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GENERALIZED TORUS IN R^3 WITH PRESCRIBED MEAN CURVATURE

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Abstract

Let H be a given function satisfying two conditions on some region N in R^3. It is proved that there exists a closed surface in N which is homeomorphic to a torus such that its mean curvature is equal to H.

§ 0. Introduction

In 1980, S. T. yau[1] raised the following problem:

Given function H in R^3, can you find suitable condition on H such that there is a closed surface in R^3 whose mean curvature is given by H?

In 1983, A. E. Treibergs and S. W. Wei[2] set up some conditions on H, where the closed surface is homeomorphic to a unit sphere.

In this paper, we consider the mean curvature function of a closed surface which is homeomorphic to a torus in R^3, and obtain an existence theorem.

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§ 1. Mean Curvature Equation

In R^3, set $\boldsymbol{\rho}=(b\cos\theta,\ b\sin\theta,\ 0)$, $\boldsymbol{r}=(\cos\varphi\cos\theta,\ \cos\varphi\sin\theta,\ \sin\varphi)$. We get a torus T_1 in R^3 whose position vector field

$$\boldsymbol{y}=\boldsymbol{\rho}+\boldsymbol{r}, \tag{1.1}$$

where $(\varphi,\ \theta)\in R/2\pi Z\times R/2\pi Z$ (Z is an integer group). Constant $b>1$.

In this paper, we introduce generalized torus $M=\boldsymbol{x}(T_1)$ whose position vector field

$$\boldsymbol{x}=\boldsymbol{\rho}+\frac{be^u}{1+e^u}\boldsymbol{r}, \tag{1.2}$$

where u is a differentiable function on T_1. It is obvious that $\boldsymbol{x}$ is one to one, and $\boldsymbol{x}$ is a homeomorphism from T_1 onto M. Evidently.

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$$\boldsymbol{x}_\varphi=\frac{be^u}{1+e^u}\left(\boldsymbol{r}_\varphi+\frac{u_\varphi}{1+e^u}\boldsymbol{r}\right), \tag{1.3}$$

$$\boldsymbol{x}_\theta=\boldsymbol{\rho}_\theta+\frac{be^u}{1+e^u}\left(\boldsymbol{r}_\theta+\frac{u_\theta}{1+e^u}\boldsymbol{r}\right). \tag{1.4}$$

Then, the unit normal vector $\boldsymbol{n}$ at a point on M satisfies

$$\boldsymbol{n}=\frac{\boldsymbol{x}_\varphi\times\boldsymbol{x}_\theta}{|\boldsymbol{x}_\varphi\times\boldsymbol{x}_\theta|}=\{(1+e^u+e^u\cos\varphi)^2[u_\varphi^2+(1+e^u)^2]+e^{2u}u_\theta^2\}^{-\frac{1}{2}}$$
$$\times\left\{(1+e^u+e^u\cos\varphi)[u_\varphi\boldsymbol{r}_\varphi-(1+e^u)\boldsymbol{r}]+\frac{1}{b}e^u u_\theta\boldsymbol{\rho}_\theta\right\}. \tag{1.5}$$

Secondly, we can see

$$E^*=\boldsymbol{x}_\varphi^2=\frac{b^2e^{2u}}{(1+e^u)^2}\left[1+\frac{u_\varphi^2}{(1+e^u)^2}\right], \tag{1.6}$$

$$F^*=\boldsymbol{x}_\varphi\cdot\boldsymbol{x}_\theta=\frac{b^2e^{2u}}{(1+e^u)^4}u_\theta u_\varphi, \tag{1.7}$$

$$G^*=\boldsymbol{x}_\theta^2=b^2\left[\left(1+\frac{e^u\cos\varphi}{1+e^u}\right)^2+\frac{e^{2u}u_\theta^2}{(1+e^u)^4}\right]. \tag{1.8}$$

By a calculation, we can see

$$L^*=\boldsymbol{n}\cdot\boldsymbol{x}_{\varphi\varphi}=be^u(1+e^u)^{-1}(1+e^u+e^u\cos\varphi)\cdot[u_\varphi^2+(1+e^u)-u_{\varphi\varphi}]$$
$$\times\{(1+e^u+e^u\cos\varphi)^2[u_\varphi^2+(1+e^u)^2]+e^{2u}u_\theta^2\}^{-\frac{1}{2}}, \tag{1.9}$$

$$M^*=\boldsymbol{n}\cdot\boldsymbol{x}_{\theta\varphi}=be^u(1+e^u)^{-1}[e^u(1+\cos\varphi)u_\theta u_\varphi-(1+e^u+e^u\cos\varphi)u_{\theta\varphi}$$
$$-e^u u_\theta\sin\varphi]\{(1+e^u+e^u\cdot\cos\varphi)^2\cdot[u_\varphi^2+(1+e^u)^2]+e^{2u}u_\theta^2\}^{-\frac{1}{2}}, \tag{1.10}$$

$$N^*=\boldsymbol{n}\cdot\boldsymbol{x}_{\theta\theta}=be^u(1+e^u)^{-1}\{(1+e^u+e^u\cos\varphi)^2\cdot e^{-u}[(1+e^u)\cos\varphi+u_\varphi\sin\varphi]$$
$$+(e^u\cos\varphi+e^u-1)u_\theta^2-(1+e^u+e^u\cos\varphi)u_{\theta\theta}\}$$
$$\times\{(1+e^u+e^u\cos\varphi)^2\cdot[u_\varphi^2+(1+e^u)^2]+e^{2u}u_\theta^2\}^{-\frac{1}{2}}, \tag{1.11}$$

$$E^*G^*-F^{*2}=\frac{b^4e^{2u}}{(1+e^u)^6}\{(1+e^u+e^u\cos\varphi)^2[(1+e^u)^2+u_\varphi^2]+e^{2u}u_\theta^2\}. \tag{1.12}$$

Using the local orthogonal frame e_1, e_2 on T_1, we know the first fundamental form of T_1 in (1.1) is

$$ds^2=(d\varphi)^2+(b+\cos\varphi)^2(d\theta)^2. \tag{1.13}$$

Set

$$\omega^1=d\varphi,\ \omega^2=(b+\cos\varphi)d\theta. \tag{1.14}$$

Obviously,

$$u_1=u_\varphi,\ u_2=(b+\cos\varphi)^{-1}u_\theta,$$
$$\omega_2^1=\sin\varphi\, d\theta,\ u_{\varphi\varphi}=u_{11},$$
$$u_{\theta\theta}=(b+\cos\varphi)^2u_{22}+u_1\sin\varphi(b+\cos\varphi),$$
$$u_{\theta\varphi}=u_{12}(b+\cos\varphi)-u_2\sin\varphi, \tag{1.15}$$

where subscript $i(i=1, 2)$ denotes the covariant derivative along e_i. Utilizing (1.15) and the mean curvature formula $H=\frac{1}{2}\frac{L^*G^*-2M^*F^*+N^*E^*}{E^*G^*-F^{*2}}$, we obtain the mean curvature equation of M

$$[e^{-2u}(1+e^u)^2(b+\cos\varphi)^{-2}(1+e^u+e^u\cos\varphi)^2+u_2^2]u_{11}-2u_1u_2u_{12}+[(1+e^u)^2+u_1^2]u_{22}$$
$$+2(b+\cos\varphi)^{-1}\cdot\sin\varphi(1+e^u+e^u\cos\varphi)^{-1}(1+e^u-be^u)u_1u_2^2$$
$$+(b+\cos\varphi)^{-2}\cdot(b-1-e^{-u})\sin\varphi[(1+e^u)^2+u_1^2]u_1$$
$$-(b+\cos\varphi)^{-2}\cdot(1+e^u+e^u\cos\varphi)(1+e^u)[(1+e^{-u})^2+\cos\varphi(1+2e^{-u})]u_1^2$$
$$-e^u(1+e^u)(1+e^u+e^u\cos\varphi)^{-1}[(1+e^u)+(2+e^u)\cos\varphi]u_2^2$$
$$-(b+\cos\varphi)^{-2}(1+e^u)^3(1+e^u+e^u\cos\varphi)(e^{-2u}+e^{-u}+2e^{-u}\cos\varphi)$$
$$+2b(b+\cos\varphi)^{-2}e^{-u}(1+e^u+e^u\cos\varphi)^{-1}(1+e^u)^{-1}\{(1+e^u+e^u\cos\varphi)^2$$
$$\times[(1+e^u)^2+u_1^2]+e^{2u}(b+\cos\varphi)^2u_2^2\}^{3/2}\cdot H\left(\boldsymbol{\rho}+\frac{be^u}{1+e^u}\boldsymbol{r}\right)=0. \tag{1.16}$$

Equation (1.16) is a quasilinear elliptic equation on T_1, where H is a given function.

§ 2. The Main Theorem

In R^3, we have a variety of torus T_s:

$$\boldsymbol{y}_s=\boldsymbol{\rho}+s\boldsymbol{r}, \tag{2.1}$$

where s is a constant and $0<s<b$. The mean curvature $H_{T_s}(\boldsymbol{y}_s)=\dfrac{1}{2s}+\dfrac{\cos\varphi}{2(b+s\cos\varphi)}$. Set $N=\bigcup\limits_{s\in(0,b)}\boldsymbol{T}_s$. N is a connected open subset in R^3. The given function $H(\boldsymbol{x})$ lies in $C^{k,\alpha}(N)$ ($k\geqslant 1$, an integer, $0<\alpha<1$).

We introduce a new linear equation: $\forall v\in C^{k,\alpha}(T_1)$, $\forall t\in[0, 1]$,

$$[e^{-2v}(1+e^v)^2(b+\cos\varphi)^{-2}(1+e^v+e^v\cos\varphi)^2+v_2^2]u_{11}$$
$$-2v_1v_2u_{12}+[(1+e^v)^2+v_1^2]u_{22}-u=t[B(x, v, \nabla v)-v], \tag{2.2}$$

where

$$B(x, v, \nabla v)=-2(b+\cos\varphi)^{-1}\sin\varphi(1+e^v+e^v\cos\varphi)^{-1}\cdot(1+e^v-be^v)v_1v_2^2$$
$$-(b+\cos\varphi)^{-2}(b-1-e^{-v})\sin\varphi\cdot[(1+e^v)^2+v_1^2]v_1$$
$$+(b+\cos\varphi)^{-2}(1+e^v+e^v\cos\varphi)\cdot(1+e^v)[(1+e^{-v})^2$$
$$+\cos\varphi(1+2e^{-v})]v_1^2+e^v\cdot(1+e^v)$$
$$\times(1+e^v+e^v\cos\varphi)^{-1}[(1+e^v)+(2+e^v)\cos\varphi]v_2^2$$
$$+(b+\cos\varphi)^{-2}(1+e^v)^3(1+e^v+e^v\cos\varphi)(e^{-2v}+e^{-v}+2e^{-v}\cos\varphi)$$
$$-2b(b+\cos\varphi)^{-2}e^{-v}(1+e^v+e^v\cos\varphi)^{-1}(1+e^v)^{-1}$$
$$\times\{(1+e^v+e^v\cos\varphi)^2[(1+e^v)^2+v_1^2]+e^{2v}\cdot(b+\cos\varphi)^2v_2^2\}^{3/2}$$
$$\times H\left(\boldsymbol{\rho}+\frac{be^v}{1+e^v}\boldsymbol{r}\right), \tag{2.3}$$

where x is a point of T_1, its position vector is $\boldsymbol{y}$ in (1.1).

Lemma 2.1. $\forall v\in C^{k,\alpha}(T_1)$, $\forall t\in[0, 1]$, *equation* (2.2) *always hase a unique solution* $u\in C^{k+1,\alpha}(T_1)$.

Proof Set the matrix

$$(a_{ij}(x,\ v,\ \nabla v))=\begin{pmatrix} e^{-2v}(1+e^v)^2(b+\cos\varphi)^{-2}(1+e^v+e^v\cos\varphi)^2+v_2^2, & -v_1v_2 \\ -v_1v_2, & (1+e^v)^2+v_1^2 \end{pmatrix}, \tag{2.4}$$

where $1\leqslant i,\ j\leqslant 2$. $(a_{ij}(x,\ v,\ \nabla v))$ is a positive definite symmetric matrix on T_1.

We consider the corresponding homogeneous equation on T_1

$$\sum a_{ij}(x,\ v,\ \nabla v)u_{ij}-u=0. \tag{2.5}$$

If u is a solution of (2.5), there is a point $x_1\in T_1$, $u(x_1)=\max\limits_{x\in T_1}u(x)$. It is obvious that

$$\{\sum_{i,j}a_{ij}(x,\ v,\ \nabla v)u_{ij}\}(x_1)\leqslant 0. \tag{2.6}$$

From (2.5), we can see $u(x_1)\leqslant 0$. In a way similar to the above we can obtain $u(x_2)=\inf\limits_{x\in T_1}u(x)\geqslant 0$. In sum, $\forall x\in T_1$, $u(x)=0$. By using the alternative theorem and regular theorem of elliptic equation on T_1, the lemma is obtained.

We consider a map T: $C^{k,\alpha}(T_1)\times[0,\ 1]\to C^{k,\alpha}(T_1)$. $T(v,\ t)=u$, where u is a solution of linear equation (2.2). And T is a compact map. $\forall v\in C^{k,\alpha}(T_1)$, $T(v,\ 0)=0$.

We consider the following equation

$$\sum_{i,j}a_{ij}(x,\ u,\ \nabla u)u_{ij}-u=t[B(x,\ u,\ \nabla u)-u], \tag{2.7}$$

where $t\in(0,\ 1]$.

If the solution of (2.7) exists, and there is a constant C, where C is independent of t, such that $\|u\|_{C^1(T_1)}\leqslant C$ for all u satisfying (2.7). Then we can make use of Leray-Schauder fixed theorem, and we can find a solution $u\in C^{k+2,\alpha}(T_1)$ satisfying

$$\sum_{i,j}a_{ij}(x,u,\ \nabla u)u_{ij}=B(x,\ u,\ \nabla u). \tag{2.8}$$

Equation (2.8) is just as equation (1.16).

Lemma 2.2. *Suppose $H(\boldsymbol{X})$ in $C^{k,\alpha}(N)$ satisfies the following condition:*

There are two constants s_1, s_2, where $b>s_1\geqslant\frac{1}{2}b\geqslant s_2>0$. When $s>s_1$, $H(\boldsymbol{\rho}+s\boldsymbol{r})<H_{T_s}(\boldsymbol{\rho}+s\boldsymbol{r})$. When $s<s_2$, $H(\boldsymbol{\rho}+s\boldsymbol{r})>H_{T_s}(\boldsymbol{\rho}+s\boldsymbol{r})$. Then, $\ln\frac{s_2}{b-s_2}\leqslant u\leqslant\ln\frac{s_1}{b-s_1}$.

Remark. Arbitrary point in N has unique expression $\boldsymbol{\rho}+s\boldsymbol{r}$, where $0<s<b$.

Proof $u(x_1)=\max\limits_{x\in T_1}u(x)$, and we can see

$$u_1(x_1)=u_2(x_1)=0,\ \{\sum_{i,j}a_{ij}(x,\ u,\ \nabla u)u_{ij}\}\ (x_1)\leqslant 0. \tag{2.9}$$

From (2.3), we know

$$B(x,\ u,\ \nabla u)(x_1)=(b+\cos\varphi)^{-2}\Big\{(1+e^u)^2(1+e^u+e^u\cos\varphi)\cdot\Big[(1+e^u)(e^{-2u}+e^{-u}+2e^{-u}\cos\varphi)$$
$$-2be^{-u}(1+e^u+e^u\cos\varphi)H\Big(\boldsymbol{\rho}+\frac{be^u}{1+e^u}\ \boldsymbol{r}\Big)\Big]\Big\}(x_1). \tag{2.10}$$

Assume $u(x_1)>\ln\frac{s_1}{b-s_1}\geqslant 0$. Then $\frac{be^u}{1+e^u}(x_1)>s_1$, and

$$H\left(\rho+\frac{be^u}{1+e^u}\boldsymbol{r}\right)(x_1)<H_{T_s}\left(\sigma+\frac{be^u}{1+e^u}\boldsymbol{r}\right)(x_1)$$
$$=\left[\frac{1+e^u}{2be^u}+\frac{(1+e^u)\cos\varphi}{2b(1+e^u+e^u\cos\varphi)}\right](x_1). \tag{2.11}$$

Inserting (2.11) into (2.10), we can see

$$B(x,\ u,\ \nabla u)(x_1)>0. \tag{2.12}$$

So

$$t[B(x,\ u,\ \nabla u)(x_1)-u(x_1)]>-tu(x_1)\geqslant -u(x_1)$$
$$\geqslant\{\sum_{i,j}a_{ij}(x,\ u,\ \nabla u)u_{ij}\}(x_1)-u(x_1). \tag{2.13}$$

It is impossible because of equation (2.7). Then

$$u(x_1)\leqslant\ln\frac{s_1}{b-s_1}. \tag{2.14}$$

Similarly, we can see

$$u(x_2)=\min_{x\in T_1}u(x)\geqslant\ln\frac{s_2}{b-s_2}. \tag{2.15}$$

Secondly, we shall estimate $|\nabla u|^2$. Set

$$a_1(x,\ u)=e^{-2u}(1+e^u)^2(b+\cos\varphi)^{-2}(1+e^u+e^u\cos\varphi)^2,$$
$$a_2(x,\ u)=(1+e^u)^2. \tag{2.16}$$

So, equation (2.7) becomes

$$[a_1(x,\ u)+u_2^2]u_{11}-2u_1u_2u_{12}+[a_2(x,\ u)+u_1^2]u_{22}$$
$$=tB(x,\ u,\ \nabla u)+(1-t)u. \tag{2.17}$$

Set function

$$\varphi=e^{2u}\ln(|\nabla u|^2+1). \tag{2.18}$$

Computing at the point $x_0\in T_1$, where φ attains its maximum, we have

$$\varphi_1(x_0)=\varphi_2(x_0)=0,$$
$$\{[a_1(x,\ u)+u_2^2]\varphi_{11}-2u_1u_2\varphi_{12}+[a_2(x,\ u)+u_1^2]\varphi_{22}\}(x_0)\leqslant 0. \tag{2.19}$$

Generally, we assume $|\nabla u|(x_0)>1$. When $|\nabla u|(x_0)\leqslant 1$, it is obvious that $\forall x\in T_1$, $|\nabla u|^2(x)\leqslant 2e^m-1$, where $m=\frac{s_1^2(b-s_2)^2}{s_2^2(b-s_1)^2}$. In this paper, $1\leqslant i,\ j,\ k\leqslant 2$. By calculation, we can see at point x_0

$$\sum_j u_ju_{ji}=-(|\nabla u|^2+1)u_i\ln(|\nabla u|^2+1), \tag{2.20}$$

$$[a_1(x,\ a)+u_2^2]\varphi_{11}=2e^{2u}[a_1(x,\ u)+u_2^2]$$
$$\times\{\ln(|\nabla u|^2+1)[u_{11}-2u_1^2(1+\ln(|\nabla u|^2+1))]$$
$$+(|\nabla u|^2+1)^{-1}(\sum_j u_{j1}^2+\sum_j u_ju_{j11})\}, \tag{2.21}$$

$$-2u_1u_2\varphi_{12}=-4e^{2u}u_1u_2\{\ln(|\nabla u|^2+1)[u_{12}-2u_1u_2$$
$$\times(1+\ln(|\nabla u|^2+1))]+(|\nabla u|^2+1)^{-1}(\sum_j u_{j2}u_{j1}+\sum_j u_ju_{j12})\}, \tag{2.22}$$

$$[a_2(x,u)+u_1^2]\varphi_{22}=2e^{2u}[a_2(x,u)+u_1^2]$$
$$\times\{\ln(|\nabla u|^2+1)[u_{22}-2u_2^2(1+\ln(|\nabla u|^2+1))]$$
$$+(|\nabla u|^2+1)^{-1}(\sum_j u_{j2}^2+\sum_j u_j u_{j22})\}. \tag{2.23}$$

Substituting (2.21)—(2.23) into (2.19), we can see at point x_0

$$0\geqslant[a_1(x,u)+u_2^2]u_{11}-2u_1u_2u_{12}+[a_2(x,u)+u_1^2]\cdot u_{22}$$
$$-2(1+\ln(|\nabla u|^2+1))[a_1(x,u)u_1^2+a_2(x,u)u_2^2]$$
$$+[(|\nabla u|^2+1)\ln(|\nabla u|^2+1)]^{-1}\{[a_1(a,u)+u_2^2](\sum_j u_{j1}^2+\sum_j u_j\cdot u_{j11})$$
$$-2u_1u_2(\sum_j u_{j2}u_{j1}+\sum_j u_ju_{j12})+[a_2(x,u)+u_1^2](\sum_j u_{j2}^2+\sum_j u_ju_{j22})\}. \tag{2.24}$$

We differentiate formula (2.17), and have

$$\sum_k\{[a_1(x,u)+u_2^2]u_{11}\}_k u_k-2\sum_k(u_1u_2u_{12})_k u_k$$
$$+\sum_k\{[a_2(x,u)+u_1^2]u_{22}\}_k u_k$$
$$=t\sum_k[B(x,u,\nabla u)]_k u_k+(1-t)|\nabla u|^2. \tag{2.25}$$

By a calculation, we can see at point x_0

$$\sum_k\{[a_1(x,u)+u_2^2]u_{11}\}_k u_k=\sum_k(a_1(x,u))_k u_k u_{11}$$
$$-2(|\nabla u|^2+1)u_2^2\ln(|\nabla u|^2+1)u_{11}+[a_1(x,u)+u_2^2]\sum_k u_k\cdot u_{11k}, \tag{2.26}$$

$$-2\sum_k(u_1u_2u_{12})_k u_k=4u_1u_2u_{12}(|\nabla u|^2+1)\ln(|\nabla u|^2+1)-2u_1u_2\sum_k u_{12k}u_k \tag{2.27}$$

$$\sum_k\{[a_2(x,u)+u_1^2]u_{22}\}_k u_k=\sum_k(a_2(x,u))_k u_k u_{22}$$
$$-2(|\nabla u|^2+1)u_1^2\ln(|\nabla u|^2+1)u_{22}+[a_2(x,u)+u_1^2]\sum_k u_k u_{22k}. \tag{2.28}$$

By a straight calculation, we have Ricci formulas

$$u_{112}-u_{121}=-u_2\cos\varphi(b+\cos\varphi)^{-1},$$
$$u_{212}-u_{221}=u_1\cos\varphi(b+\cos\varphi)^{-1}. \tag{2.29}$$

Using (2.25)—(2.29), we can see at point x_0

$$[a_1(x,u)+u_2^2]\sum_j u_ju_{j11}-2u_1u_2\sum_j u_ju_{j12}+[a_2(x,u)+u_1^2]\sum_j u_ju_{j22}$$
$$\geqslant t\sum_k[B(x,u,\nabla u)]_k u_k-\sum_k(a_1(x,u))_k u_k u_{11}$$
$$-\sum_k(a_2(x,u))_k u_k u_{22}+2(|\nabla u|^2+1)\ln(|\nabla u|^2+1)$$
$$\times[u_2^2u_{11}-2u_1u_2u_{12}+u_1^2u_{22}]-C_1|\nabla u|^4. \tag{2.30}$$

Substituting (2.30) into (2.24), and making use of (2.17), at point x_0, we can see

$$0\geqslant 3tB(x,u,\nabla u)-2[a_1(x,u)u_{11}+a_2(x,u)u_{22}]$$
$$-2\ln(|\nabla u|^2+1)[a_1(x,u)u_1^2+a_2(x,u)u_2^2]$$
$$+[(|\nabla u|^2+1)\ln(|\nabla u|^2+1)]^{-1}\{[a_1(x,u)+u_2^2]\sum_j u_{j1}^2$$
$$-2u_1u_2\sum_j u_{j2}u_{j1}+[a_2(x,u)+u_1^2]\sum_j u_{j2}^2+t\sum_k[B(x,u,\nabla u)]_k u_k$$
$$-\sum_k[a_1(x,u)]_k u_k u_{11}-\sum_k[a_2(x,u)]_k u_k u_{22}\}-C_2|\nabla u|^2, \tag{2.31}$$

where C_1, C_2 are positive constants. They rely only on s_1, s_2 and b.

From (2.20), we can see

$$u_2u_{12} = -u_1u_{11} - (|\nabla u|^2+1)u_1\ln(|\nabla u|^2+1), \tag{2.32}$$

$$u_1u_{12} = -u_2u_{22} - (|\nabla u|^2+1)u_2\ln(|\nabla u|^2+1). \tag{2.33}$$

From (2.17)、(2.32) and (2.33), we obtain

$$\begin{aligned} u_{11} = {} & [1+|\nabla u|^{-4}(a_1(x, u)u_2^2+a_2(x, u)u_1^2)]^{-1} \\ & \times\{-(1+|\nabla u|^{-2})\ln(|\nabla u|^2+1)[u_1^2+|\nabla u|^{-2}a_2(x, u)\cdot(u_1^2-u_2^2)] \\ & +|\nabla u|^{-4}u_2^2[tB(x, u, \nabla u)+(1-t)u]\}, \end{aligned} \tag{2.34}$$

$$\begin{aligned} u_{22} = {} & [1+|\nabla u|^{-4}(a_1(x, u)u_2^2+a_2(x, u)u_1^2)]^{-1} \\ & \times\{-(1+|\nabla u|^{-2})\ln(|\nabla u|^2+1)[u_2^2+|\nabla u|^{-2}a_1(x, u)\cdot(u_2^2-u_1^2)] \\ & +|\nabla u|^{-4}u_1^2[tB(x, u, \nabla u)+(1-t)u]\}. \end{aligned} \tag{2.35}$$

Evidently,

$$-\sum_k[a_1(x, u)]_k u_k u_{11} \geqslant -C_3|\nabla u|^2|u_{11}|, \tag{2.36}$$

$$-\sum_k[a_2(x, u)]_k u_k u_{22} \geqslant -C_4|\nabla u|^2|u_{22}|, \tag{2.37}$$

where constants C_3 and C_4 depend on s_1, s_2 and b. We square (2.32) and (2.33), and then add them up. We can obtain

$$\begin{aligned} u_{12}^2 = {} & |\nabla u|^{-2}\{u_1^2u_{11}^2+u_2^2u_{22}^2+(|\nabla u|^2+1)^2|\nabla u|^2\cdot[\ln(|\nabla u|^2+1)]^2 \\ & +2(|\nabla u|^2+1)\ln(|\nabla u|^2+1)(u_1^2u_{11}+u_2^2u_{22})\}. \end{aligned} \tag{2.38}$$

Using (2.32)—(2.38), by a long calculation, we can see, at point x_0, (2.31) reduces to

$$\begin{aligned} 0 \geqslant {} & 3tB(x, u, \nabla u)+[a_1(x, u)u_1^2+a_2(x, u)u_2^2]\cdot\ln(|\nabla u|^2+1) \\ & +t[(|\nabla u|^2+1)\ln(|\nabla u|^2+1)]^{-1}\sum_k[B(x, u, \nabla u)]_k u_k - C_5|\nabla u|^2, \end{aligned} \tag{2.39}$$

where C_5 is positive constant, and it depends on s_1, s_2, b and $\max_{X\in\bar{U}}|H(\boldsymbol{X})|$. The region

$$\bar{U} = \{\boldsymbol{X} = \boldsymbol{\rho}+s\boldsymbol{r}\in N \mid s_2 \leqslant s \leqslant s_1\}.$$

From (2.3), by a straight calculation, we can see

$$\begin{aligned} (1)\quad 3B(x, u, \nabla u) \geqslant {} & -6(b+\cos\varphi)^{-1}\sin\varphi(1+e^u+e^u\cos\varphi)^{-1} \\ & \times(1+e^u-be^u)u_1u_2^2-3u_1^3\sin\varphi(b-1-e^{-u})(b+\cos\varphi)^{-2} \\ & -6b(b+\cos\varphi)^{-2}e^{-u}(1+e^u+e^u\cos\varphi)^{-1}(1+e^u)^{-1}\{(1+e^u+e^u\cos\varphi)^2 \\ & \times[(1+e^u)^2+u_1^2]+e^{2u}(b+\cos\varphi)^2u_2^2\}^{3/2}H\left(\boldsymbol{\rho}+\frac{be^u}{1+e^u}\boldsymbol{r}\right)-C_6|\nabla u|^2, \end{aligned} \tag{2.40}$$

where we calculate at point x_0.

Next, at point x_0, we estimate $\sum_k[B(x, u, \nabla u)]_k\cdot u_k$

$$\begin{aligned} (2)\quad & -2\sum_k[(b+\cos\varphi)^{-1}\sin\varphi(1+e^u+e^u\cos\varphi)^{-1}(1+e^u-be^u)u_1u_2^2]_k u_k \\ & \geqslant 2\sin\varphi e^u(1+e^u+e^u\cos\varphi)^{-2}|\nabla u|^2u_1u_2^2+6(b+\cos\varphi)^{-1} \\ & \quad\times\sin\varphi(1+e^u+e^u\cos\varphi)^{-1}(1+e^u-be^u)u_1u_2^2(|\nabla u|^2+1) \\ & \quad\times\ln(|\nabla u|^2+1)-C_7|\nabla u|^4. \end{aligned} \tag{2.41}$$

(3) $-\sum_k \{(b+\cos\varphi)^{-2}(b-1-e^{-u})\sin\varphi[(1+e^u)^2+u_1^2]\cdot u_1\}_k u_k$

$$\geqslant 3(b+\cos\varphi)^{-2}\sin\varphi(b-1-e^{-u})u_1^3(|\nabla u|^2+1)\ln(|\nabla u|^2+1)$$
$$-(b+\cos\varphi)^{-2}\sin\varphi e^{-u}u_1^3|\nabla u|^2-C_8|\nabla u|^4\ln(|\nabla u|^2+1). \tag{2.42}$$

(4) $\sum_k\{(b+\cos\varphi)^{-2}(1+e^u+e^u\cos\varphi)(1+e^u)[(1+e^{-u})^2+\cos\varphi(1+2e^{-u})]u_1^2$

$$+e^u(1+e^u)(1+e^u+e^u\cos\varphi)^{-1}\cdot[(1+e^u)+(2+e^u)\cos\varphi]u_2^2$$
$$+(b+\cos\varphi)^{-2}(1+e^u)^3\cdot(1+e^u+e^u\cos\varphi)(e^{-2u}+e^{-u}+2e^{-u}\cos\varphi)\}_k u_k$$
$$\geqslant -C_9|\nabla u|^4\ln(|\nabla u|^2+1). \tag{2.43}$$

(5) $-2b\sum_k\{(b+\cos\varphi)^{-2}e^{-u}(1+e^u+e^u\cos\varphi)^{-1}(1+e^u)^{-1}$

$$\times\Big\{(1+e^u+e^u\cos\varphi)^2[(1+e^u)^2+u_1^2]+e^{2u}(b+\cos\varphi)^2u_2^2\}^{3/2}$$
$$\times H\left(\boldsymbol{\rho}+\frac{be^u}{1+e^u}\boldsymbol{r}\right)\Big\}_k u_k$$
$$\geqslant 2b(b+\cos\varphi)^{-2}e^{-u}(1+e^u+e^u\cos\varphi)^{-1}(1+e^u)^{-1}$$
$$\times\{(1+e^u+e^u\cos\varphi)^2[(1+e^u)^2+u_1^2]+e^{2u}(b+\cos\varphi)^2u_2^2\}^{3/2}$$
$$\times|\nabla u|^2\{1+(1+e^u+e^u\cos\varphi)^{-1}(e^u+e^u\cos\varphi)$$
$$+e^u(1+e^u)^{-1}\}H\left(\boldsymbol{\rho}+\frac{be^u}{1+e^u}\boldsymbol{r}\right)-3b(b+\cos\varphi)^{-2}e^{-u}(1+e^u+e^u\cos\varphi)^{-1}$$
$$\times(1+e^u)^{-1}\{(1+e^u+e^u\cos\varphi)^2[(1+e^u)^2+u_1^2]$$
$$+e^{2u}(b+\cos\varphi)^2u_2^2\}^{1/2}\{2e^u(1+\cos\varphi)(1+e^u+e^u\cos\varphi)|\nabla u|^2u_1^2$$
$$+2e^{2u}(b+\cos\varphi)^2|\nabla u|^2u_2^2-2(|\nabla u|^2+1)\ln(|\nabla u|^2+1)$$
$$\times[(1+e^u+e^u\cos\varphi)^2u_1^2+e^{2u}(b+\cos\varphi)^2u_2^2]\}H\left(\boldsymbol{\rho}+\frac{be^u}{1+e^u}\boldsymbol{r}\right)$$
$$-2b(b+\cos\varphi)^{-2}e^{-u}(1+e^u+e^u\cos\varphi)^{-1}(1+e^u)^{-1}$$
$$\times\{(1+e^u+e^u\cos\varphi)^2[(1+e^u)^2+u_1^2]+e^{2u}(b+\cos\varphi)^2u_2^2\}^{3/2}$$
$$\times\sum_k\left[H\left(\boldsymbol{\rho}+\frac{be^u}{1+e^u}\boldsymbol{r}\right)\right]_k u_k-C_{10}|\nabla u|^4. \tag{2.44}$$

Substituting (2.16). (1)—(5) into (2.39), we can see

$$0\geqslant[e^{-2u}(1+e^u)^2(b+\cos\varphi)^{-2}(1+e^u+e^u\cos\varphi)^2u_1^2$$
$$+(1+e^u)^2u_2^2]\ln(|\nabla u|^2+1)+t[\ln(|\nabla u|^2+1)]^{-1}$$
$$\times\Big\{2\sin\varphi e^u(1+e^u+e^u\cos\varphi)^{-2}u_1u_2^2-(b+\cos\varphi)^{-2}\sin\varphi e^{-u}\cdot u_1^3$$
$$+be^{-u}(1+e^u+e^u\cos\varphi)^{-1}(1+e^u)^{-1}\{(1+e^u+e^u\cos\varphi)^2\cdot[(1+e^u)^2+u_1^2]$$
$$+e^{2u}(b+\cos\varphi)^2u_2^2\}^{1/2}[2(b+\cos\varphi)^{-2}(1+e^u+e^u\cos\varphi)(1+e^u)^{-1}(1+e^u-e^u\cos\varphi)u_1^2$$
$$-2e^{2u}u_2^2(1+e^u)^{-1}(1+e^u+e^u\cos\varphi)^{-1}(2+2e^u+e^u\cos\varphi)]H\left(\boldsymbol{\rho}+\frac{be^u}{1+e^u}\boldsymbol{r}\right)$$
$$-2b(b+\cos\varphi)^{-2}e^{-u}(1+e^u+e^u\cos\varphi)^{-1}(1+e^u)^{-1}$$
$$\times\{(1+e^u+e^u\cos\varphi)^2[(1+e^u)^2+u_1^2]+e^{2u}(b+\cos\varphi)^2u_2^2\}^{3/2}$$
$$\times(|\nabla u|^2+1)^{-1}\sum_k\left[H\left(\boldsymbol{\rho}+\frac{be^u}{1+e^u}\boldsymbol{r}\right)\right]_k u_k\Big\}-C_{11}|\nabla u|^2, \tag{2.45}$$

where $C_\alpha(6\leqslant\alpha\leqslant 11)$ is positive constant, and it is just as C_5.

By a calculation, we can see

(6) $$-(|\nabla u|^2+1)^{-1}\sum_k\left(H\left(\rho+\frac{be^u}{1+e^u}\boldsymbol{r}\right)\right)_k u_k$$
$$\geqslant -be^u(1+e^u)^{-2}\frac{\partial}{\partial s}H(\rho+s\boldsymbol{r})\Big|_{s=\frac{be^u}{1+e^u}}$$
$$-(|\nabla u|^2+1)^{-1}(C_{12}|\nabla u|+C_{13}), \tag{2.46}$$
where constants C_{12} and C_{13} depend on b, s_1, s_2 and $\max_{X\in U}|\nabla H(\boldsymbol{X})|$.

(7) Let
$$A=2\sin\varphi e^u(1+e^u+e^u\cos\varphi)^{-2}u_1u_2^2-(b+\cos\varphi)^{-2}\sin\varphi e^{-u}u_1^3. \tag{2.47}$$
Then when $u_1\sin\varphi\geqslant 0$,
$$A\geqslant-(b+\cos\varphi)^{-2}\sin\varphi e^{-u}u_1^3$$
$$>-(b+\cos\varphi)^{-2}e^{-u}(1+e^u+e^u\cos\varphi)^{-3}\{(1+e^u+e^u\cos\varphi)^2[(1+e^u)^2+u_1^2]$$
$$+e^{2u}(b+\cos\varphi)^2u_2^2\}^{3/2}; \tag{2.48}$$
when $u_1\sin\varphi<0$,
$$A>2\sin\varphi e^u(1+e^u+e^u\cos\varphi)^{-2}u_1u_2^2. \tag{2.49}$$
Using
$$2(1+e^u+e^u\cos\varphi)^2u_1^2\cdot e^{2u}(b+\cos\varphi)^2u_2^2\cdot e^{2u}(b+\cos\varphi)^2\cdot u_2^2$$
$$\leqslant\left\{\frac{2}{3}[(1+e^u+e^u\cos\varphi)^2u_1^2+e^{2u}(b+\cos\varphi)^2u_2^2]\right\}^3, \tag{2.50}$$
we obtain
$$|u_1u_2^2|<\frac{2}{3\sqrt{3}}e^{-2u}(b+\cos\varphi)^{-2}(1+e^u+e^u\cos\varphi)^{-1}$$
$$\times\{(1+e^u+e^u\cos\varphi)^2[(1+e^u)^2+u_1^2]+e^{2u}(b+\cos\varphi)^2u_2^2\}^{3/2}. \tag{2.51}$$
So, we can see (2.49) becomes
$$A>-\frac{4}{3\sqrt{3}}e^{-u}(b+\cos\varphi)^{-2}(1+e^u+e^u\cos\varphi)^{-3}$$
$$\times\{(1+e^u+e^u\cos\varphi)^2[(1+e^u)^2+u_1^2]+e^{2u}(b+\cos\varphi)^2u_2^2\}^{3/2}. \tag{2.52}$$
Combining (2.48) with (2.52), we have
$$A>-(b+\cos\varphi)^{-2}e^{-u}(1+e^u+e^u\cos\varphi)^{-3}$$
$$\times\{(1+e^u+e^u\cos\varphi)^2[(1+e^u)^2+u_1^2]+e^{2u}(b+\cos\varphi)^2u_2^2\}^{3/2}. \tag{2.53}$$

(8) We estimate
$$B=[2(b+\cos\varphi)^{-2}(1+e^u+e^u\cos\varphi)(1+e^u)^{-1}(1+e^u-e^u\cos\varphi)u_1^2$$
$$-2e^{2u}(1+e^u)^{-1}(1+e^u+e^u\cos\varphi)^{-1}\cdot(2+2e^u+e^u\cos\varphi)u_2^2]H\left(\rho+\frac{be^u}{1+e^u}\boldsymbol{r}\right). \tag{2.54}$$

When $H\left(\rho+\frac{be^u}{1+e^u}\boldsymbol{r}\right)\leqslant 0$, we can see
$$B\geqslant 2(b+\cos\varphi)^{-2}(1+e^u+e^u\cos\varphi)^{-1}(1+e^u)^{-1}(1+e^u-e^u\cos\varphi)$$
$$\times\{(1+e^u+e^u\cos\varphi)^2[(1+e^u)^2+u_1^2]+e^{2u}(b+\cos\varphi)^2u_2^2\}$$
$$\times H\left(\rho+\frac{be^u}{1+e^u}\boldsymbol{r}\right). \tag{2.55}$$

When $H\left(\rho+\frac{be^u}{1+e^u}\boldsymbol{r}\right)>0$, we have

$$B>-2(b+\cos\varphi)^{-2}(1+e^u+e^u\cos\varphi)^{-1}(1+e^u)^{-1}\cdot(2+2e^u+e^u\cos\varphi)$$
$$\times\{(1+e^u+e^u\cos)^2[(1+e^u)^2+u_1^2]+e^{2u}(b+\cos\varphi)^2u_2^2\}$$
$$\times H\left(\rho+\frac{be^u}{1+e^u}\boldsymbol{r}\right). \tag{2.56}$$

Combining (2.55) with (2.56), we obtain

$$B\geqslant-2(b+\cos\varphi)^{-2}(1+e^u+e^u\cos\varphi)^{-1}(1+e^u)^{-1}$$
$$\times(2+3e^u)\{(1+e^u+e^u\cos\varphi)^2[(1+e^u)^2+u_1^2]+e^{2u}(b+\cos\varphi)^2u_2^2\}$$
$$\times\left|H\left(\rho+\frac{be^u}{1+e^u}\boldsymbol{r}\right)\right|. \tag{2.57}$$

Inserting (6)—(8) into (2.45), we reduce them and we can see

$$0>e^{-2u}(1+e^u)^2(b+\cos\varphi)^{-2}\ln(|\nabla u|^2+1)$$
$$-t[\ln(|\nabla u|^2+1)]^{-1}\cdot\{(1+e^u+e^u\cos\varphi)^2[(1+e^u)^2+u_1^2]$$
$$+e^{2u}(b+\cos\varphi)^2u_2^2\}^{1/2}e^{-u}(b+\cos\varphi)^{-2}\cdot(1+e^u+e^u\cos\varphi)^{-1}\Big\{(1+e^u+e^u\cos\varphi)^{-2}$$
$$+2b(1+e^u)^{-2}\cdot(1+e^u+e^u\cos\varphi)^{-1}(2+3e^u)\left|H\left(\rho+\frac{be^u}{1+e^u}\boldsymbol{r}\right)\right|$$
$$+2b^2e^u(1+e^u)^{-3}\frac{\partial}{\partial s}H(\rho+s\boldsymbol{r})\Big|_{s=\frac{be^u}{1+e^u}}\Big\}-C_{14}, \tag{2.58}$$

where we use the inequality $(1+e^u+e^u\cos\varphi)^2\cdot[(1+e^u)^2+u_1^2]+e^{2u}(b+\cos\varphi)^2u_2^2\geqslant\beta|\nabla u|^2$. Because of Lemma 2.2, β is a positive constant, and independent of t. Constant C_{14} relies only on constants b、s_1、s_2、$\max_{X\in\bar U}|H(\boldsymbol{X})|$ and $\max_{X\in\bar U}|\nabla H(\boldsymbol{X})|$.

Now, we establish the following theorem:

Theorem. *Let* T_s: $\boldsymbol{y}_s=\rho+s\boldsymbol{r}=(b\cos\theta, b\sin\theta, 0)+(s\cos\varphi\cos\theta, \cos\varphi\sin\theta, \sin\varphi)$ *be a variety of torus in* R^3, *where* s *is a constant and* $0<s<b$. *Constant* $b>1$, $(\varphi, \theta)\in R/2\pi Z\times R/2\pi Z$ (*Z is an integer group*). *Set* $N=\bigcup_{s\in(0,b)}T_s$. *Function* $H(\boldsymbol{X})\in C^{k,\alpha}(N)$ ($k\geqslant1$, *an integer*, $0<\alpha<1$) *satisfies the following two conditions:*

(1) *There are two constants* s_1, s_2, *where* $b>s_1\geqslant\frac{1}{2}b\geqslant s_2>0$.

When $s>s_1$, $H(\rho+s\boldsymbol{r})<H_{T_s}(\rho+s\boldsymbol{r})$.

When $0<s<s_2$, $H(\rho+s\boldsymbol{r})>H_{T_s}(\rho+s\boldsymbol{r})$,

where $H_{T_s}(\rho+s\boldsymbol{r})$ *is the mean curvature of torus* T_s *in* R^3.

(2) Let

$$\bar U=\{\boldsymbol{X}=\rho+s\boldsymbol{r}\in N\,|\,s_2\leqslant s\leqslant s_1\}-s(b+s\cos\varphi)\frac{\partial}{\partial s}H(\rho+s\boldsymbol{r})$$
$$\geqslant\frac{1}{2}b(b+s\cos\varphi)^{-1}+(2b+s)|H(\rho+s\boldsymbol{r})| \qquad \textit{in } \bar U.$$

Then, there is a closed surface which is homeomorphic to a torus T_1, *its mean curvature* *is given by* $H(X)$.

Proof Under the conditions of the theorem, all the above statement is valid. By virtue of condition (2), set $s=\frac{be^u}{1+e^u}$, we can see

$$-b^2e^u(1+e^u)^{-2}(1+e^u+e^u\cos\varphi)\frac{\partial}{\partial s}H(\rho+s\boldsymbol{r})\Big|_{s=\frac{be^u}{1+e^u}}$$
$$\geqslant\frac{1}{2}(1+e^u)(1+e^u+e^u\cos\varphi)^{-1}+b(1+e^u)^{-1}(2+3e^u)\cdot\left|H\left(\rho+\frac{be^u}{1+e^u}\boldsymbol{r}\right)\right|. \tag{2.59}$$

Inserting (2.59) into (2.58), we can see at x_0

$$|\nabla u|^2<C_{15}. \tag{2.60}$$

C_{15} is a positive constant, it is independent of t. From (2.18), we can see $\forall x\in T_1$

$$|\nabla u|^2(x)<e^{C_{16}}-1, \tag{2.61}$$

where $C_{16}=\left[\frac{s_1(b-s_2)}{s_2(b-s_1)}\right]^2\ln(C_{15}+1)$. So we obtain the theorem.

Remark 1. There are many functions satisfying the conditions of the theorem. For example, set $b=2$,

$$H(\rho+s\boldsymbol{r})=(2s)^{-\gamma}+(2-s)^{\beta}-(2-s)^{-\beta}, \tag{2.62}$$

where $0<s<2$, β、γ are constants, $\beta\geqslant 11$, $\gamma\geqslant 11$. We can find that the two conditions of the theorem are satisfied for $s_1=\frac{3}{2}$ and $s_2=\frac{1}{2}$.

Remark 2. In this paper, the result of estimating $|\nabla u|^2$, for example, inequality (2.39), is useful for similar problem.

References

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