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CONVEXITY THEOREM OF PARAMETRIC TRIANGULAR BÉZIER SURFACES

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Abstract

The author introduces twist vectors of control nets of parametric triangular Bèzier surfaces. With twist vectors and other geometric magnitudes, a sufficient condition for parametric triangular Bèzier surfaces to be convex is offered. Incidently, a necessary and sufficient condition for triangular Bézier surfaces of parametric form to degenerate to that of functional form is given. A less stringent sufficient condition is proposed for quadratic Bézier surfaces to be convex.

§ 1. Introduction

Bézier curves have got extensive application since they were introduced into CAGD. Because of the intimate relation between Bézier curves and their control nets, people anticipated naturally that the convexity of a Bézier curve could be guaranteed by the convexity of its control polygon. That is true[2].

As an extention of Bézier curves we have triangular Bézier surfaces. Chang and Davis have studied the convexity of functional triangular Bézier surfaces and they got a significont convexity theorem which claims that the surface will be convex if its control net is convex. But, unfortunately, the theorem fails to hold if the surface is of parametric form[5] (also refer to ex. 2 of this paper). In this paper the author tries to give a sufficint condition for parametric triangular Bézier surfaces. Incidently, the author has found an essential difference between triangular Bézier surfaces of functional form and parametric form.

§ 2. Notations and Lemmas

Parametric triangular Bézier surfaces of degree n are of the form

$$\boldsymbol{r}(u, v, w)=\sum_{i+j+k=n} P_{i,j,k}B^n_{i,j,k}(u, v, w), \tag{1}$$

where $P_{i,j,k}\in R^3$, $u+v+w=1$, $B^n_{i,j,k}(u, v, w)=\begin{pmatrix} & n & \\ i & j & k\end{pmatrix} u^i v^j w^k$.

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For convenience, following notations are introduced:

$$\boldsymbol{s}_{i,j,k}=P_{i+1,j,k}-P_{i,j,k+1},$$
$$\boldsymbol{q}_{i,j,k}=P_{i,j+1,k}-P_{i,j,k+1},$$
$$U_{i-1,j,k}=(P_{i+1,j,k}+P_{i-1,j+1,k+1}-P_{i,j+1,k}-P_{i,j,k+1})/2,$$
$$V_{i-1,j,k}=(P_{i-1,j+2,k}+P_{i,j,k+1}-P_{i,j+1,k}-P_{i-1,j+1,k+1})/2,$$
$$W_{i-1,j,k}=(P_{i,j+1,k}+P_{i-1,j,k+2}-P_{i-1,j+1,k+1}-P_{i,j,k+1})/2,$$

where $\quad i+j+k=n-1. \qquad (2)$

$U_{i,j,k}$, $V_{i,j,k}$, $W_{i,j,k}$ $(i+j+k=n-2)$ control the twist of the control net. We call them twist vectors with respect to u-direction, v-direction and w-direction respectively. It is clear that

$$\boldsymbol{s}_{i,j,k}=\boldsymbol{s}_{i-1,j+1,k}+2U_{i-1,j,k}, \tag{3}$$
$$\boldsymbol{s}_{i,j,k}=\boldsymbol{s}_{i,j-1,k+1}+2W_{i,j-1,k}, \tag{4}$$
$$\boldsymbol{q}_{i,j,k}=\boldsymbol{q}_{i-1,j,k+1}+2W_{i-1,j,k}, \tag{5}$$
$$\boldsymbol{q}_{i,j,k}=\boldsymbol{q}_{i+1,j-1,k}+2V_{i,j-1,k}. \tag{6}$$

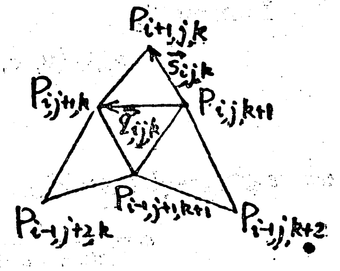

Figure 1.

Regarding $\boldsymbol{r}$ as a vector valued function of u and $v(w=1-u-v)$, differentiating expression (1) and assuming the notations that $\boldsymbol{r}_u=\dfrac{\partial}{\partial u}\boldsymbol{r}$, $\boldsymbol{r}_v=\dfrac{\partial}{\partial v}\boldsymbol{r}$, etc., we have

$$\boldsymbol{r}_u=n\sum_{i+j+k=n-1}\boldsymbol{s}_{i,j,k}B^{n-1}_{i,j,k}, \tag{7}$$
$$\boldsymbol{r}_v=n\sum_{i+j+k=n-1}\boldsymbol{q}_{i,j,k}B^{n-1}_{i,j,k}, \tag{8}$$
$$\begin{aligned}\boldsymbol{r}_{uu}&=n(n-1)\sum(\boldsymbol{s}_{i+1,j,k}-\boldsymbol{s}_{i,j,k+1})B^{n-2}_{i,j,k}\\&=2n(n-1)\sum_{i+j+k=n-2}(U_{i,j,k}+W_{i,j,k})B^{n-2}_{i,j,k},\end{aligned} \tag{9}$$
$$\boldsymbol{r}_{uv}=2n(n-1)\sum W_{i,j,k}B^{n-2}_{i,j,k}, \tag{10}$$
$$\boldsymbol{r}_{vv}=2n(n-1)\sum(V_{i,j,k}+W_{i,j,k})B^{n-2}_{i,j,k}. \tag{11}$$

Given u, v, w, we define

$$\boldsymbol{s}^m_{i,j,k}=u\boldsymbol{s}^{m-1}_{i+1,j,k}+v\boldsymbol{s}^{m-1}_{i,j+1,k}+w\boldsymbol{s}^{m-1}_{i,j,k+1}, \tag{12}$$
$$\boldsymbol{q}^m_{i,j,k}=u\boldsymbol{q}^{m-1}_{i+1,j,k}+v\boldsymbol{q}^{m-1}_{i,j+1,k}+w\boldsymbol{q}^{m-1}_{i,j,k+1}, \tag{13}$$

where $i+j+k+m=n-1$, $\boldsymbol{s}^0_{i,j,k}=\boldsymbol{s}_{i,j,k}$, $\boldsymbol{q}^0_{i,j,k}=\boldsymbol{q}_{i,j,k}$, and

$$U^m_{i,j,k}=uU^{m-1}_{i+1,j,k}+vU^{m-1}_{i,j-1,k}+wU^{m-1}_{i,j,k+1}, \tag{14}$$
$$V^m_{i,j,k}=uV^{m-1}_{i+1,j,k}+vV^{m-1}_{i,j+1,k}+wV^{m-1}_{i,j,k+1}, \tag{15}$$
$$W^m_{i,j,k}=uW^{m-1}_{i+1,j,k}+vW^{m-1}_{i,j+1,k}+wW^{m-1}_{i,j,k+1}, \tag{16}$$

where $i+j+k+m=n-2$, $U^0_{i,j,k}=U_{i,j,k}$, $V^0_{i,j,k}=V_{i,j,k}$, $W^0_{i,j,k}=W_{i,j,k}$.

Let

$$\boldsymbol{v}^m_{i,j,k}=\boldsymbol{s}^m_{i,j,k}\times\boldsymbol{q}^m_{i,j,k} \quad (i+j+k+m=n-1). \tag{17}$$

Obviously, $\boldsymbol{v}^0_{i,j,k}=\boldsymbol{s}_{i,j,k}\times\boldsymbol{q}_{i,j,k}$ is the normal vector of the plane on which lies $\Delta P_{i+1,j,k}P_{i,j+1,k}P_{i,j,k+1}$, and its absolute value is twice the area of $\Delta P_{i+1,j,k}P_{i,j+1,k}P_{i,j,k+1}$. From the recursive algorithm of Bernstein polynomial, we have

$$\boldsymbol{s}_{0,0,0}^{n-1}=\sum_{i+j+k=n-1}\boldsymbol{s}_{i,j,k}B_{i,j,k}^{n-1},$$
$$\boldsymbol{q}_{0,0,0}^{n-1}=\sum_{i+j+k=n-1}\boldsymbol{q}_{i,j,k}B_{i,j,k}^{n-1},$$

and so, by (7)—(8),

$$\boldsymbol{v}_{0,0,0}^{n-1}=\boldsymbol{s}_{0,0,0}^{n-1}\times\boldsymbol{q}_{0,0,0}^{n-1}=n^{-2}\boldsymbol{r}_u\times\boldsymbol{r}_v. \tag{18}$$

$\boldsymbol{v}_{0,0,0}^{n-1}$ and $\boldsymbol{v}_{i,j,k}=\boldsymbol{v}_{i,j,k}^{0}$ have, therefore, direct geometric meaning, while $\boldsymbol{v}_{i,j,k}^{m}(0<m<n-1)$ have not. The following lemma tries to correlate $\boldsymbol{v}_{0,0,0}^{n-1}$ with $\boldsymbol{v}_{i,j,k}$.

Lemma 1.

$$\boldsymbol{v}_{0,0,0}^{n-1}(u,v,w)=\sum_{i+j+k=n-1}\boldsymbol{v}_{i,j,k}B_{i,j,k}^{n-1}(u,v,w)+2\delta\sum_{m=0}^{n-2}\sum_{i+j+k=m}\boldsymbol{g}_{i,j,k}^{n-m-2}B_{i,j,k}^{m}(u,v,w), \tag{19}$$

where $\boldsymbol{g}_{i,j,k}^{m}=\alpha U_{i,j,k}^{m}\times V_{i,j,k}^{m}+\beta V_{i,j,k}^{m}\times W_{i,j,k}^{m}+\gamma W_{i,j,k}^{m}\times U_{i,j,k}^{m}$, *while*

$$\alpha=\frac{uv}{uv+vw+wu},\quad\beta=\frac{vw}{uv+vw+wu},\quad\gamma=\frac{wu}{uv+vw+wu},$$
$$\delta=1-u^2-v^2-w^2=2(uv+vw+wu).$$

Proof

$$\begin{aligned}\boldsymbol{v}_{i,j,k}^{1}&=\boldsymbol{s}_{i,j,k}^{1}\times\boldsymbol{q}_{i,j,k}^{1}\\&=(u\boldsymbol{s}_{i+1,j,k}+v\boldsymbol{s}_{i,j+1,k}+w\boldsymbol{s}_{i,j,k+1})\times(u\boldsymbol{q}_{i+1,j,k}+v\boldsymbol{q}_{i,j+1,k}+w\boldsymbol{q}_{i,j,k+1})\\&=u^2\boldsymbol{s}_{i+1,j,k}\times\boldsymbol{q}_{i+1,j,k}+v^2\boldsymbol{s}_{i,j+1,k}\times\boldsymbol{q}_{i,j+1,k}+w^2\boldsymbol{s}_{i,j,k+1}\times\boldsymbol{q}_{i,j,k+1}\\&\quad+uv(\boldsymbol{s}_{i+1,j,k}\times\boldsymbol{q}_{i,j+1,k}+\boldsymbol{s}_{i,j+1,k}\times\boldsymbol{q}_{i+1,j,k})+vw(\boldsymbol{s}_{i,j+1,k}\times\boldsymbol{q}_{i,j,k+1}\\&\quad+\boldsymbol{s}_{i,j,k+1}\times\boldsymbol{q}_{i,j+1,k})+wu(\boldsymbol{s}_{i,j,k+1}\times\boldsymbol{q}_{i+1,j,k}+\boldsymbol{s}_{i+1,j,k}\times\boldsymbol{q}_{i,j,k+1}).\end{aligned}$$

By (3)—(6)

$$\begin{aligned}&\boldsymbol{s}_{i+1,j,k}\times\boldsymbol{q}_{i,j+1,k}+\boldsymbol{s}_{i,j+1,k}\times\boldsymbol{q}_{i+1,j,k}\\&=\boldsymbol{s}_{i,j+1,k}\times\boldsymbol{q}_{i,j+1,k}+2U_{i,j,k}\times\boldsymbol{q}_{i,j+1,k}+\boldsymbol{s}_{i+1,j,k}\times\boldsymbol{q}_{i+1,j,k}-2U_{i,j,k}\times\boldsymbol{q}_{i+1,j,k}\\&=\boldsymbol{v}_{i,j+1,k}+\boldsymbol{v}_{i+1,j,k}+4U_{i,j,k}\times V_{i,j,k}.\end{aligned}$$

Similarly

$$\boldsymbol{s}_{i,j+1,k}\times\boldsymbol{q}_{i,j,k+1}+\boldsymbol{s}_{i,j,k+1}\times\boldsymbol{q}_{i,j+1,k}=\boldsymbol{v}_{i,j,k+1}+\boldsymbol{v}_{i,j+1,k}+4V_{i,j,k}\times W_{i,j,k}$$
$$\boldsymbol{s}_{i,j,k+1}\times\boldsymbol{q}_{i+1,j,k}+\boldsymbol{s}_{i+1,j,k}\times\boldsymbol{q}_{i,j,k+1}=\boldsymbol{v}_{i+1,j,k}+\boldsymbol{v}_{i,j,k+1}+4W_{i,j,k}\times U_{i,j,k}.$$

So,

$$\begin{aligned}\boldsymbol{v}_{i,j,k}^{1}&=u^2\boldsymbol{v}_{i+1,j,k}+v^2\boldsymbol{v}_{i,j+1,k}+w^2\boldsymbol{v}_{i,j,k+1}+uv(\boldsymbol{v}_{i+1,j,k}+\boldsymbol{v}_{i,j+1,k}+4U_{i,j,k}\times V_{i,j,k})\\&\quad+vw(\boldsymbol{v}_{i,j+1,k}+\boldsymbol{v}_{i,j,k+1}+4V_{i,j,k}\times W_{i,j,k})+wu(\boldsymbol{v}_{i,j,k+1}+\boldsymbol{v}_{i+1,j,k}+4W_{i,j,k}\times U_{i,j,k})\\&=u\boldsymbol{v}_{i+1,j,k}+v\boldsymbol{v}_{i,j+1,k}+w\boldsymbol{v}_{i,j,k+1}+4(uvU_{i,j,k}\times V_{i,j,k}+vwV_{i,j,k}\times W_{i,j,k}\\&\quad+wuW_{i,j,k}\times U_{i,j,k}).\end{aligned}$$

Now we prove

$$\boldsymbol{v}_{i,j,k}^{m}=\boldsymbol{v}1_{i,j,k}^{m}+2\delta\sum_{p=0}^{m-1}\sum_{a+b+c=p}\boldsymbol{g}_{i+a,j+b,k+c}^{m-1-p}B_{a,b,c}^{p}, \tag{20}$$

where $i+j+k+m=n-1$, $\boldsymbol{v}1_{i,j,k}^{m}=u\boldsymbol{v}1_{i+1,j,k}^{m-1}+v\boldsymbol{v}1_{i,j+1,k}^{m-1}+w\boldsymbol{v}1_{i,j,k+1}^{m-1}$, $\boldsymbol{v}1_{i,j,k}^{0}=\boldsymbol{v}_{i,j,k}$. From the deduction above we know (20) holds for $m=1$. Assuming that (20) holds for $m<n-1$, and taking in mind that (3)—(6) remain true if superscript m is attached

to each term, we have

$$\begin{aligned}\boldsymbol{v}^{m+1}_{i,j,k}&=\boldsymbol{s}^{m+1}_{i,j,k}\times\boldsymbol{q}^{m+1}_{i,j,k}\\&=u\boldsymbol{v}^m_{i+1,j,k}+v\boldsymbol{v}^m_{i,j+1,k}+w\boldsymbol{v}^m_{i,j,k+1}+2\delta(\alpha U^m_{i,j,k}\times V^m_{i,j,k}+\beta V^m_{i,j,k}\times W^m_{i,j,k}\\&\quad+\gamma W^m_{i,j,k}\times U^m_{i,j,k})\\&=u\Big(\boldsymbol{v}1^m_{i+1,j,k}+2\delta\sum_{p=0}^{m-1}\sum_{a+b+c=p}\boldsymbol{g}^{m-1-p}_{i+1+a,j+b,k+c}B^p_{abc}\Big)\\&\quad+v\Big(\boldsymbol{v}1^m_{i,j+1,k}+2\delta\sum_{p=0}^{m-1}\sum_{a+b+c=p}\boldsymbol{g}^{m-1-p}_{i+a,j+1+b,k+c}B^p_{a,b,c}\Big)\\&\quad+w\Big(\boldsymbol{v}1^m_{i,j,k+1}+2\delta\sum_{p=0}^{m-1}\sum_{a+b+c=p}\boldsymbol{g}^{m-1-p}_{i+a,j+b,k+1+c}B^p_{a,b,c}\Big)+2\delta\boldsymbol{g}^m_{i,j,k}\\&=\boldsymbol{v}1^{m+1}_{i,j,k}+2\delta\sum_{p=0}^{m}\sum_{a+b+c=p}\boldsymbol{g}^{m-p}_{i+a,j+b,k+c}B^p_{a,b,c}.\end{aligned}$$

Hence (20) is true by induction. The lemma is proved simply by setting $m=n-1$ and $(i, j, k)=(0, 0, 0)$ in the expression (20).

§ 3. Convexity Theorem of Parametric Triangular Bézier Surfaces

Let $\Delta=\{(u, v)\in R^2;\ u>0, v>0,\ u+v<1\}$. For any (u_0, v_0), $(u_1, v_1)\in\Delta$, there are linear functions $u(t)$, $v(t)$ such that $u(0)=u_0$, $v(0)=v_0$, $u(1)=u_1$, $v(1)=v_1$. With $(u(t), v(t))$,

$$\begin{aligned}\boldsymbol{r}(t)&=\boldsymbol{r}(u(t),\ v(t),\ 1-u(t)-v(t))\\&=\boldsymbol{r}(0)+\boldsymbol{r}'(0)t+\int_0^t\boldsymbol{r}''(\xi)(t-\xi)d\xi.\end{aligned}$$

Let $\boldsymbol{r}(u, v)=\boldsymbol{r}(u, v, 1-u-v)$ and $t=1$. Then

$$\begin{aligned}\boldsymbol{r}(u_1, v_1)=\boldsymbol{r}(u_0, v_0)+\boldsymbol{r}_u u'+\boldsymbol{r}_v v'+\int_0^1(\boldsymbol{r}_{uu}(\xi)u'^2\\+2\boldsymbol{r}_{uv}(\xi)u'v'+\boldsymbol{r}_{vv}v'^2)(1-\xi)d\xi.\end{aligned}\tag{21}$$

With (21) we have the following lemma.

Lemma 2. *Bézier surface* (1) *is convex in Δ if for any $(u_0, v_0)\in\Delta$ the following inequalities hold:*

$$(U_{i,j,k}+V_{i,j,k})\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)\geqslant0,\tag{22}$$

$$(V_{i,j,k}+W_{i,j,k})\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)\geqslant0,\tag{23}$$

$$(W_{i,j,k}+U_{i,j,k})\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)\geqslant0,\tag{24}$$

$$\begin{aligned}U_{i,j,k}\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)V_{i,j,k}\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)+V_{i,j,k}\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)W_{i,j,k}\cdot\boldsymbol{r}_u\\\times\boldsymbol{r}_v(u_0, v_0)+W_{i,j,k}\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)U_{i,j,k}\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)\geqslant0,\end{aligned}\tag{25}$$

where $i+j+k=n-2$.

Proof At first we prove that (22)—(25) are hereditary with respect to recursive algorithm, i.e., (22)—(25) remain true if superscript m is attached to twist vectors in the expressions. It suffices to give a proof for the case where superscript is one.

Obviously, (22)—(24) remain true when $U^1_{i,j,k}$, $V^1_{i,j,k}$, $W^1_{i,j,k}$ are substituted for $U_{i,j,k}$, $V_{i,j,k}$, $W_{i,j,k}$. Let $\bar{u}_{i,j,k}=U_{i,j,k}\cdot \boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)$, $\bar{v}_{i,j,k}=V_{i,j,k}\cdot \boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)$, $\bar{w}_{i,j,k}=W_{i,j,k}\cdot \boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)$, $\bar{u}^1_{i,j,k}=U^1_{i,j,k}\cdot \boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)$, etc.

$$\begin{aligned}
&\bar{u}^1_{i,j,k}\bar{v}^1_{i,j,k}+\bar{v}^1_{i,j,k}\bar{w}^1_{i,j,k}+\bar{w}^1_{i,j,k}\bar{u}^1_{i,j,k}\\
&\quad=(u\bar{u}_{i+1,j,k}+v\bar{u}_{i,j+1,k}+w\bar{u}_{i,j,k+1})(u\bar{v}_{i+1,j,k}+v\bar{v}_{i,j+1,k}+w\bar{v}_{i,j,k+1})\\
&\qquad+(u\bar{v}_{i+1,j,k}+v\bar{v}_{i,j+1,k}+w\bar{v}_{i,j,k+1})(u\bar{w}_{i+1,j,k}+v\bar{w}_{i,j+1,k}+w\bar{w}_{i,j,k+1})\\
&\qquad+(u\bar{w}_{i+1,j,k}+v\bar{w}_{i,j+1,k}+w\bar{w}_{i,j,k+1})(u\bar{u}_{i+1,j,k}+v\bar{u}_{i,j+1,k}+w\bar{u}_{i,j,k+1})\\
&\quad=u^2(\bar{u}_{i+1,j,k}\bar{v}_{i+1,j,k}+\bar{v}_{i+1,j,k}\bar{w}_{i+1,j,k}+\bar{w}_{i+1,j,k}\bar{u}_{i+1,j,k})+v^2C_{vv}+w^2C_{ww}\\
&\qquad+uv(\bar{u}_{i+1,j,k}\bar{v}_{i,j+1,k}+\bar{u}_{i,j+1,k}\bar{v}_{i+1,j,k}+\bar{v}_{i+1,j,k}\bar{w}_{i,j+1,k}+\bar{v}_{i,j+1,k}\bar{w}_{i+1,j,k}\\
&\qquad+\bar{w}_{i+1,j,k}\bar{u}_{i,j+1,k}+\bar{w}_{i,j+1,k}\bar{u}_{i+1,j,k})+vwC_{vw}+wuC_{wu},
\end{aligned}$$

where C_{vv}, C_{ww}, C_{vw}, C_{wu} are corresponding coefficients. By (25), the coefficients of u^2, v^2, w^2 are nonnegative. Let C_{uv} denote the coefficient of uv. From (22)—(24), among $\bar{u}_{i+1,j,k}$, $\bar{v}_{i+1,j,k}$, $\bar{w}_{i+1,j,k}$, there are at least two which are nonnegative. Assume that $\bar{u}_{i+1,j,k}\geqslant 0$, $\bar{v}_{i+1,j,k}\geqslant 0$. Similarly we may assume $\bar{u}_{i,j+1,k}\geqslant 0$, $\bar{v}_{i,j+1,k}\geqslant 0$. If $\bar{u}_{i+1,j,k}+\bar{v}_{i+1,j,k}=0$, i.e., $\bar{u}_{i+1,j,k}=\bar{v}_{i+1,j,k}=0$, then $\bar{w}_{i+1,j,k}\geqslant 0$, and it is clear that $C_{uv}\geqslant 0$. Now if $\bar{u}_{i+1,j,k}+\bar{v}_{i+1,j,k}>0$, $\bar{u}_{i,j+1,k}+\bar{v}_{i,j+1,k}>0$, we have

$$\begin{aligned}
&\bar{u}_{i,j+1,k}\bar{v}_{i,j+1,k}((\bar{u}_{i+1,j,k})^2+2\bar{u}_{i+1,j,k}\bar{v}_{i+1,j,k}+(\bar{v}_{i+1,j,k})^2)\\
&\qquad+\bar{u}_{i+1,j,k}\bar{v}_{i+1,j,k}(\bar{u}_{i,j+1,k})^2+2\bar{u}_{i,j+1,k}\bar{v}_{i,j+1,k}+(\bar{v}_{i,j+1,k})^2)\\
&\quad\leqslant(\bar{u}_{i+1,j,k})^2\bar{v}_{i,j+1,k}\bar{u}_{i,j+1,k}+(\bar{u}_{i,i+1,k})^2\bar{v}_{i+1,j,k}\bar{u}_{i+1,j,k}\\
&\qquad+\bar{u}_{i+1,j,k}\bar{v}_{i,j+1,k}\bar{u}_{i,j+1,k}\bar{v}_{i+1,j,k}+(\bar{u}_{i,j+1,k}\bar{v}_{i+1,j,k})^2+(\bar{u}_{i+1,j,k}\bar{v}_{i,j+1,k})^2\\
&\qquad+\bar{u}_{i,j+1,k}\bar{v}_{i,j+1,k}\bar{v}_{i+1,j,k}\bar{u}_{i+1,j,k}+\bar{u}_{i+1,j,k}\bar{v}_{i+1,j,k}(\bar{v}_{i,j+1,k})^2\\
&\qquad+\bar{u}_{i,j+1,k}\bar{v}_{i,j+1,k}(\bar{v}_{i+1,i,k})^2,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\bar{u}_{i,j+1,k}\bar{v}_{i,+1,k}(\bar{u}_{i+1,j,k}+\bar{v}_{i+1,j,k})^2+\bar{u}_{i+1,j,k}\bar{v}_{i+1,j,k}(\bar{u}_{i,j+1,k}+\bar{v}_{i,j+1,k})^2\\
&\quad\leqslant(\bar{u}_{i+1,j,k}\bar{v}_{i,j+1,k}+\bar{u}_{i,j+1,k}\bar{v}_{i+1,j,k})(\bar{u}_{i+1,j,k}+\bar{v}_{i+1,j,k})(\bar{u}_{i,j+1,k}+\bar{v}_{i,j+1,k}).
\end{aligned}\tag{26}$$

From (25)

$$\begin{aligned}
\bar{w}_{i+1,j,k}&\geqslant-\bar{u}_{i+1,j,k}\bar{v}_{i+1,j,k}/(\bar{u}_{i+1,j,k}+\bar{v}_{i+1,j,k}),\\
\bar{w}_{i,j+1,k}&\geqslant-\bar{u}_{i,j+1,k}\bar{v}_{i,j+1,k}/(\bar{u}_{i,j+1,k}+\bar{v}_{i,j+1,k}).
\end{aligned}$$

So

$$\begin{aligned}
C_{uv}&=\bar{u}_{i+1,j,k}\bar{v}_{i,j+1,k}+\bar{u}_{i,j+1,k}\bar{v}_{i+1,j,k}+\bar{w}_{i+1,j,k}(\bar{u}_{i,j+1,k}+\bar{v}_{i,j+1,k})\\
&\quad+\bar{w}_{i,j+1,k}(\bar{u}_{i+1,j,k}+\bar{v}_{i+1,j,k})\\
&\geqslant\bar{u}_{i+1,j,k}\bar{v}_{i,j+1,k}+\bar{u}_{i,j+1,k}\bar{v}_{i+1,j,k}-\bar{u}_{i+1,j,k}\bar{v}_{i+1,j,k}(\bar{u}_{i,j+1,k}+\bar{v}_{i,j+1,k})/\\
&\quad(\bar{u}_{i+1,j,k}+\bar{v}_{i+1,j,k})-\bar{u}_{i,j+1,k}\bar{v}_{i,j+1,k}(\bar{u}_{i+1,j,k}+\bar{v}_{i+1,j,k})/(\bar{u}_{i,j+1,k}+\bar{v}_{i,j+1,k}).
\end{aligned}$$

It follows from (26) that $C_{uv}\geqslant 0$. From the symmetry it is inferred that C_{vw}, C_{wu} are nonnegative. Thus we have proved that (22)—(25) are hereditary. In particular,

$$(U^{n-2}_{0,0,0}+V^{n-2}_{0,0,0})\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)\geqslant 0,\tag{27}$$

$$(V^{n-2}_{0,0,0}+W^{n-2}_{0,0,0})\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)\geqslant 0,\tag{28}$$

$$(W^{n-2}_{0,0,0}+U^{n-2}_{0,0,0})\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0, v_0)\geqslant 0,\tag{29}$$

$$U^{n-2}_{0,0,0}\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)V^{n-2}_{0,0,0}\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)+V^{n-2}_{0,0,0}\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)W^{n-2}_{0,0,0}\cdot\boldsymbol{r}_u$$
$$\times\boldsymbol{r}_v(u_0,v_0)+W^{n-2}_{0,0,0}\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)U^{n-2}_{0,0,0}\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)\geqslant 0. \tag{30}$$

$\forall(u_1, v_1)\in\Delta$, there is a line $(u(\xi), v(\xi))(0\leqslant\xi\leqslant 1)$ which connects (u_0, v_0) and (u_1, v_1). Since Δ is a convex region, $(u(\xi), v(\xi))\in\Delta$. From (21),

$$(\boldsymbol{r}(u_1,v_1)-\boldsymbol{r}(u_0,v_0))\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)$$
$$=\int_0^1(1-\xi)(\boldsymbol{r}_{uu}(u(\xi),v(\xi))\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)u'^2+2\boldsymbol{r}_{uv}(u(\xi),v(\xi))\cdot$$
$$\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)u'v'+\boldsymbol{r}_{vv}(u(\xi),v(\xi))\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)v'^2)d\xi,$$

where the first factor in the integrand $1-\xi\geqslant 0$, and the second factor is a quadratic form. From (9)—(11), (27)—(30) and noticing that $\sum U_{i,j,k}B^{n-2}_{i,j,k}=U^{n-2}_{0,0,0}$, etc., we know the quadratic form is non-negative, i.e.,

$$(\boldsymbol{r}(u_1,v_1)-\boldsymbol{r}(u_0,v_0))\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)\geqslant 0.$$

Since (u_0, v_0), (u_1, v_1) are arbitrary in Δ, Bézier surfaces (1) are convex in Δ if (22)—(25) are satisfied.

Corollary. *Bézier surfaces* (1) *are convex in* Δ *if* $\forall(u_0, v_0)$

$$U_{i,j,k}\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)\geqslant 0,$$
$$V_{i,j,k}\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)\geqslant 0,$$
$$W_{i,j,k}\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)\geqslant 0,$$

where $i+j+k=n-2$.

The condition of Lemma 2 or its corollary guarantees the convexity of Bézier surfaces, but the relation between the condition and Bézier control nets is not very clear. We try to describe the convexity theorem with $\boldsymbol{v}_{i,j,k}=\boldsymbol{s}_{i,j,k}\times\boldsymbol{q}_{i,j,k}$ and twist vectors. For this purpose, we prove in advance the following lemma.

Lemma 3. *Let* $\boldsymbol{\zeta}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ *be unit vectors. If*

$$\boldsymbol{\zeta}\cdot\boldsymbol{a}\geqslant\cos\varepsilon,\ \boldsymbol{\zeta}\cdot\boldsymbol{b}\geqslant\cos\varepsilon,\ \boldsymbol{\zeta}\cdot\boldsymbol{c}\geqslant\cos\varepsilon,$$

where $0\leqslant\varepsilon\leqslant\pi/4$, *then*

$$|\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}|\leqslant\frac{3\sqrt{3}}{2}\cos\varepsilon\sin^2\varepsilon.$$

Proof We may assume that $\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}\geqslant 0$ without detriment to the proof. We define the ε-cone of $\boldsymbol{\zeta}$ as

$$\{\boldsymbol{r}\in R^3: \boldsymbol{\zeta}\cdot\boldsymbol{r}\geqslant|\boldsymbol{r}|\cos\varepsilon\}.$$

Extending the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ until they touch the boundary of the ε-cone of $\boldsymbol{\zeta}$, we get $\boldsymbol{a}'$, $\boldsymbol{b}'$.

Obviously

$$\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}\leqslant\boldsymbol{a}'\times\boldsymbol{b}'\cdot\boldsymbol{c}.$$

Similarly we have $\boldsymbol{c}'$ such that

$$\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}\leqslant\boldsymbol{a}'\times\boldsymbol{b}'\cdot\boldsymbol{c}',$$

where $\boldsymbol{\zeta}\cdot\boldsymbol{a}'=\cos\varepsilon$, $\boldsymbol{\zeta}\cdot\boldsymbol{b}'=\cos\varepsilon$, $\boldsymbol{\zeta}\cdot\boldsymbol{c}'=\cos\varepsilon$.

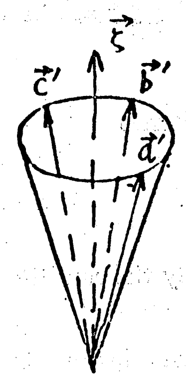

Fig. 2.

Let

$$\boldsymbol{a}'=\cos\varepsilon\boldsymbol{\zeta}+\sin\varepsilon\boldsymbol{a}'^{\perp},$$
$$\boldsymbol{b}'=\cos\varepsilon\boldsymbol{\zeta}+\sin\varepsilon\cdot\boldsymbol{b}'^{\perp},$$
$$\boldsymbol{c}'=\cos\varepsilon\boldsymbol{\zeta}+\sin\varepsilon\cdot\boldsymbol{c}'^{\perp},$$

where $\boldsymbol{a}'^{\perp}$, $\boldsymbol{b}'^{\perp}$, $\boldsymbol{c}'^{\perp}$ are unit vectors perpendicular to $\boldsymbol{\zeta}$.

$$\boldsymbol{a}'\times\boldsymbol{b}'\cdot\boldsymbol{c}'=\cos\varepsilon\sin^2\varepsilon(\boldsymbol{\zeta}\times\boldsymbol{b}'^{\perp}\cdot\boldsymbol{c}'^{\perp}+\boldsymbol{\zeta}\times\boldsymbol{c}'^{\perp}\cdot\boldsymbol{a}'^{\perp}+\boldsymbol{\zeta}\times\boldsymbol{a}'^{\perp}\cdot\boldsymbol{\zeta}'^{\perp}).$$

Let θ_1, θ_2, θ_3 be angles between $\boldsymbol{b}'^{\perp}$ and $\boldsymbol{c}'^{\perp}$, $\boldsymbol{c}'^{\perp}$ and $\boldsymbol{a}'^{\perp}$, $\boldsymbol{a}'^{\perp}$ and $\boldsymbol{b}'^{\perp}$ respectively. Then

$$\boldsymbol{a}'\times\boldsymbol{b}'\cdot\boldsymbol{c}'=\cos\varepsilon\sin^2\varepsilon(\sin\theta_1+\sin\theta_2+\sin\theta_3)\leqslant 3\frac{\sqrt{3}}{2}\cos\varepsilon\sin^2\varepsilon.$$

Now we set forth a sufficient condition of convexity described by geometric magnitudes $\boldsymbol{v}_{i,j,k}$, $U_{i,j,k}$, $V_{i,j,k}$, $W_{i,j,k}$.

Theorem 1. *Let $\boldsymbol{\zeta}$ be a unit vector, $G=\max\{|U_{i,j,k}|, |V_{i,i,k}|, |W_{i,j,k}|: i+j+k=n-2\}$ and ε, $\bar{\alpha}$ be two real numbers such that $0\leqslant\bar{\alpha}\leqslant\pi/2$, $0\leqslant\varepsilon\leqslant\pi/4$ and*

$$\boldsymbol{v}_{i,j,k}\cdot\boldsymbol{\zeta}\geqslant|\boldsymbol{v}_{i,j,k}|\cos\bar{\alpha},\ U_{i,j,k}\cdot\boldsymbol{\zeta}\geqslant|U_{i,j,k}|\cos\varepsilon,$$
$$V_{i,j,k}\cdot\boldsymbol{\zeta}\geqslant|V_{i,j,k}|\cos\varepsilon,\ W_{i,j,k}\cdot\boldsymbol{\zeta}\geqslant|W_{i,j,k}|\cos\varepsilon.$$

Then parametric triangular Bézier surfaces (1) *are convex in Δ if*

$$\cos(\bar{\alpha}+\varepsilon)\geqslant\frac{2\sqrt{3}(n-1)G^2\cos\varepsilon\sin^2\varepsilon}{\min\limits_{\substack{u+v+w=1\\u\geqslant0,v\geqslant0,w\geqslant0}}\sum\limits_{i+j+k=n-1}|\boldsymbol{v}_{i,j,k}|B^{n-1}_{i,j,k}(u,v,w)}.$$

Proof From (19), for any vector U satisfying $U\cdot\boldsymbol{\zeta}\geqslant|U|\cos\varepsilon$,

$$n^{-2}\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)\cdot U=\sum\boldsymbol{v}_{i,j,k}\cdot UB^{n-1}_{i,j,k}(u_0,v_0,1-u_0-v_0)$$
$$+2\delta\sum_{m=0}^{n-2}\sum\boldsymbol{g}^{n-2-m}_{i,j,k}\cdot UB^m_{i,j,k}(u_0,v_0,1-u_0-v_0),$$

$\delta=1-u_0^2-v_0^2-w_0^2\leqslant2/3$, where $w_0=1-u_0-v_0$. Since $\boldsymbol{v}_{i,j,k}$ is in $\bar{\alpha}$-cone of $\boldsymbol{\zeta}$ and U in ε-cone of $\boldsymbol{\zeta}$,

$$\boldsymbol{v}_{i,j,k}\cdot U\geqslant|\boldsymbol{v}_{i,j,k}|\,|U|\cos(\alpha+\varepsilon)$$

and

$$\boldsymbol{g}^{n-2-m}_{i,j,k}\cdot U=\alpha U^{n-2-m}_{i,j,k}\times V^{n-2-m}_{i,j,k}\cdot U+\beta V^{n-2-m}_{i,j,k}\times W^{n-2-m}_{i,j,k}\cdot U+\gamma W^{n-2-m}_{i,j,k}\times U^{n-2-m}_{i,j,k}\cdot U.$$

$U^{n-2-m}_{i,j,k}$ is convex combination of $U_{i,j,k}$, therefore it is in ε-cone of $\boldsymbol{\zeta}$. Similarly $V^{n-2-m}_{i,j,k}$, $W^{n-2-m}_{i,j,k}$ are in ε-cone of $\boldsymbol{\zeta}$. From Lemma 3

$$\boldsymbol{g}^{n-2-m}_{i,j,k}\cdot U\leqslant\frac{3\sqrt{3}}{2}G^2|U|\cos\varepsilon\sin^2\varepsilon.$$

Hence

$$n^{-2}\cdot\boldsymbol{r}_u\times\boldsymbol{r}_v\cdot U\geqslant\cos(\bar{\alpha}+\varepsilon)\sum|\boldsymbol{v}_{i,j,k}|B^{n-1}_{i,j,k}(u_0,v_0,w_0)|U|$$
$$-\frac{4}{3}(n-1)\frac{3\sqrt{3}}{2}G^2\cos\varepsilon\sin^2\varepsilon|U|.$$

By the condition of the theorem

$$\boldsymbol{r}_u\times\boldsymbol{r}_v(u_0,v_0)\cdot U\geqslant0.$$

The theorem is proved with the corollary to Lemma 2.

The geometric meaning is conspicuous in the theorem. If a Bézier surface is wanted to be convex, for some direction ζ, the less is the cone spanned by twist vectors the larger is the cone in which normal vectors $\nu_{i,j,k}$ of triangle patches of the Bézier control net are allowed.

§4. Functional Bézier Surfaces

Functional Bézier surface is a special case of parametric Bézier surface. It is clear that twist vectors of a functional Béier surface are collinear, say, with z-axis, while those of parametric Bezier surface are generally not collinear. Does a parametric Bézier surface degenerate to a functional Bézier surface when its twist vectors become collinear? The following theorem answers that question.

Theorem 2. *If a parametric Bézier surface has control net not lying on a plane, it degenerates to a functional Bézier surface if and only if its twist vectors are collinear.*

Proof. The necessity of the condition is obvious. We prove only that the condition is sufficient.

Let ζ be the unit vector with which the twist vectors are collinear, α be the plane whose normal vector is ζ, ACD, BDC be two adjacent triangle patches of the control net, A', B', C', D' be projections on α of A, B, C, D respectively, M be the mean point of A and B, M' be its projection on α (refer to Fig. 3.). Obviously, M' is the mean point of C' and D' as well as A' and B'.

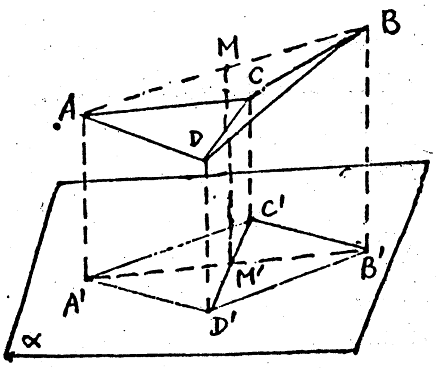

Fig. 3.

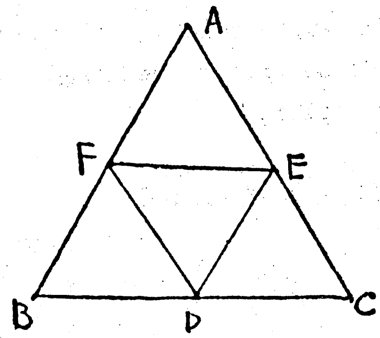

Fig. 4.

Therefore, the projection of the control net on the plane α is some parallelograms. For the case $n=2$ (Fig. 4.), these parallelograms are $AFDE$, $EFDC$, $EFBD$ as shows in Fig. 4. Obviously, ABC is a triangle with E, F, D being mean points of its edges. Take $\triangle ABC$ as domain triangle of barycentric coordinate and define the barycentric coordinate so that (1, 0, 0) is the point to which P_{200} is projected, etc. Let f_{200} be the directed distance from (1, 0, 0) to P_{200} and $f_{ijk}(i+j+k=2)$ are similarly defined. From the figure-making theorem of Bézier surface, we know the point determined by (1) coincides with the point whose projection on $\triangle ABC$ has (u, v, w) as its barycentric coordinate and from which to the plane α the distance is

$$f(u, v, w)=\sum f_{i,j,k}B^2_{i,j,k}(u, v, w).$$

So the parametric Bézier surface degenerates to the functional Bézier surface. The proof for the case $n>2$ is similar.

Theorem 2 signifies the particular importance of twist vectors. If ζ is assigned unit vector $(0, 0, 1)$, the 3-direction convex condition of Bézier control net offered by Chang and Davis implies that the ε in Theorem 1 is zero, the $\bar{\alpha}$ in Theorem 1 satisfies $0\leqslant\bar{\alpha}\leqslant\pi/2$, and hence the condition of Theorem 1 is satisfied. So Theorem 1 may be considered as an extension of the convexity theorem given by Chang and Davis.

Chang and Fong offered recently a new sufficint convexity condition which reads as follows.

Theorem 3 (Chang & Fong, 1984). *For functional Bézier surface, let* $\triangle^{(1)}_{i,j,k}=U_{i,j,k}\cdot z^0$, $\triangle^{(2)}_{i,j,k}=V_{i,j,k}\cdot z^0$, $\triangle^{(3)}_{i,j,k}=W_{i,j,k}\cdot z^0$, *where* $z^0=(0, 0, 1)$. *Then functional Bézier surfaces will be convex if*

$$\triangle^{(1)}_{i,j,k}+\triangle^{(2)}_{i,j,k}\geqslant 0, \tag{31}$$

$$\triangle^{(2)}_{i,j,k}+\triangle^{(3)}_{i,j,k}\geqslant 0, \tag{32}$$

$$\triangle^{(3)}_{i,j,k}+\triangle^{(1)}_{i,j,k}\geqslant 0, \tag{33}$$

$$\triangle^{(1)}_{i,j,k}\triangle^{(2)}_{i,j,k}+\triangle^{(2)}_{i,j,k}\triangle^{(3)}_{i,j,k}+\triangle^{(3)}_{i,j,k}\triangle^{(1)}_{i,j,k}\geqslant 0, \tag{34}$$

where $i+j+k=n-2$, n *is degree of the surface.*

For proof of the theorem, refer to [4].

Theorem 3 can be derived from Lemma 2. It is clear that for functional surface, $\boldsymbol{r}_u\times\boldsymbol{r}_v\cdot\boldsymbol{z}^0$ does not change its sign. We may as well assume that $\boldsymbol{r}_u\times\boldsymbol{r}_v\cdot\boldsymbol{z}^0\geqslant 0$. Under this assumption,

$$\boldsymbol{r}_u\times\boldsymbol{r}_v=\lambda\boldsymbol{z}^0+\mu\boldsymbol{h}$$

where $\lambda\geqslant 0$, $\boldsymbol{h}\cdot\boldsymbol{z}^0=0$, and

$$\boldsymbol{r}_u\times\boldsymbol{r}_v\cdot U_{i,j,k}=\lambda\triangle^{(1)}_{i,j,k},$$

$$\boldsymbol{r}_u\times\boldsymbol{r}_v\cdot V_{i,j,k}=\lambda\triangle^{(2)}_{i,j,k},$$

$$\boldsymbol{r}_u\times\boldsymbol{r}_v\cdot W_{i,j,k}=\lambda\triangle^{(3)}_{i,j,k}.$$

So (22)—(25) are equivalent to (31)—(34).

§5. Special Case

For $n=2$ we can get a less stringent sufficient condition.

Theorem 4. *For Bézier surface of degree 2, let* $U=U_{0,0,0}$, $V=V_{0,0,0}$, $W=W_{0,0,0}$. *Assume that* $\boldsymbol{v}_{1,0,0}\cdot U\geqslant 0$, $\boldsymbol{v}_{1,0,0}\cdot V\geqslant 0$, $\boldsymbol{v}_{1,0,0}\cdot W\geqslant 0$, $\boldsymbol{v}_{0,1,0}\cdot U\geqslant 0$, $\boldsymbol{v}_{0,1,0}\cdot V\geqslant 0$, $\boldsymbol{v}_{0,1,0}W\geqslant 0$, $\boldsymbol{v}_{0,0,1}\cdot U\geqslant 0$, $\boldsymbol{v}_{0,0,1}\cdot V\geqslant 0$, $\boldsymbol{v}_{0,0,1}\cdot W\geqslant 0$. *Then the Bézier surface is convex in* Δ *if*

$$(U, V, W)=U\times V\cdot W\geqslant 0$$

or following three conditions are satisfied.

$$(1)\begin{cases}\left|\dfrac{(\boldsymbol{v}_{0,1,0}-\boldsymbol{v}_{0,0,1})\cdot U}{(U,V,W)}\right|\geqslant 4,\\ or\ (\boldsymbol{v}_{0,1,0}+\boldsymbol{v}_{0,0,1})\cdot U/2\geqslant -\dfrac{((\boldsymbol{v}_{0,1,0}-\boldsymbol{v}_{0,0,1})\cdot U)^2}{16(U,V,W)}-(U,V,W);\end{cases}$$

$$(2)\begin{cases}\left|\dfrac{(\boldsymbol{v}_{0,0,1}-\boldsymbol{v}_{1,0,0})\cdot V}{(U,V,W)}\right|\geqslant 4,\\ or\ (\boldsymbol{v}_{0,0,1}+\boldsymbol{v}_{1,0,0})\cdot V/2\geqslant -\dfrac{((\boldsymbol{v}_{0,0,1}-\boldsymbol{v}_{1,0,0})\cdot V)^2}{16(U,V,W)}-(U,V,W);\end{cases}$$

$$(3)\begin{cases}\left|\dfrac{(\boldsymbol{v}_{1,0,0}-\boldsymbol{v}_{0,1,0})\cdot W}{(U,V,W)}\right|\geqslant 4,\\ or\ (\boldsymbol{v}_{1,0,0}+\boldsymbol{v}_{0,1,0})\cdot W/2\geqslant -\dfrac{((\boldsymbol{v}_{1,0,0}-\boldsymbol{v}_{0,1,0})\cdot W)^2}{16(U,V,W)}-(U,V,W).\end{cases}$$

Proof For $n=2$, that the surface is convex is equivalent to that Guass curvature is nonnegative. Let

$$a=\boldsymbol{r}_u\times\boldsymbol{r}_v\cdot U,\ b=\boldsymbol{r}_u\times\boldsymbol{r}_v\cdot V,\ c=\boldsymbol{r}_u\times\boldsymbol{r}_v\cdot W.$$

The sign of Guass curvature is the same as sign of

$$K=ab+bc+ca,$$

$$\boldsymbol{r}_u\times\boldsymbol{r}_v=u\boldsymbol{v}_{1,0,0}+v\boldsymbol{v}_{0,1,0}+w\boldsymbol{v}_{0,0,1}+4(uvU\times V+vwV\times W+wuW\times U),$$

$$a=u\boldsymbol{v}_{1,0,0}\cdot U+v\boldsymbol{v}_{0,1,0}\cdot U+w\boldsymbol{v}_{0,0,1}\cdot U+4vw(U,V,W).$$

If $(U,V,W)\geqslant 0$, then $a\geqslant 0$ in Δ, and similarly $b\geqslant 0$, $c\geqslant 0$ in Δ, and so $K\geqslant 0$. If $(U,V,W)<0$,

$$a=u(\boldsymbol{v}_{1,0,0}-\boldsymbol{v}_{0,0,1})\cdot U+v(\boldsymbol{v}_{0,1,0}-\boldsymbol{v}_{0,0,1})\cdot U+4v(U,V,W)$$
$$-4v^2(U,V,W)-4uv(U,V,W)+\boldsymbol{v}_{0,0,1}\cdot U.$$

Hesse matrix of a with respect to u, v is clearly negative and so a cannot reach its minimum in the interior of Δ. a is clearly nonnegative along $v=0$ and $w=0$. When $u=0$,

$$a=-4(U,V,W)v^2+v((\boldsymbol{v}_{0,1,0}-\boldsymbol{v}_{0,0,1})\cdot U+4(U,V,W))+\boldsymbol{v}_{0,0,1}\cdot U,$$

$a\geqslant 0$ at $v=0$ and $v=1$. If $v_0\in(0,1)$ is minimum point of a, then

$$\frac{da(v_0)}{dv}=-8(U,V,W)v_0+(\boldsymbol{v}_{0,1,0}-\boldsymbol{v}_{0,0,1})\cdot U+4(U,V,W)=0,$$

$$v_0=1/2+(\boldsymbol{v}_{0,1,0}-\boldsymbol{v}_{0,0,1})\cdot U/8(U,V,W),$$

$$a|_{u=0,v=v_0}=(\boldsymbol{v}_{0,1,0}+\boldsymbol{v}_{0,0,1})\cdot U/2+(U,V,W)$$
$$+((\boldsymbol{v}_{0,1,0}-\boldsymbol{v}_{0,0,1})\cdot U)^2/16(U,V,W).$$

If the condition (1) of the theorem is satisfied, either a has no minimum point in (0, 1) or its minimum is nonnegative, and consequently a is nonnegative in $\triangle$. For the same reason, b and c are nonnegative in $\triangle$ if (2) and (3) are satisfiea. Hence the surface is convex in $\triangle$.

The following example shows that non-convex control net may generate a convex Bézier surface.

Ex. 1 $P_{2,0,0}=(0, 0, 1)$, $P_{0,0,2}=(-1, 1, t)$, $P_{0,2,0}=(1, 1, h)$, $P_{0,1,1}=(0, 0, 0)$, $P_{1,1,0}=(1, 0, 0)$, $P_{0,1,1}=(0, 1, 0)$ (Fig. 5.), $U=(-1, 1, 1)/2$, $V=(0, 0, h)/2$, $W=(0, 0, t)/2$, $(U, V, W)=0$.

$$K=(h+ht+t)v^2/4+(h+t-t^2)W^2/4+(h+t)uv/4$$
$$+(2h+th+2t-t^2)vw/4+(h+t)wu/4.$$

When $0<t<1$, $0>h>\max(-t/(t+1), t^2-t, -t, (t^2-2t)/(2+t))$, K is nonnegative, while the control net is non-convex.

In the following example is constructed a non-convex surface generated by a convex control net for which $(U, V, W)<0$.

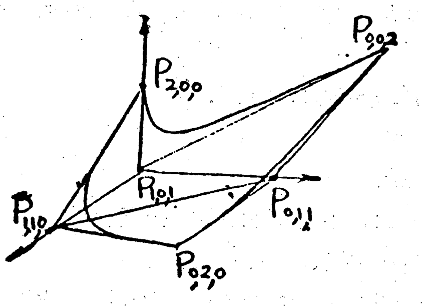

Fig. 5.

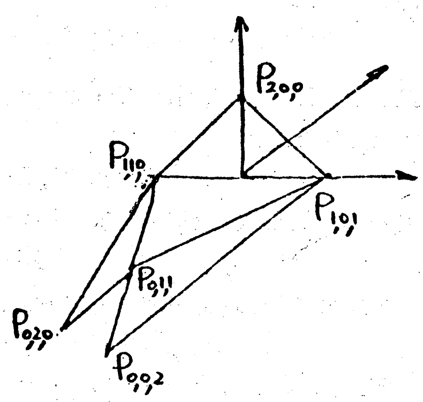

Fig. 6.

Ex. 2 $P_{2,0,0}=(0, 0, 1)$, $P_{0,2,0}=(1/2. -3, 0)$, $P_{0,0,2}=(1/2, -h, 0)$, $P_{0,1,1}=(0, -2, 0)$, $P_{1,0,1}=(1, 0, 0)$, $P_{1,1,0}=(-1, 0, 0)$ (Fig. 6.), $U=(0, -2, 1)/2$, $V=(3/2, -1, 0)/2$, $W=(-3/2, 2-h, 0)/2$.

When $v=0$,

$$K=(-3w^2u/8-3wu^2/4+wu/2)h^2+O(h).$$

The coefficient of h^2

$$-3w^2u/8-3wu^2/4+wu/2=wu(2-3w/2-3u)/4$$
$$=wu(1/2-3u/2)/4<0$$

if $1/3<u<1$. So K becomes negative if h is sufficiently large.

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