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FOLIATION ON A SURFACE OF CONSTANT CURVATURE AND SOME NONLINEAR EVOLUTION EQUATIONS

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Abstract

In this paper, the author explains the solutions of Sine-Gordon equation and KdV equation as the geodesic curvature of the leaves of a foliation on a surface of constant curvature and negative constant curvature respectively. Therefore, a question which was asked in a paper of S. S. Chern and K. Tenenblat is answered.

In [1], S. S. Chern and K. Tenenblat established the connection between the foliation on a surface of constant curvature and MKdV equation. They explained the solution of MKdV equation as the geodesic curvature of the leaves of a foliation on a surface of constant curvature. In that paper, they asked if there is a similar explanation for KdV equation. In this paper, we'll give a similar explanation for Sine-Gordon equation and for KdV equation on a surface of negative constant curvature.

Consider a surface M endowed with a C^∞-Riemannian metric of constant Gaussian curvature K, and a foliation on M given by curves. Suppose that both M and the foliation are oriented. At a point x, we take e_1 to be unit tangent vector to the leaf of the foliation through x. Since M is oriented, turning e_1 on $\frac{\pi}{2}$ we obtain e_2. Thus, we obtain an orthonormal frame field e_1, e_2 and its dual frame field ω_1, ω_2. Then we have the structure equations

$$d\omega_1=\omega_{12}\wedge\omega_2,\ d\omega_2=-\omega_{12}\wedge\omega_1,\ d\omega_{12}=-K\omega_1\wedge\omega_2. \tag{1}$$

Under the choice of the frame field the foliation is defined by $\omega_2=0$ and ω_1 is the element of arc on the leaves. We write

$$\omega_{12}=p\omega_1+q\omega_2,$$

then p is the geodesic curvature of the leaves.

We coordinatize M by the coordinates x, t, such that

$$\omega_1=\eta\,dx+A\,dt,\ \omega_2=B\,dt,\ \omega_{12}=u_x\,dx+C\,dt,$$

where η is an arbitrary constant, u is a function of x and t. A, B and C are functions

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of x, t, u and the partial derivatives of u. Thus the leaves are given by $t=$const. and ηx is the arc length of the leaves. Substituting (2) into (1), we get

$$A_x=u_xB,\ B_x=\eta C-u_xA,\ C_x-u_{xt}=-K\eta B. \tag{3}$$

Elimination of B and C gives

$$u_{xt}=C_x+K\eta B=\frac{1}{\eta}\left(\frac{A_{xx}u_x-A_xu_{xx}+Au_x^3}{u_x^2}\right)_x+K\eta\frac{A_x}{u_x}. \tag{4}$$

If we take $A=F(u)/\eta$, F is an arbitrary function of u, then (4) is reduced to

$$u_{xt}=\frac{1}{\eta^2}(F'(u)+F'''(u))u_x^2+\frac{1}{\eta^2}(F(u)+F''(u))u_{xx}+KF'(u). \tag{5}$$

Comparing

$$\omega_{12}=p\omega_1+q\omega_2=\eta p\,dx+(Ap+Bq)dt$$

with

$$\omega_{12}=u_x\,dx+C\,dt,$$

we have

$$u_x=\eta p,\ C=Ap+Bq.$$

Substituting $A=F(u)/\eta$, $B=F'(u)/\eta$ and $C=(F(u)+F''(u))u_x/\eta^2$ into them, we obtain

$$p=\frac{1}{\eta}u_x,\ q=\frac{1}{\eta}(\ln F')_x,$$

where p is the geodesic curvature of the leaves and u is the solution of equation (5). Since, for any C^∞ function f on M,

$$df=f_1\omega_1+f_2\omega_2,$$

we have $f_x=f_1\eta$. Then $p=u_1$, $q=(\ln F')$, where f_1, f_2, u_1, $(\ln F')_1$ represent the covariant derivative.

In particular, if we take $F(u)=-\cos u/K$ $(K\neq 0)$, then (5) is reduced to $u_{xt}=\sin u$. This is Sine-Gordon equation and (x, t) are Tchebyshev coordinates. Thus, we have proved the following theorem.

Theorem. *Suppose thae M is a surface with constant Gaussian curvature K and coordinatized by x, t such that (2) is established $(A=-\cos u/\eta K)$. Then $p=\frac{1}{\eta}u_x$, where p is the geodesic curvature of $t=$constant, and u satisfies Sine-Gordon equation $u_{xt}=\sin u$.*

Since $p/q=\tan u$ and

$$p_x=p_{,}\eta,\ q_x=q_{,}\eta;$$

$$p_x=q_x\tan u+q\sec^2 u\,u_x,$$

we have

$$p_1\eta=q_1\eta\frac{p}{q}+q\left(1+\frac{p^2}{q^2}\right)p\eta,$$

i.e.

$$q_1=-(p^2+q^2)+\frac{q}{p}p_1.$$

If we take $F(u)=\eta^2$, (5) is reduced to $u_{xt}=u_{xx}$, and q is infinitive.

In the following, we suppose that M is a surface of negative constant curvature K. We set

$$\omega_1=\frac{1+\eta^2-u}{\sqrt{-K}}\,dx+\frac{B+C}{\sqrt{-K}}\,dt,$$
$$\omega_2=\frac{-2A}{\sqrt{-K}}\,dt, \tag{6}$$
$$\omega_{12}=(1-\eta^2+u)dx+(B-C)dt,$$

and substituting them into the structure equation (1), we have

$$u_t+B_x+C_x=-2A(1-\eta^2+u), \tag{7}$$
$$A_x=C-(\eta^2-u)B, \tag{8}$$
$$u_t+B_x-C_x=-2A(1+\eta^2-u). \tag{9}$$

From (7) and (9), we have $B_x=-2A$, and then

$$C=-\frac{B_{xx}}{2}+(\eta^2-u)B,$$
$$C_x=-\frac{B_{xxx}}{2}-u_xB+(\eta^2-u)B_x.$$

Therefore, (7), (8) and (9) are reduced to

$$u_t=\frac{B_{xxx}}{2}+u_xB-2(\eta^2-u)B_x. \tag{10}$$

If we take $B=-(2u+4\eta^2)$, we obtain the KdV equation

$$u_t+u_{xxx}+6uu_x=0.$$

Comparing

$$\omega_{12}=p\omega+q\omega_2=\frac{p(1+\eta^2-u)}{\sqrt{-K}}\,dx+\frac{1}{\sqrt{-K}}(p(B+C)-2Aq)dt$$

with

$$\omega_{12}=(1-\eta^2+u)dx+(B-C)dt,$$

we have

$$p=\frac{1-\eta^2+u}{1+\eta^2-u}\sqrt{-K}, \tag{11}$$

and

$$q=\frac{p(B+C)-\sqrt{-K}(B-C)}{2A}. \tag{12}$$

Therefore, we obtain the following theorem.

Theorem. *Suppose M is a surface with negative constant Gaussian curvature K and coordinatized by x, t such that (6) is established $(B=-(2u+4\eta^2))$. Then $p=\frac{1-\eta^2+u}{1+\eta^2-u}\sqrt{-K}$, where p is the geodesic curvature of $t=$*constant *and u satisfies the* KdV *equation $u_t+u_{xxx}+6uu_x=0$.*

Simply, we assume $K=-1$. Then (11) and (12) are reduced to

$$p=\frac{1-\eta^2+u}{1+\eta^2-u},\quad q=\frac{(p+1)u_{xx}}{2u_x}, \tag{13}$$

$(A=u_x,\ B=-(2u+4\eta^2),\ C=-(\eta^2-u)(2u+4\eta^2)+u_{xx})$. Since

$$dp=p_1\omega_1+p_2\omega_2=p_xdx+p_tdt,$$

we have

$$p_x = p_1(1+\eta^2-u),$$

and

$$p_{xx} = -u_x p_1 + p_{11}(1+\eta^2-u)^2.$$

Again, since

$$u = \eta^2 + \frac{p-1}{p+1}$$

and

$$1+\eta^2-u = \frac{2}{p+1},$$

we have

$$u_x = \frac{2p_x}{(1+p)^2} = \frac{2p_1(1+\eta^2-u)}{(p+1)^2} = 4p_1/(p+1)^3.$$

Hence

$$u_{xx} = 4(p_1/(p+1)^3)_x = 8p_{11}/(p+1)^4 - 24p_1^2/(p+1)^5.$$

Subtituting them into (13), we obtain the relationship between p and q of this foliation

$$q = p_{11}/p_1 - 3p_1/(p+1).$$

The above discussion suits the KdV equations of higher order. In fact, if we take $B=-1$, we obtain the equation $u_t = u_x$; if we take $B = -(2u+4\eta^2)$, we obtain the KdV equation; if we take $B = -\left(\frac{1}{2}u_{xx} + \frac{3}{2}u^2 + 2u\eta^2 + 4\eta^4\right)$, we obtain the equation

$$u_t + \frac{1}{4}u_{xxxxx} + \frac{5}{2}uu_{xxx} + 5u_x u_{xx} + \frac{15}{2}u^2 u_x = 0$$

and so on.

If $K>0$, we change $\sqrt{-K}$ to $\sqrt{K}$ in ω_1 and ω_2, then (7) and (8) are invariant but (9) is changed to

$$-u_t + B_x - C_x = 2A(1+\eta^2-u).$$

Thus, we obtain

$$B_x = 2A(\eta^2-u),\ C = A_x + (\eta^2-u)B,$$
$$C_x = A_{xx} - u_x B + (\eta^2-u)B_x,$$

and the structure equation is reduced to

$$u_t + A_{xx} + 2A - u_x B + 2(\eta^2-u)^2 A = 0, \qquad (14)$$

where $A = \frac{1}{2(\eta^2-u)}B_x$. But we could not obtain the KdV equation from (14).

The discussion on KdV equation can be expanded to the general KdV equation too. We consider η^2 as a function of x and t:

$$\eta^2 = xf(t) + \lambda g(t),$$

where λ is an arbitrary constant, $f(t)$ is an arbitrary function of t and $g(t) = e^{-\int 12f(t)dt}$. Then (10) is changed to

$$u_t = \frac{1}{2}B_{xxx} - (\eta^2-u)_x B - 2(\eta^2-u)B_x + (\eta^2)_t.$$

Substituting $\eta^2 = xf(t) + \lambda g(t)$ and $B = -(2u+4\eta^2) = -(2u+4xf(t)+4\lambda g(t))$ into it, we obtain

$$u_t+u_{xxx}+6uu_x+6f(t)u-(f^1+12f^2)x=0, \tag{15}$$

we call it the general KdV equation.

When $f=0$, (15) is reduced to the KdV equation

$$u_t+u_{xxx}+6uu_x=0.$$

When $f=\frac{1}{12t}$, (15) is reduced to the cylindrical equation

$$u_t+u_{xxx}+6uu_x+\frac{u}{2t}=0.$$

When $f=1$, (15) is reduced to equation

$$u_t+u_{xxx}+6uu_x+6u-12x=0,$$

and so on.

Reference

[1] Chern, S. S. and Tenenblat, Foliation on a surface of constant curvature and the Modified Korteweg-de Vries equations, *J. Diff. Geo.*, **16** (1981), 347—349.