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JULIA'S LEMMA FOR ANALYTIC OPERATOR FUNCTIONS

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Abstract

The aim of this note is to generalize the Julia lemma in the classical theory of functions to analytic operator functions.

§1. Introduction

Julia's lemma[1] in the classical theory of functions has been extended in several cases [2,3,4]. The aim of this note is to generalize Fan's result[3] to analytic operator functions.

We follow [5] for notation. Let H be a complex Hilbert space and $L(H)$ the Banach algebra of all bounded linear operators on H. Δ will always stand for the open disc in the complex plane, i.e., $\Delta=\{z: |z|<1\}$. We recall that $N_H(\Delta)$ represents the set of all such analytic functions f with domain Δ that $\{f(z): z\in\Delta\}$ is a family of bounded normal operators on H, commuting pairwise. As usual, for two selfadjoint operators A and B in $L(H)$, by $A\geqslant B$ we mean that $A-B$ is positive. The notation $A>B$ means that $A-B$ is both positive and invertible.

§2. Two Lemmas

Lemma 1. *Let H be a complex Hilbert space. Suppose A and B are elements of $L(H)$ with $\|AB^{-1}\|<1$. If $N\in L(H)$ is a normal proper contraction* (i.e., $\|N\|<1$) *commuting with both A and B, and if $0<\rho<1$, then the inequality*

$$\|(A-NB)(B-N^*A)^{-1}\|<\rho \tag{1.1}$$

is equivalent to

$$(B^*-A^*N)(B-N^*A)<(1-\rho^2)^{-1}(I-N^*N)(B^*B-A^*A). \tag{1.2}$$

Proof From $\|N^*AB^{-1}\|<1$, it follows that both $I-N^*AB^{-1}$ and $B-N^*A=(I-N^*AB^{-1})B$ are invertible. By Lemma 5.1 in [5], inequality (1.1) is equivalent to

$$(A^*-B^*N^*)(A-NB)<\rho^2(B^*-A^*N)(B-N^*A)$$

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which may be written in

$$\begin{aligned}&(1-\rho^2)(B^*-A^*N)(B-N^*A)\\&\quad<(B^*-A^*N)(B-N^*A)-(A^*-B^*N^*)(A-NB)\\&\quad=B^*B-A^*A+A^*NN^*A-B^*N^*NB.\end{aligned}\tag{1.3}$$

By Fuglede–Putnam's theorem[6], that N commutes with both A and B implies $AN^*=N^*A$, $BN^*=N^*B$. Then

$$B^*B-A^*A+A^*NN^*A-B^*N^*NB=(I-N^*N)(B^*B-A^*A).$$

Thus inequality (1.3) (and hence (1.1)) is equivalent to

$$(B^*-A^*N)(B-N^*A)<(1-\rho^2)^{-1}(I-N^*N)(B^*B-A^*A),$$

which completes the proof.

Lemma 2. *Suppose A is a proper contraction on a complex Hilbert space and b is a positive number. Then the following inequalities are equivalent.*

(1) $\|(I-A)(I-A^*A)^{-1}(I-A^*)\|<b$,

(2) $(I-A^*)(I-A)<b(I-A^*A)$,

(3) $\|A-(1+b)^{-1}I\|<b(1+b)^{-1}$.

Proof See Lemma 3 in [4].

§ 3. The Main Result

Now we are ready to show the following Julia's lemma for analytic operator functions, which generalizes the one for operators of K. Fan[4].

Theorem 3. *Let H be a complex Hilbert space, and let $f\in N_H(\Delta)$ with $\|f(z)\|<1$ for all $z\in\Delta$. Suppose $\{N_n\}$ is a sequence of normal proper contraction on H such that*

(1) *each N_n commutes with f, i.e., $N_nf(z)=f(z)N_n$ for all z in Δ,*

(2) $\lim\limits_{n\to\infty}\|I-N_n\|=0$,

(3) $\lim\limits_{n\to\infty}\|I-f(N_n)\|=0$,

(4) $\lim\limits_{n\to\infty}\|aI-\{I-f(N_n)^*f(N_n)\}\{I-N_n^*N_n\}^{-1}\|=0$ *for some real number a,*

(5) $c=\liminf\limits_{n\to\infty}\{\|I-N_n^*N_n\|\,\|(I-N_n^*N_n)^{-1}\|\}<\infty$.

Suppose $T\in L(H)$ is a proper contraction (not necessarily normal) commuting with both N_n' s and f. Then we have

(a)
$$\begin{aligned}&\|\{I-f(T)\}\{I-f(T)^*f(T)\}^{-1}\{I-f(T)^*\}\|\\&\quad\leqslant ac\|(I-T)(I-T^*T)^{-1}(I-T^*)\|;\end{aligned}\tag{3.1}$$

(b) *If* $(I-T^*)(I-T)<b(I-T^*T)$ *for some* $b>0$, (3.2)

then

$$\{I-f(T)^*\}\{I-f(T)\}=abc\{I-f(T)^*f(T)\};\tag{3.3}$$

(c) *If* $\|T-(1+b)^{-1}I\|<b(1+b)^{-1}$, $(b>0)$, (3.4) *then*

$$\|f((T)-(1+abc)^{-1}I\|<abc(1+abc)^{-1}. \qquad (3.5)$$

Proof (a) Suppose $N\in L(H)$ is normal. Then

$$\|N\|=\sup\{|z|: z\in\sigma(N)\}=\sup\{|\langle Nx, x\rangle|: \|x\|=1\} \quad \text{(see [6])}.$$

If N is invertible as well, then $\sigma(N^{-1})=\{z: z^{-1}\in\sigma(N)\}$ and hence $\|N^{-1}\|^{-1}=\min\{|z|: z\in\sigma(N)\}$. Since N_n is normal with $\|N_n\|<1$, it follows that $I-N_n^*N_n$ is normal and invertible, and thus

$$\begin{aligned}\|(I-N_n^*N_n)^{-1}\|^{-1}&=\min\{|z|: z\in\sigma(I-N_n^*N_n)\}\\ &=\inf\{\langle(I-N_n^*N_n)x, x\rangle: \|x\|=1\}\\ &=1-\sup\{\langle N_n^*N_nx, x\rangle: \|x\|=1\}=1-\|N_n\|^2.\end{aligned}$$

Set $m_n=1-\|N_n\|^2$, $M_n=\|I-N_n^*N_n\|$. Then it is readily seen that

$$0<m_nI\leqslant I-N_n^*N_n\leqslant M_nI \qquad (3.6)$$

and

$$\liminf_{n\to\infty}\frac{M_n}{m_n}=c.$$

Suppose $\eta>c$. Without loss of generality, we may assume that

$$\frac{M_n}{m_n}<\eta \text{ for all } n, \qquad (3.7)$$

since $\{N_n\}$ may be replaced by a subsequence if necessary. Let b be a positive number such that

$$\|(I-T)(I-T^*T)^{-1}(I-T^*)\|<b. \qquad (3.8)$$

Noting that $N_n\to I$ and hence $m_n\to 0$ as $n\to\infty$, one may choose $0<a_n<1$ so that $b(1-a_n^2)=m_n$ (Cast away finite N_n's if necessary). Then, by (3.6) and (3.7),

$$bI\leqslant(1-a_n^2)^{-1}(I-N_n^*N_n)\leqslant\eta bI. \qquad (3.9)$$

An application of Theorem 3.1 in [5] shows that $\|f(N_n)\|<1$ and hence $I-f(N_n)^*f(N_n)$ is invertible. From the fact that N_n is normal and commutes with f, we deduce that $I-f(N_n)^*f(N_n)$ commutes with $(I-N_n^*N_n)^{-1}$ (F. P. Theorem, [6], p. 300), and thus $\{I-f(N_n)^*f(N_n)\}(I-N_n^*N_n)^{-1}$ is positive and invertible.

Put

$$\varepsilon_n=\|aI-\{I-f(N_n)^*f(N)\}(I-N_n^*N_n)^{-1}\|,$$

i.e.,

$$\varepsilon_n=\sup\{|\langle\{I-f(N_n)^*f(N_n)\}(I-N_n^*N_n)^{-1}x, x\rangle-a|: \|x\|=1\}.$$

Then $\varepsilon_n\to 0$ by our assumption, and hence $a\geqslant 0$. Thus

$$(a-\varepsilon_n)I\leqslant\{I-f(N_n)^*f(N_n)\}(I-N_n^*N_n)^{-1}\leqslant(a+\varepsilon_n)I. \qquad (3.10)$$

Combining (3.10) with (3.9), we have

$$b(a-\varepsilon_n)I\leqslant(1-a_n^2)^{-1}\{I-f(N_n)^*f(N_n)\}\leqslant\eta b(a+\varepsilon_n)I, \qquad (3.11)$$

since $I-N_n^*N_n$ and $I-f(N_n)^*f(N_n)$ are commutative and both of them are invertible.

Observe that, by Lemma 2

$$\|(I-T)(I-T^*T)^{-1}(I-T^*)\|<b$$

is equivalent to

$$(I-T^*)(I-T)<b(I-T^*T).$$

Thus inequality (3.8) implies that there exists a $\delta>0$ such that

$$b(I-T^*T)-(I-T^*)(I-T)\geqslant\delta_I.$$

Since $\|N_n-I\|\to 0$ as $n\to\infty$, we may assume that

$$b(I-T^*T)-(I-T^*N_n)(I-N_n^*T)\geqslant\frac{1}{2}\delta_I$$

or

$$(I-T^*N_n)(I-N_n^*T)<b(I-T^*T)$$

for all n, by throwing away finite N_n 's again if necessary. From (3.9), it follows that

$$(I-T^*N_n)(I-N_n^*T)<(1-a_n^2)^{-1}(I-N_n^*N_n)(I-T^*T)$$

since T commutes with N_n (and hence with N_n^*). Then we have

$$\|(T-N_n)(I-N_n^*T)^{-1}\|<a_n$$

by Lemma 1. On the other hand, Pick's theorem for operator functions, i.e., Theorem 5.2 in [5], shows that

$$\|\{f(T)-f(N_n)\}\{I-f(N_n)^*f(T)\}^{-1}\|\leqslant\|(T-N_n)(I-N_n^*T)^{-1}\|.$$

Thus

$$\|\{f(T)-f(N_n)\}\{I-f(N_n)^*f(T)\}^{-1}\|<a_n. \tag{3.12}$$

Observing that $f(N_n)$ is also a normal proper contraction, $\|f(T)\|<1$ and $f(T)f(N_n)=f(N_n)f(T)$, one can easily see by Lemma 1 again that (3.12) is equivalent to

$$\{I-f(T)^*f(N_n)\}\{I-f(N_n)^*f(T)\}$$
$$<(1-a_n^2)^{-1}\{I-f(N_n)^*f(N_n)\}\{I-f(T)^*f(T)\},$$

which implies

$$\{I-f(T)^*f(N_n)\}\{I-f(N_n)^*f(T)\}$$
$$<b\eta(a+\varepsilon_n)\{I-f(T)^*f(T)\}$$

by (3.11). Since $f((N_n)\to I$ and $\varepsilon_n\to 0$ as $n\to\infty$, it follows that

$$\{I-f(T)^*\}\{I-f(T)\}\leqslant\eta ab\{I-f(T)^*f(T)\},$$

which, by Lemma 2, is equivalent to

$$\|\{I-f(T)\}\{I-f(T)^*f(T)\}^{-1}\{I-f(T)^*\}\|\leqslant ab\eta.$$

It is clear now that assertion (a) holds since b, η are any positive numbers satisfying the conditions:

$$\|(I-T)\cdot(I-T^*T)^{-1}(I-T^*)\|<b,\ \eta>c.$$

(b) First, from $\|f(0)\|<1$, we have

$$ac\geqslant\|\{I-f(0)\}\{I-f(0)^*f(0)\}^{-1}\{I-f(0)^*\}\|>0, \tag{3.13}$$

by setting $T=0$ in (3.1). Thus $a>0$ for c is not less than 1. Lemma 2 tells that

(3.2) is equivalent to (3.8) which implies, according to (3.1),

$$\|\{I-f(T)\}\{I-f(T)^*f(T)\}^{-1}\{I-f(T)^*\}\|<abc.$$

This leads to

$$\{I-f(T)^*\}\{I-f(T)\}<abc\{I-f(T)^*f(T)\} \tag{3.14}$$

by one more use of Lemma 2. Assertion (b) is proved.

(c) In fact, we have shown in the above proof that (3.4), which is equivalent to (3.8) by Lemma 2, implies (3.14). Moreover, (3.5) is a consequence of the latter, again by Lemma 2. This amounts to saying that assertion (c) holds. The proof is complete.

Remark. (1) If $N_n=\lambda_n I$, $\lambda_n\in\Delta$, then the coefficient c in Theorem 3 is precisely equal to 1.

(2) (3.13) may be regarded as an extension of the inequality (296.4) in [7] (also [1], p. 88).

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