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A CHARACTERIZATION OF SUBMETACOMPACTNESS

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Abstract

The following results are obtained:

1. A space is submetacompact iff it is almost discretely θ-expandable and strictly quasi-paracompact.

2. A space is paracompact iff it is θ-expandable and strictly quasi-paracompact.

§ 1. Introduction

In order to generalize the classes of metacompact spaces and subparacompact spaces simultaneously, Liu Yingming introduced strictly quasi-paracompact spaces[6]. In 1984, Zhu Jun [11] and Long Bing [7] proved independently that every submetacompact space is strictly quasi-paracompact but not conversely.

In this paper, we give a necessary and sufficient condition for a strictly quasi-paracompact space to be submetacompact. We also characterize paracompactness by conditions which are similar to that of the definition of submetacompactness.

Our terminology follows that of [3]; we do not, however, require paracompact or metacompact spaces to satisfy any separation axioms. Let $\mathscr{U}$ be a family of subsets of X. For each $A\subset X$, the family $\{U\cap A: U\in\mathscr{U}\}$ is denoted by $\mathscr{U}|A$. The letter N denotes the set of positive integers. If $n\in N$ and $(n_1, \cdots, n_k)\in N^k$ for some $k\in N$, then $(n_1, \cdots, n_k)\oplus n$ denotes the element $(n_1, \cdots, n_k, n)$ of the set N^{k+1}.

Definition 1.1. [10] *A space X is submetacompact ($=\theta$-refinable) iff every open cover of X has a refinement $\bigcup_{i=1}^{\infty}\mathscr{G}_i$ satisfying* (a) *each $\mathscr{G}_i$ is an open cover of X, and* (b) *for each $x\in X$, there exists $n\in N$ such that $\mathscr{G}_n$ is point finite at x.*

Definition 1.2. [6] *A space X is strictly quasi-paracompact iff every open cover of X has a refinement $\bigcup_{i=1}^{\infty}\mathscr{F}_i$ such that $\mathscr{F}_1$ is a discrete family of closed subsets of X and $\mathscr{F}_n|(X\setminus\bigcup_{i=1}^{n-1}(\bigcup\mathscr{F}_i))$ is a discrete family of closed subsets of subspace $X\setminus\bigcup_{i=1}^{n-1}$*

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$(\bigcup \mathscr{F}_i)$ for each $n \geqslant 2$.

Remark 1.3. Let $\bigcup_{i=1}^{\infty} \mathscr{F}_i$ be a cover of X satisfying the conditions in Definition 1.2. It is easy to show by induction that $\bigcup_{i=1}^{n} (\bigcup \mathscr{F}_i)$ is a closed subset of X for each $n \in N$.

Recall that an expansion of a family $\mathscr{F}$ of subsets of X is a family $\{G(F): F \in \mathscr{F}\}$ of subsets of X such that $F \subset G(F)$ for each $F \in \mathscr{F}$.

Definition 1.4.[8] *A space X is θ-expandable (almost discretely θ-expandable) iff every locally finite (discrete) family $\mathscr{F}$ of closed subsets of X has a sequence $\langle \mathscr{V}_n = \{V_n(F): F \in \mathscr{F}\} \rangle_{n=1}^{\infty}$ of open expansions such that for each $x \in X$ there exists $n \in N$ such that $\mathscr{V}_n$ is locally (point) finite at x.*

The following results will be used in the proof of our theorems.

Theorem 1.5.[11,7] *Every submetacompact space is strictly quasi-paracompact but not conversely.*

Theorem 1.6.[2] *A space X is subparacompact iff it is collectionwise subnormal and submetacompact.*

Theorem 1.7.[1] (i) *A space is metacompact iff it is almost discretely expandable and submetacompact.*

(ii) *A Hausdorff space is paracompact iff it is collec-tionwise normal and submetacompact.*

Theorem 1.8.[4] *A space is paracompact iff it is θ-expandable and submetacompact.*

§ 2. Characterizations

Theorem 2.1. *A space is submetacompact iff it is almost discretely θ-expandable and strictly quasi-paracompact.*

Proof Necessity follows from Theorem 1.5 and the fact that every submetacompact space is almost θ-expandable (see [5, Theorem 1.5]). To prove sufficiency, let $\mathscr{W}$ be any open cover of an almost discretely θ-expandable and strictly quasi-paracompact space X. Let $\bigcup_{i=1}^{\infty} \mathscr{F}_i$ be a refinement of $\mathscr{W}$ satisfying the conditions of Definition 1.2. Let $\mathscr{F}_i = \{F(i, s): s \in S_i\}$ for each $i \in N$. For each $i \in N$ and each $s \in S_i$, there exists $W(i, s) \in \mathscr{W}$ such that $F(i, s) \subset W(i, s)$. By induction, for each $t \in \bigcup_{i=1}^{\infty} N^i$, we can find a family $\mathscr{V}(t)$ of subsets of X such that:

(a) $\mathscr{V}(t) = \{V(t, s): s \in S_n\}$ is a family of open subsets of X for each $n \in N$ and each $t \in N^n$;

(b) $\mathscr{V}(t)$ is a partial refinement of $\mathscr{W}$ for each $t \in \bigcup_{i=1}^{\infty} N^i$;

(c) $\bigcup_{i=1}^{n}(\bigcup\mathscr{F}_i)\subset\bigcup_{i=1}^{n}(\bigcup\mathscr{V}(t_1, \cdots, t_i))$ for each $n\in N$ and each $t=(t, \cdots, t_n)\in N^n$;

(d) $(\bigcup\mathscr{V}(t))\cap\bigcup_{i=1}^{n-1}(\bigcup\mathscr{F}_i)=\varnothing$ for each $n\geqslant 2$ and each $t\in N^n$;

(e) For each $x\in X$, each $n\geqslant 2$ and each $t\in N^{n-1}$, there exists $t_n(x)\in N$ such that $\mathscr{V}(t\oplus t_n(x))$ is point finite at x.

Now we prove this fact. Since $\mathscr{F}_1$ is a discrete family of closed subsets of X and X is almost discretely θ-expandable, there exists a sequence $\langle\mathscr{V}(t_1)\rangle_{t_1=1}^{\infty}$ of open expansion of $\mathscr{F}_1$ such that for each $x\in X$, there exists $t_1(x)\in N$ such that $\mathscr{V}(t_1(x))$ is point finite at x. We may suppose that $\mathscr{V}(t_1)$ is a partial refinement of $\mathscr{W}$ for each $t_1\in N$. Let $n\in N$ and assume that we have defined the family $\mathscr{V}(t)$ for each $t\in\bigcup_{i=1}^{n}N^i$ such that the conditions (a)—(e) are satisfied. By the induction hypothesis, we have

$$\bigcup_{i=1}^{n}(\bigcup\mathscr{F}_i)\subset\bigcup_{i=1}^{n}(\bigcup\mathscr{V}(t_1, \cdots, t_i)) \text{ for each } t=(t_1, \cdots, t_n)\in N^n. \quad (1)$$

It follows from Definition 1.2 that $\mathscr{F}'_{n+1}=\mathscr{F}_{n+1}\Big|\Big(X\setminus\bigcup_{i=1}^{n}\bigcup\mathscr{V}(t_1, \cdots, t_i)\Big)$ is a discrete family of closed subsets in the closed subspace $X\setminus\bigcup_{i=1}^{n}(\bigcup\mathscr{V}(t_1, \cdots, t_i))$. Since $\mathscr{F}'_{n+1}$ is a discrete family of closed subsets of X, there exists a sequence $\langle\mathscr{D}(t\oplus t_{n+1})=\{D(t\oplus t_{n+1}, s): s\in S_{n+1}\}\rangle_{t_{n+1}=1}^{\infty}$ of open expansion of $\mathscr{F}'_{n+1}$ such that for each $x\in X$ there exists $t_{n+1}(x)\in N$ such that $\mathscr{D}(t\oplus t_{n+1}(x))$ is point finite at x. For each $s\in S_{n+1}$, set

$$V(t\oplus t_{n+1}, s)=D(t\oplus t_{n+1}, s)\cap W(n+1, s)\cap\Big(X\setminus\bigcup_{i=1}^{n}(\bigcup\mathscr{F}_i)\Big).$$

By Remark 1.3, $V(t\oplus t_{n+1}, s)$ is an open set for each $s\in S_{n+1}$. Then $\mathscr{V}(t\oplus t_{n+1})=\{V(t\oplus t_{n+1}, s): s\in S_{n+1}\}$ is an open expansion of $\mathscr{F}'_{n+1}$ for each $t_{n+1}\in N$ and

$$(\bigcup\mathscr{V}(t\oplus t_{n+1}))\cap\bigcup_{i=1}^{n}(\bigcup\mathscr{F}_i)=\varnothing.$$

Thus the collection $\Big\{\mathscr{V}(t): t\in\bigcup_{i=1}^{n+1}N^i\Big\}$ meets the conditions (a), (b), (d), (e). To see (c), by (1) and

$$F(n+1, s)\setminus\bigcup_{i=1}^{n}\bigcup\mathscr{V}(t_1, \cdots, t_i)\subset V(t\oplus t_{n+1}, s)$$

for each $t_{n+1}\in N$ and each $s\in S_{n+1}$ we have

$$\begin{aligned}\bigcup_{i=1}^{n+1}(\bigcup\mathscr{F}_i)&\subset\Big(\bigcup_{i=1}^{n}\bigcup\mathscr{V}(t_1, \cdots, t_i)\Big)\cup\Big(\bigcup\mathscr{F}_{n+1}\setminus\bigcup_{i=1}^{n}\bigcup\mathscr{V}(t_1, \cdots, t_i)\Big)\\&\subset\Big(\bigcup_{i=1}^{n}\bigcup\mathscr{V}(t_1, \cdots, t_i)\Big)\cup\bigcup\mathscr{V}(t\oplus t_{n+1})\\&=\bigcup_{i=1}^{n+1}\bigcup\mathscr{V}(t_1, \cdots, t_i).\end{aligned}$$

We have now defined the family $\mathscr{V}(t)$, $t\in\bigcup_{i=1}^{\infty}N^i$. By (a) and (e), $\mathscr{H}(t)=\bigcup_{n=1}^{\infty}\mathscr{V}(t_1, \cdots, t_n)$ is an open refinement of $\mathscr{W}$ for each $t=(t_1, t_2, \cdots)\in N^N$. For each i, $n\in N$, set

$$T_{in}=\{t\in N^N: t_k=i \text{ for each } k>n\}.$$

Then $T=\bigcup_{i=1}^{\infty}\bigcup_{n=1}^{\infty}T_{in}$ is countable and $\{\mathscr{H}(t): t\in T\}$ is a countable family of open refinements of $\mathscr{W}$. For each $x\in X$, there exists $n\in N$ such that $x\in\bigcup\mathscr{F}_n$. By (e), there exists $(t_1(x), \cdots, t_n(x))\in N^n$ such that $\mathscr{V}(t_1(x), \cdots, t_n(x))$ is point finite at x. Let $t_k(x)=t_n(x)$ for each $k>n$. Then $t(x)=\langle t_1(x), \cdots, t_n(x), t_{n+1}(x), \cdots\rangle\in T$. By (d), $x\notin\bigcup\mathscr{V}(t_1(x), \cdots, t_{m+1}(x))$ for each $m\geqslant n$. Therefore, $\mathscr{H}(t(x))$ is point finite at x. Thus X is submetacompact.

Definition 2.2.[2] *A space X is collectionwise subnormal iff for each discrete collection $\mathscr{F}$ of closed subsets of X, there exists a countable family $\langle\mathscr{V}_n\rangle_{n=1}^{\infty}$ of open expansions of $\mathscr{F}$ such that $X=\bigcup_{n=1}^{\infty}E_n$, where E_n is the set of all $x\in X$ that are not contained in two different elements of $\mathscr{V}_n$.*

Every collectionwise subnormal space is evidently almost discretely θ-expandable. By Theorem 1.6, we have the following corollary to Theorem 2.1.

Corollary 2.3. *A space is subparacompact iff it is collectionwise subnormal and strictly quasi-paracompact.*

By Theorem 1.7, we also have the following corollaries.

Corollary 2.4.[6] *A Hausdorff space is paracompact iff it is collectionwise normal and strictly quasi-paracompact.*

Corollary 2.5.[7,11] *A space X is metacompact iff it is almost discretely expandable and strictly quasi-paracompact.*

Recall that a cover $\mathscr{U}$ of X is semiopen if $x\in \operatorname{Int}St(x, \mathscr{U})$ for each $x\in X$.

Theorem 2.6. *The following are equivalent for a space X:*

(i) *X is paracompact;*

(ii) *X is strictly quasi-paracompact and every open cover of X has a refinement $\bigcup_{i=1}^{\infty}\mathscr{G}_i$ satisfying*

(a) *each $\mathscr{G}_i$ is a semi-open cover of X, and*

(b) *for each $x\in X$, there exists $n\in N$ such that $\mathscr{G}_n$ is locally finite at x;*

(iii) *X is strictly quasi-paracompact and θ-expandable.*

Proof Clearly (i)⇒(ii). (iii)⇒(i) follows from Theorems 1.8 and 2.1. To prove (ii)⇒(iii), let $\mathscr{F}$ be a locally finite family of closed subsets of X. Set

$$S=\{\mathscr{B}: \mathscr{B} \text{ is a finite subfamily of } \mathscr{F}\}.$$

For each $\mathscr{B}\in S$, set $V(\mathscr{B})=X\setminus\bigcup(\mathscr{F}\setminus\mathscr{B})$. Then $\mathscr{V}=\{V(\mathscr{B}): \mathscr{B}\in S\}$ is an open cover of X such that each element of $\mathscr{V}$ intersects only finitely many elements of $\mathscr{F}$. By hypothesis (ii), there exists a refinement $\bigcup_{i=1}^{\infty} \mathscr{G}_i$ of $\mathscr{V}$ satisfying the conditions of (ii). For each $F \in \mathscr{F}$ and each $i \in N$, set

$$H(F, i) = \operatorname{Int} St(F, \mathscr{G}_i),$$
$$\mathscr{H}_i = \{H(F, i): F \in \mathscr{F}\}.$$

It is easy to prove that $\langle \mathscr{H}_i \rangle_{i=1}^{\infty}$ is a sequence of open expan-sion of $\mathscr{F}$ such that for each $x \in X$, there exists $n \in N$ such that $\mathscr{H}_n$ is locally finite at x. Thus X is θ-expandable.

Corollary 2.7. *A space X is paracompact iff every open cover of X has a refinement $\bigcup_{i=1}^{\infty} \mathscr{G}_i$ such that*

(a) *each $\mathscr{G}_i$ is an open cover of X, and*

(b) *for each $x \in X$, there exists $n \in N$ such that $\mathscr{G}_n$ is locally finite at x.*

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