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A SORT OF POLYNOMIAL IDENTITIES OF $M_n(F)$ WITH CHAR $F\neq 0$

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Abstract

Let F denote a field, finite or infinite, with characteristic $p\neq 0$. In this paper, the author obtains the following result: *The symmetric polynomial on t letters*

$$S_{sym(t)}(x_1, x_2, \cdots, x_t)=\sum_{\pi\in sym(t)} X_{\pi 1}X_{\pi 2}\cdots X_{\pi t}$$

is a polynomial identity of $M_n(F)$ when $t\geqslant pn$, and this is sharp in the sense that if $t\leqslant pn-1$, it is not a polynomial identity of $M_n(F)$.

All terminologies used in this paper are agreeable to those in [1, 2, 3]. F is a field of characteristic $p\neq 0$, sym (m) is the symmetric group on m symbols which is the group of permutations of $(1, 2, \cdots, m)$.

The standard polynomial of degree t is

$$S_t(X_1, X_2, \cdots, X_t)=\sum_{\pi\in sym(t)}(\mathrm{sg}\,\pi)X_{\pi 1}X_{\pi 2}\cdots X_{\pi t}. \tag{1}$$

The symmetric polynomial on t letters is

$$S_{sym(t)}(X_1, X_2, \cdots, X_t)=\sum_{\pi\in sym(t)} X_{\pi 1}X_{\pi 2}\cdots X_{\pi t}. \tag{2}$$

Obviously, the symmetric polynomial of any degree is not a polynomial identity of $M_n(Z)$. In this paper we seek the symmetric polynomials which are polynomial identities of $M_n(F)$.

Lemma 1. *Let*

$$e_{i_1j_1}, e_{i_2j_2}, \cdots, e_{i_tj_t} \tag{3}$$

be t matrix units of $M_n(F)$, and assume $e_{i_1j_1}$, $e_{i_2j_2}\cdots e_{i_kj_k}$ are all distinct matrix units contained in (3) *with $e_{i_xj_x}$ occuring m_x times in* (3), $x=1, 2, \cdots, k$. *Then*

$$S_{sym(t)}(e_{i_1j_1}, e_{i_2j_2}, \cdots, e_{i_tj_t})=\Big(\prod_{x=1}^{k} m_x!\Big)A, \tag{4}$$

for some suitable $A\in M_n(F)$.

Proof Let $X_d=e_{i_dj_d}$ in (2) for $d=1, 2, \cdots, t$. Since $e_{i_1j_1}$ occurs m_1 times in (3), we may assume $X_{y_1}=X_{y_2}\cdots=X_{y_{m_1}}=e_{i_1j_1}$, where $1\leqslant y_1, y_2, \cdots, y_{m_1}\leqslant t$. If

$$X_{i_x}\cdots X_{y_1}\cdots X_{y_2}\cdots X_{y_{m_1}}\cdots X_{i_y} \tag{5}$$

is a summand of (2), then

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$$X_{i_x} \cdots X_{y_{\sigma 1}} \cdots X_{y_{\sigma 2}} \cdots X_{y_{\sigma m_1}} \cdots X_{i_y} \tag{5'}$$

is also a summand of (2), where $\sigma \in \mathrm{sym}(m_1)$. If

$$e_{i_x j_x} \cdots e_{i_1 j_1} \cdots e_{i_1 j_1} \cdots e_{i_1 j_1} \cdots e_{i_y j_y} \tag{5''}$$

is the summand of the left of (4) corresponding to (5), with the $m_1!$ permutations of $X_{y_1}, X_{y_2}, \cdots, X_{y_{m_1}}$ there produce $m_1!$ terms of form (5′) in (2), corresponding to these $m_1!$ terms of form (5′) in (2), there are $M_1!$ summands of form (5″) in the left of (4) due to $e_{i_1 j_1}$. Using the same consideration used for $e_{i_2 j_2}, \cdots, e_{i_k j_k}$, we see that there are $(m_1!)(m_2!) \cdots (m_k!)$ summands of form (5″) appearing in the left of (4). Hence we see that each summand of the left of (4) appears exactly $\prod_{x=1}^{k}(m_x!)$ times. So we obtain (4) and the lemma holds.

Corollary *If $t=(p-1)n^2+1$, then (2) is a polynomial identity of $M_n(F)$.*

Proof Since (2) is multilinear polynomial, we may prove the corollary by taking $X_1, X_2, \cdots, X_t$ to bo matrix units. $M_n(F)$ has only n^2 distinct matrix units. Given any $(p-1)n^2+1$ matrix units, by pigeon-hole principle, there exists (at least) one which occurs at least p times in those given $(p-1)n^2+1$ matrix units. By Lemma 1 and note that char $F=P$, our corollary holds immediately.

Now we delve the symmetric polynomial of the least degree which is a polynomial identity of $M_n(F)$. First some preparation. Let

$$e_{i_1 j_1} e_{i_2 j_2} \cdots e_{i_u j_u} \tag{6}$$

be a product of u matrix units. If $j_x = i_{x+1}$ for $x=1, 2, \cdots, u-1$, then we say (6) is a path. If

$$e_{x_1 y_1} e_{x_2 y_2} \cdots e_{x_d y_d} \tag{7}$$

is another path of d matrix units, we say that (6) and (7) are equal if $d=u$ and for all $k=1, 2, \cdots, u$, $e_{x_k y_k}=e_{i_k j_k}$, otherwise we say that they are different. Obviously (6) is not zero if and only if it is a path. If (6) is a path and $i_1 = j_u$, then we say that it is an i_1-cycle and denote it by O_{i_1}. A single matrix unit e_{ij} is a path, which is a cycle if and only if $i=j$. If (6) is an i_1-cycle and cannot be written as product of two i_1-cycles, we say that it is a simple i_1-cycle. If (6) is a path and contains m i_1-cycles, we denote the v-th i_1-cycle by $\overset{v}{O}_{i_1}$, $1 \leqslant v \leqslant m$. Obviously any permutation of the m i_1-cycles gives another path, the new and the old have the same product value. If (6) is a path (or a cycle), then we call u the length of the path (cycle).

For example, in $M_n(Z)$

$$e_{12}e_{23}e_{31}e_{11}e_{12}e_{21}e_{13} \tag{8}$$

is a path, which contains 3 simple 1-cycles, i. e., $\overset{1}{O}_1 = e_{12}e_{23}e_{31}$, $\overset{2}{O}_1 = e_{11}$, $\overset{3}{O}_1 = e_{12}e_{21}$, and we can write (8) as: $\overset{1}{O}1\ \overset{2}{O}1\ \overset{3}{O}1\ e_{13}$. If $\sigma \in \mathrm{sym}(3)$, then $\overset{\sigma 1}{O1}\ \overset{\sigma 2}{O2}\ \overset{\sigma 3}{O3}\ e_{13}$ is also a path and $\overset{1}{O}1\ \overset{2}{O}1\ \overset{3}{O}1\ e_{13} = \overset{\sigma 1}{O1}\ \overset{\sigma 2}{O1}\ \overset{\sigma 3}{O1} e_{13} = e_{13}$. Moreover, it is easy to see that the i_1-cycles appear one by one in a path.

Remark 1. Use the above argument, we see that the matrix A in (4) is, in fact, the sum of distinct paths consisting of $e_{i_1j_1}, e_{i_2j_2}, \cdots, e_{i_tj_t}$.

Lemma 2. *If*

$$e_{i_{\pi1}j_{\pi1}} \cdots e_{i_{\pi k_1}j_{\pi k_1}} \cdots \overset{\sigma1}{O}_{i_x} \overset{\sigma2}{O}_{i_x} \cdots \overset{\sigma(u-1)}{O}_{i_x} e_{i_{\pi k_u}j_{\pi k_u}} \cdots$$

$$e_{i_{\pi t}j_{\pi t}} = e_{i_{\tau1}j_{\tau1}} \cdots e_{i_{\tau k'_1}j_{\tau k'_1}} \overset{\rho1}{O'}_{i_x} \overset{\rho2}{O'}_{i_x} \cdots \overset{\rho(u-1)}{O'}_{i_x} e_{i_{\tau k'_u}j_{\tau k_u}} \cdots e_{i_{\tau t}j_{\tau t}}$$

as paths, $e_{i_{\pi1}j_{\pi1}} \cdots e_{i_{\pi k_1}j_{\pi k_1}}$, $e_{i_{\tau1}j_{\tau1}} \cdots e_{i_{\tau k'_1}j_{\tau k'_1}}$ *contains no* i_x*-cycle, and* $\overset{\sigma d}{O}_{i_x}$, $\overset{\rho d}{O}_{i_x}$, $d=1, 2, \cdots, u-1$ *are simple* i_x*-cycles,* π, $\tau \in \mathrm{sym}(t)$, σ, $\rho \in \mathrm{sym}(u-1)$, *then* $k_1=k'_1$, $k_u=k'_u$, $e_{i_{\pi d}j_{\pi d}} = e_{i_{\tau d}j_{\tau d}}$, $1 \leqslant d \leqslant k_1$, $k_u \leqslant d \leqslant t$, *and* $\overset{\sigma d}{O}_{i_x} = \overset{\sigma d}{O'}_{i_x}$ *for* $d=1, 2, \cdots, u-1$.

Proof Compare the matrix units of the two sides of the above equality, the result is obvious.

Lemma 3. *Let*

$$e_{i_1j_1}e_{i_2j_2} \cdots e_{i_tj_t} \tag{9}$$

be a path, and some $i_x \in \{1, 2, \cdots, n\}$ *appears* u *times in*

$$i_1, i_2, \cdots, i_t. \tag{10}$$

Then (9) *exactly contains* $u-1$ *simple* i_x*-cycles if* $j_t \neq i_x$, *and* (9) *exactly contains* u *simple* i_x*-cycles if* $j_t = i_x$.

Proof Since i_x appears u times in (10), we may write (9) precisely as:

$$\underset{\substack{\vdots\\ \text{the first } i_x}}{e_{i_1j_1} \cdots e_{i_{k_1}j_{k_1}}e_{i_xj_{k_1+1}}} \cdots \underset{\substack{\vdots\\ \text{the second } i_x}}{e_{i_{k_2}j_{k_2}}e_{i_xj_{k_2+1}}} \cdots \underset{\substack{\vdots\\ \text{the } u\text{-th } i_x}}{e_{i_{k_u}j_{k_u}} e_{i_xj_{k_u+1}}} \cdots e_{i_tj_t}$$

It is easy to see that between the first and the second times that i_x occurs, there is a simple i_x-cycle, and between the d-th and $(d+1)$-th times that i_x appears, there exists one i_x-cycle, $d=2, \cdots, u-1$. Hence if $j \neq i_x$, there exist exactly $u-1$ simple i_x-cycles in (9), and if $j_t = i_x$ from the u-th time that i_x appears to the terminate of (10) forms another i_x-cycle. So (9) contains exactly u simple i_x-cycles.

Remark 2. In Lemma 3, we may write (9) as

$$e_{i_1j_1} \cdots e_{i_{k_1}j_{k_1}} \overset{1}{O}_{i_x} \overset{2}{O}_{i_x} \cdots \overset{u-1}{O}_{i_x} e_{i_xj_{k_u+1}} \cdots e_{i_tj_t} \text{ if } i_x \neq j_t;$$

$$e_{i_1j_1} \cdots e_{i_{k_1}j_{k_1}} \overset{1}{O}_{i_x} \overset{2}{O}_{i_x} \cdots \overset{u}{O}_{i_x} \text{ if } i_x = j_t.$$

Obviously $e_{i_1j_1} \cdots e_{i_{k_1}j_{k_1}}$ contains no i_x-cycle.

Lemma 4. *If* $t=pn$, *then* (2) *is a polynomial identity of* $M_n(F)$.

Proof As in the proof of the Corollary to Lemma 1, we may show the lemma by taking

$$x_k = e_{i_kj_k}, \ k=1, 2, \cdots, t, \tag{11}$$

in (2) and proving

$$S_{\mathrm{sym}(t)}(e_{i_1j_1}, \cdots, e_{i_tj_t}) \tag{12}$$

vanishes for any $t=pn$ matrix units $e_{i_kj_k}$, $k=1, 2, \cdots, t$.

Now consider (10) $i_1, i_2, \cdots, i_t$.

Case 1. If some i_x appears $u \geqslant p+1$ times in (10), and

$$e_{i_{\pi 1}j_{\pi 1}} e_{i_{\pi 2}j_{\pi 2}} \cdots e_{i_{\pi t}j_{\pi t}},\ \pi \in \mathrm{sym}(t), \tag{13}$$

is a path of (12), under the subsitition (11), (13) corresponds to the term

$$X_{\pi 1} X_{\pi 2} \cdots X_{\pi t} \tag{14}$$

of (2). By Lemma 3, there are $u-1$ simple i_x-cycles contained in (13) if $i_x \neq j_t$, and by Remark 2, we can write (13) as

$$e_{i_{\pi 1}j_{\pi 1}} \cdots e_{i_{\pi k_1}j_{\pi k_1}j_{\pi k_1}} \overset{1}{O}_{i_x} \cdots \overset{u-1}{O}_{i_x} e_{i_{\pi(k_u+1)}j_{\pi(k_u+1)}} \cdots e_{i_{\pi t}j_{\pi t}}$$

where

$$\overset{v}{O}_{i_x} = e_{i_{\pi(k_v+1)}j_{\pi(k_v+1)}} \cdots e_{i_{\pi k_{v+1}}j_{\pi k_{v+1}}}$$

is a simple i_x-cycle. Denote $\overset{v}{O}_\pi = X_{\pi(k_v+1)} X_{\pi(k_v+2)} \cdots X_{\pi k_{v+1}}$. Obviously $\overset{v}{O}_\pi$ corresponds to $\overset{v}{O}_{i_x}$ under the substitution (11), so we have

$$X_1 X_2 \cdots X_t = X_1 X_2 \cdots X_{k_1} \overset{1}{O}_\pi \overset{2}{O}_\pi \cdots \overset{u-1}{O}_\pi X_{\pi(k_u+1)} \cdots X_{\pi t}.$$

Let
$$D_\pi^x = \sum_{\sigma \in \mathrm{sym}(u-1)} X_{\pi 1} X_{\pi 2} \cdots X_{\pi k_1} \overset{\sigma 1}{O}_\pi \overset{\sigma 2}{O}_\pi \cdots \overset{\sigma(u-1)}{O}_\pi X_{\pi(k_u+1)} \cdots X_{\pi t},$$

$$D_\pi = \sum_{\sigma \in \mathrm{sym}(u-1)} e_{i_{\pi 1}j_{\pi 1}} \cdots e_{i_{\pi k_1}n_{\pi k_1}} \overset{\sigma 1}{O}_{i_x} \overset{\sigma 2}{O}_{i_x} \cdots \overset{\sigma(u-1)}{O}_{i_x} e_{i_{\pi(k_u+1)}j_{\pi(k_u+1)}} \cdots e_{i_{\pi t}j_{\pi t}},\ \sigma \in \mathrm{sym}(u-1).$$

Then the partially sum D_π^x of (2) corresponds to the partially sum D_π of (12).

Moreover by the argument preceding Remark 1, we have

$$D_\pi = (u-1)!\, e_{i_{\pi 1}j_{\pi t}}.$$

Now we shall show that, except the zero summands, (12) is a sum of such D_π's.

Let

$$S_{\mathrm{sym}(t)}(x_1, x_2, \cdots, x_t) - D_\pi^x = \Sigma\{\text{summand of (2) which is not summand of } D_\pi^x\}. \tag{15}$$

Let

$$S_{\mathrm{sym}(t)}(e_{i_1j_1}, \cdots, e_{i_tj_t}) - D_\pi \tag{16}$$

denote the partially sum of (12) which corresponds to (15) under the substitution (11).

If

$$e_{i_{\tau 1}j_{\tau 1}} \cdots e_{i_{\tau t}j_{\tau t}} \tag{17}$$

is a summand of (16) which is a path, and

$$X_{\tau 1} X_{\tau 2} \cdots X_{\tau t} \tag{18}$$

is the summand of (15) which corresponds to (17) under (11), (Note (17) may be equal to some summands of D_π as path), according to (17), (18) we may costruct D_τ, D_τ^x as above respectively. Now we show D_τ^x and D_τ are partially sum of (15), (16) respectively. If D_τ^x and D_π^x have a summand in common, say

$$X_{\pi 1}\cdots X_{\pi k_1}\overset{\sigma 1}{O}_{\pi}\overset{\sigma 2}{O}_{\pi}\cdots\overset{\sigma(u-1)}{O}_{\pi}X_{\pi(k_u+1)}\cdots X_{\pi t}$$
$$=X_{\tau 1}\cdots X_{\tau k_1'}\overset{\rho 1}{O}_{\tau}\overset{\rho 2}{O}_{\tau}\cdots\overset{\rho(u-1)}{O}_{\tau}X_{\tau(k_u'+1)}\cdots X_{\tau t},\tag{18'}$$

$\sigma,\ \rho\in \text{sym}(u-1)$, correspondingly we have

$$e_{i_{\pi 1}j_{\pi 1}}\cdots e_{i_{\pi k_1}j_{\pi k_1}}\overset{\sigma 1}{O}_{i_x}\cdots\overset{\sigma(u-1)}{O}_{i_x}e_{i_{\pi(k_u+1)}j_{\pi(k_u+1)}}\cdots e_{i_{\pi t}j_{\pi t}}$$
$$=e_{i_{\tau 1}j_{\tau 1}}\cdots e_{i_{\tau k_1'}j_{\tau k_1'}}\overset{\rho 1}{O'_{i_x}}\overset{\rho 2}{O'_{i_x}}\cdots\overset{\rho(u-1)}{O'_{i_x}}e_{i_{\tau(k_u'+1)}j_{\tau(k_u'+1)}}\cdots e_{i^{\tau(t)}j_{\tau(t)}},$$

where $\overset{\sigma d}{O}_{i_x}$, $\overset{\rho d}{O'_{i_x}}$ are simple i_x-cycles contained in (13), (17) respectively, $d=1,\ 2,\ \cdots,\ u-1$ and $e_{i_{\pi 1}j_{\pi 1}}\cdots e_{i_{\pi k_1}j_{\pi k_1}}$, $e_{i_{\tau 1}j_{\tau 1}}\cdots e_{i_{\tau k_1'}j_{\tau k_1'}}$ contain no i_x-cycle. By Lemma 2 we have $k_1=k_1'$, $k_u=k_u'$ and $e_{i_{\pi d}j_{\pi d}}=e_{i_{\tau d}j_{\tau d}}$ for $1\leqslant d\leqslant k_1$, $k_u+1\leqslant d\leqslant t$, and $\overset{\sigma d}{O}_{i_x}=\overset{\rho d}{O'_{i_x}}$ for $d=1,\ 2,\ \cdots,\ u-1$. So each $\overset{\sigma d}{O}_{i_x}$ has the same length as that of $\overset{\rho d}{O'_{i_x}}$. These force $X_{\pi d}=X_{\tau d}$ for $1\leqslant d\leqslant k_1$, $k_u+1\leqslant d\leqslant t$, and $\overset{\sigma d}{O}_{\pi}=\overset{\rho d}{O}_{\tau}$, for $d=1,\ 2,\ \cdots,\ u-1$. Therefore $D^x_\tau=D^x_\pi$ and (18) is a summand of D^x_π, it contradicts the fact that (18) is a summand of (15). This implies that D^x_π and D^x_τ have no summand in common, hence D^x_τ is a partially sum of (15). Correspondingly, D_τ is a partially sum of (16). Inductively we can construct $S_{\text{sym}(t)}(x_1,\ x_2,\ \cdots,\ x_t)-D^x_\pi-D^x_\tau$ and $S_{\text{sym}(t)}(e_{i_1j_1},\ \cdots,\ e_{i_tj_t})-D_\pi-D_\tau$, if the latter has a summand which is a path. Then using the above procedure, we may construct another partially sum $D_{\tau'}$ of $S_{\text{sym}(t)}(e_{i_1j_1},\ \cdots,\ e_{i_tj_t})-D_\pi-D_\tau$, and construct $S_{\text{sym}(t)}(e_{i_1j_1},\ \cdots,\ e_{i_tj_t})-D_\pi-D_\tau-D_{\tau'}$. So by finite steps we can show that $S_{\text{sym}(t)}(e_{i_1j_1},\ \cdots,\ e_{i_tj_t})-\sum\limits_{\substack{\text{suitable}\\ \pi\in\text{sym}(t)}}D_\pi$ has no summand which is a path, this implies

$$S_{\text{sym}(t)}(e_{i_1j_1},\ \cdots,\ e_{i_tj_t})=\sum_{\substack{\text{suitable}\\ \pi\in\text{sym}(t)}}D_\pi=(u-1)!\sum_{\substack{\text{suitable}\\ \pi\in\text{sym}(t)}}e_{i_{\pi 1}j_{\pi t}}.$$

Since $u-1\geqslant p$, (12) vanishes in this case. (Note if $i_x=j_t$, (13) contains $u\geqslant p+1$ simple i_x-cycles, the proof is the same as above. We omit it here.)

Case 2. If no $i_x\in\{1,\ 2,\ \cdots,\ n\}$ appears more than p times in (10), we claim that $\{i_1,\ i_2,\ \cdots,\ i_t\}=\{1,\ 2,\ \cdots,\ n\}$, and each $i_x\in\{1,\ 2,\ \cdots\ n\}$ appears exactly p times in (10). If $\{i_1,\ i_2,\ \cdots,\ i_t\}=A\subsetneqq\{1,\ 2,\ \cdots,\ n\}$, let A contains $d<n$ elements. Then $t\leqslant pd<pn$, which contradicts $t=pn$, so $A=\{1,\ 2,\ \cdots,\ n\}$. The fact that each $i_x\in\{1,\ 2,\ \cdots,\ n\}$ appears at most p times in (10) forces each $i_x\in\{1,\ 2,\ \cdots\ n\}$ to appear exactly p times in (10), our claim stands.

Now if

$$e_{i_{\pi 1}j_{\pi 1}}\cdots e_{i_{\pi t}j_{\pi t}}\tag{19}$$

is a path of (12), since $j_{\pi t}\in\{1,\ 2,\ \cdots,\ n\}$ appears exactly p times in (10) by our claim, by Lemma 3 and Remark 2 we may write (19) as

$$e_{i_{\pi 1}j_{\pi 1}}\cdots e_{i_{\pi k_1}j_{\pi k_1}}\overset{1}{O}_{j_{\pi t}}\overset{2}{O}_{j_{\pi t}}\cdots\overset{p}{O}_{j_{\pi t}}.\tag{20}$$

According to (20) we, again, construct D_π, and $D_\pi=p!e_{i_{\pi 1}j_{\pi t}}$. In a way analogue to the proof in Case 1, it is easy to show in this case that (12) vanishes. Hence the

Lemma is true.

Lemma 5. *If $t=pn-1$, then (2) is not a polynomial identity of $M_n(F)$.*

Proof Consider $pn-1$ matrix units in $M_n(F)$:

$$\underbrace{e_{11}\cdots e_{11}}_{p-1}e_{12}\underbrace{e_{22}\cdots e_{22}}_{p-1}e_{23}\cdots e_{n-1n}\underbrace{e_{nn}\cdots e_{nn}}_{p-1}$$

and calculate

$$S_{\mathrm{sym}(t)}(\underbrace{e_{11}\cdots, e_{11}}_{p-1}, e_{12}, \underbrace{e_{22}\cdots, e_{22}}_{p-1}, e_{23}, \cdots e_{n-1,n}\underbrace{e_{nn}\cdots, e_{nn}}_{p-1}). \tag{21}$$

Since $e_{12}e_{23}\cdots, e_{n-1n}$ is a "staircase"[3], (21) has a sole path $\underbrace{e_{11}\cdots e_{11}}_{p-1}e_{12}\underbrace{e_{22}\cdots e_{22}}_{p-1}e_{23}\cdots e_{n-1,n}\underbrace{e_{nn}\cdots e_{nn}}_{p-1}$. By Lemma 1 and Remark 1, it is easy to see that (21) is equal to $[(p-1)!]^n e_{1n}\neq 0$ in $M_n(F)$. Hence our lemma stands.

Theorem. *The symmetric polynomial (2) is a polynomial identity of $M_n(F)$ when $t=pn$, where F is a field (finite or infinite) of characteristic $p\neq 0$, this is sharp in the sense that if $t<pn$, (2) is not a polynomial identity of $M_n(F)$.*

Proof Trivially by Lemma 4 and Lemma 5 we can prove the theorem.

Remark 3. Obviously, for $t\geqslant pn$, (2) is a polynomial identity of $M_n(F)$. By Amitsur-Levitzki theorem[3], if $t\geqslant 2n$, (1) is a polynomial identity of $M_n(F)$. So we have the following corollary.

Corollary. *Let*

$$f_E(x_1,\cdots,x_t)=\sum_{\substack{\pi\in\mathrm{sym}(t)\\ \text{is even}}} x_{\pi 1}x_{\pi 2}\cdots x_{\pi t},$$

$$f_O(x_1,\cdots,x_t)=\sum_{\substack{\pi\in\mathrm{sym}(t)\\ \text{is odd}}} x_{\pi 1}x_{\pi 2}\cdots x_{\pi t}.$$

If Char $F=p$ is an odd prime, when $t\geqslant pn$, $f_E(x_1,\cdots,x_t)$ and $f_O(x_1\cdots,x_t)$ are polynomial identity of $M_n(F)$.

Proof Trivially by noting that under the given conditions (1) and (2) are polynomial identities of $M_n(F)$, and $2\neq 0$ in F, we see that (1) added to (2) or (1) minus (2) leads to the result.

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