surface $B(\mu_i)$ of $K(x, \mu_i)$ is the limit surface of $X(x, \mu_i)$, $i=1, 2$, then $B(\mu_1) \cap B(\mu_2)$ is composed of the singular points of $X(x, \mu)$.*

Proof Suppose that $B(\mu_1)$ is an exterior uniformly stable limit surface of $X(x, \mu_1)$ and $L = B(\mu_1) \cap B(\mu_2)$. If $a \in L$ is a regular point of $X(x, \mu)$, which is also a regular point of $X(x, \mu_1)$ (Notice that the singular points of $X(x, \mu)$ are independent of μ), there is a trajectory $\phi(p, t)$ in the exterior neighbourhood of B such that $a \in \Omega_p \subset B(\mu_1)$ and $\phi(p, t)$ is uniformly asymptotic to $\Omega_p \subset B(\mu_1)$. From [2] (Chapter 5, § 9, Theorem 40), we know that Ω_p is the minimal set which is compact. Hence the trajectory Γ of $X(x, \mu_1)$ passing a is P stable ([2], Chapter 5, § 7, Theorem 27) and can not meet $B(\mu_2)$ (Theorem 2). But this is contrary to the hypothesis $a \in B(\mu_1) \cap B(\mu_2)$. So Theorem 3 is proved.

From the above proof, we can come to the following conclusion.

Corollary. *If $X(x, \mu)$ is a rotated vector field in $D \subset R^n$, all $(n-1)$-dimensional closed integral surfaces of $K(x, \mu_0)$ do not meet each other and fill D. Then $X(x, \mu)$, $\mu \neq \mu_0$, has not any limit surface.*

§ 4. Absence of Limit Surfaces

It is difficult to prove a vector field $X(x)$ in $R^n (n>2)$ has no limit surfaces. But we can use the following theorem to assert that some rotated vector fields have no limit surfaces with property in Theorem 3.

Theorem 4. *Let $K(x)$ be a C^1 field of hyperplanes defined in a simple connected region $D \subset R^n$. There exist a C^1 function $B(x)$ and a C^1 differential $(n-1)$-form Ω in D, such that*

(i) *$\Omega(l_1(x), l_2(x) \cdots, l_{n-1}(x)) = 0$ for $x \in D$ and $l_i(x) \in K(x)$, $i=1, 2, \cdots, n-1$;*

(ii) *Let $d(B(x)\Omega) = W(x) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$. $W(x)$ does not change its sign in D and the set $\{x \mid W(x)=0, x \in D\}$ contains no n-dimensional region. Then $K(x)$ has no $(n-1)$-dimensional closed integral surface in D.*

Proof Suppose that S is an $(n-1)$-dimensional closed integral surface in D, and region $V \subset D$ has boundary S. We have (stokes formula)

$$\int_T B(x)\Omega = \int_V d(B(x)\Omega) = \int_V W(x)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n. \tag{4}$$

Condition (i) means that $\int_S B(x)\Omega = 0$. But condition (ii) means that $\int_V W(x)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \neq 0$. This is contrary to (4). The conclusion is reached.

This theorem is a generalization of Dulac's theorem in plane[1]. Let $n=2$. A closed trajectory of $X(x) = (X_1(x), X_2(x))$ is a closed integral curve of $K(x)$, where $K(x)$ is defined by

$$\Omega = -X_2(x)dx_1 + X_1(x)dx_2 = 0.$$

So we have $d(B\Omega) = \left(\frac{\partial BX_1}{\partial x_1} + \frac{\partial BX_2}{\partial x_2}\right) dx_1 \wedge dx_2$, $W(x) = \frac{\partial BX_1}{\partial x_1} + \frac{\partial BX_2}{\partial x_2}$. We obtain Dulac's theorem from Theorem 4.

Theorem 5. *Let* $X(x, \mu) = \sum_{i=1}^{3} X_i(x, \mu)\frac{\partial}{\partial x_i}$ *be a* C^1 *rotated vector field in* R^3, *where*

$$\omega(X(x, \mu)) = (a_2X_2 - a_1X_3)dx_1 + (a_3X_3 - a_2X_1)dx_2 + (a_1X_1 - a_3X_2)dx_3,$$

a_1, a_2, a_3, *are* C^1 *functions of* x. *There exists a* C^1 *function* $B(x)$ *in* R^n, *such that* $W = \sum_{i=1}^{3} \frac{\partial BX_i}{\partial x_i}$ *does not change its sign and the set* $\{x \mid W(x) = 0, x \in R^3\}$ *contains no 3-dimensional region. Then* $X(x, \mu)$ *has no limit surface consisting of 2-dimensional integral surfaces of* $K(x, \mu)$.

Proof Let $\Omega = X_3dx_1 \wedge dx_2 + X_1d_2 \wedge dx_3 + X_2dx_3 \wedge dx_1$. From § 2, we know that $X(x, \mu)$ and $(a_3(x), a_1(x), a_2(x))$ span $K(x, \mu)$. It is easy to check that $\Omega\ (X(x, \mu), (a_3(x), a_1(x), a_2(x))) = 0$. So Ω satisfies the condition (i) of Theorem 4. From the condition (ii) of Theorem 4 and $d(B\Omega) = \sum_{i=1}^{3} \frac{\partial BX_i}{\partial x_i}$ we obtain Theorem 5.

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Reference

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