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STRONG UNIFORM CONSISTENCY FOR DENSITY ESTIMATOR FROM RANDOMLY CENSORED DATA

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Abstract

Let $X_1, \cdots, X_n$ be a sequence of independent identically distributed random variables with distribution function F and density function f. The X_i are censored on the right by Y_i, where the Y_i are i. i. d. r. v. s with distribution function G and also independent of the X_i. One only observes

$$Z_i=\min(X_i, Y_i) \quad \delta_i=I_{(X_i<Y_i)}.$$

Let $S=1-F$ be survival function and $\hat{S}$ be the Kaplan-Meier estimator[3], i. e.,

$$\hat{S}(x)=\begin{cases}\prod\limits_{Z_{(i)}\le x}\left(1-\dfrac{1}{n-i+1}\right)^{\delta_{(i)}}, & x<\max Z_i,\\ 0, & x\geqslant \max Z_i,\end{cases}$$

where $Z_{(i)}$ are the order statistics of Z_i and $\delta_{(i)}$ are the corresponping censoring indicator functions. Define the density estimator of X_i by

$$f_n^*(x)=\frac{\hat{F}(x+h_n/2)-\hat{F}(x-h_n/2)}{h_n},$$

where $\hat{F}=1-\hat{S}$ and $h_n(>0)\downarrow 0$.

In this paper the author uses the strong approximations to get the strong uniform consistency of $f_n^*(x)$ under certain assumptions and also obtains better order, i.e.,

$$\sup_{-\infty<x\le T^*}|f_n^*(x)-f(x)|=O\left(\left(\frac{\log n}{n}\right)^{2/5}\right),$$

where $T^*<T=\inf\{x:H(x)=1\}$ and $H(x)=1-(1-F(x))(1-G(x))$.

§1. Introduction

Suppose that $X_1, X_2, \cdots X_n$ are i. i. d. random variables with density function $f(x)$. The empirical density can be defined as

$$\hat{f}_n(x)=\frac{F_n(x+h_n/2)-F_n(x-h_n/2)}{h_n}, \tag{1}$$

where $F_n(\cdot)$ is the empirical distribution function of X_i and h_n is the step length.

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$h_n>0$, $h_n\downarrow 0$.

The X_i are censored by Y_i, where Y_i, are i. i. d. random variables with distribution function G, and we only observe

$$Z_i=\min(X_i, Y_i);\quad \delta_i=I_{(X_i\le Y_i)}.$$

We can use the Kaplan-Meier[3] estimator instead of the empirical distribution function above, i. e., $\hat{F}(x)=1-\hat{S}(x)$.

$$\hat{S}(x)=\begin{cases}\prod\limits_{Z_{(i)}\le x}\left(1-\dfrac{1}{n-i+1}\right)^{\delta_{(i)}}, & x<\max Z_i,\\ 0, & x\ge\max Z_i,\end{cases}$$

where $Z_{(i)}$ are the order statistics of z_i and $\delta_{(i)}$ are corresponding censoring indicator functions. We define

$$f_n^*(x)=\frac{\hat{F}(x+h_n/2)-\hat{F}(x-h_n/2)}{h_n}\quad (h_n>0,\ h_n\downarrow 0). \tag{2}$$

In this paper we use strong approximations to show that $\hat{f}_n$ and f_n^* are uniformly almost sure consistent with rate $O\left(\left(\frac{\log n}{n}\right)^{2/5}\right)$. We need following lemmas.

Lemma 1.[1] *Suppose that $\{y_i\}$ is a sequence of i. i. d. random variables uniformly distributed on* [0, 1]. *Let $F_n^0(t)=\frac{1}{n}\Sigma I_{y_i<t}$ and $\alpha_n(t)=\sqrt{n}(F_n^0(t)-t)$. If the underlying probability space is rich enough, then there exists a sequence $\{B_n\}$ of Brownian bridges such that*

$$P\{\sup_{0\le t\le 1}|\alpha_n(t)-B_n(t)|>n^{-\frac{1}{2}}(A_1(\log n)+z)\}\le A_2\exp(-A_3 z) \tag{3}$$

for all z, where A_1, A_2, A_3 are absolute constants.

Taking $z(1+\delta)\log n/A_3$ $(\delta>0)$, we have

$$\sup_{0\le t\le 1}|\alpha_n(t)-B_n(t)|=O(n^{-1/2}(\log n))\quad \text{a. s.} \tag{4}$$

since $A_2\exp(-A_3 z)=A_2\exp(-(1+\delta)\log n)=A_2/n^{1+\delta}$ and then use Borel-Cantelli lemma.

Remark. For any sequence of i. i. d. random variables X_i with continuous distribution function F, let $\eta_i=F(X_i)$. Then η_i are uniformly distributed on [0, 1] and Lemma 1 can be still appilied.

Lemma 2.[2] *Let $B(t)$ be a Brownian bridge. If $0<h<1$ and v, $s>0$, then*

$$p\{\sup_{0\le t\le 1-h}\ \sup_{0\le s\le h}|B(t+s)-B(t)|>vh^{1/2}\}\le R(s)h^{-1}e^{-v^2/(2+s)},$$

where $R(s)=8+(32/s^2)$.

Remark. If we take $s=1$, then

$$p\{\sup_{0\le t\le 1-h}\ \sup_{0\le s\le h}|B(t+s)-B(t)|>vh^{1/2}\}\le 40h^{-1}e^{-v^2/3}.$$

Furthermore, supposing that $\{B_i(t)\}$ is a sequence of Brownian bridges and $h=h_n>$

$0,\ 1>h_n\downarrow 0,\ v=v_n>0$, we have

$$p\{\sup_{0\le t\le 1-h_n}\sup_{0\le s\le h_n}|B_n(t+s)-B_n(t)|>v_n h_n^{1/2}\}\le 40h_n^{-1}e^{-v_n^2/3}.$$

If $\sum h_n^{-1}e^{-v_n^2/3}<\infty$, then by Borel-Cantelli lemma,

$$\sup_{0\le t\le 1-h_n}\sup_{0\le s\le h_n}|B_n(t+s)-B_n(t)|\le v_n h_n^{1/2}\quad \text{a. s.}$$

§ 2. Strong Uniform Consistency of $\hat{f}_n(x)$

Let $X_1, \cdots, X_n$ be i. i. d. samples drawn from a population with density f in R^m. Let k be a positive integer and $\alpha>0$ a constant. Denote by $C_{k\alpha}$ the class of $C_{k\alpha}$ densities f in R^m satisfying the following condition:

$$|f^{(k)}(k_1, \cdots k_m;\ x)|\triangleq|\partial^k f/\partial x_1^{k_1}\cdots\partial x_m^{k_m}|\le\alpha$$

for all $x=(x_1, \cdots, x_m)\in R^m$ and $k_1+\cdots+k_m=k$.

Chen Xiru[5] (1983) proved that there exists a kernel estimate $f_n(x)=f_n(X_1, \cdots, X_n;\ x)$ such that

$$\lim_{a\to\infty}\lim_{n\to\infty}\{\inf_{f\in C_{k\alpha}}P_f(\sup_x|f_n(x)-f(x)|\le a(\log n/n)^{k/(2k+m)})\}=1.$$

He also got the following result[5]:

There exists a kernel estimator f_n such that

$$\lim_{n\to\infty}(\log n)^{-k(2k+2m)}n^{k/(2k+m)}\sup_x|f_n(x)-f(x)|=0\quad \text{a. s.}$$

for any $f\in C_{k\alpha}$.

In this section we will prove that, for $m=1$, $k=2$, the empirical density can reach the rate $(\log n/n)^{\frac{2}{5}}$ (Lemma 3). Since the method that we use here is strong approximation to the Brownian bridge, and is quite different from Chen's method, we still give the whole proof.

Let $X_1, X_2, \cdots X_n$ be i. i. d. random variables with distribution function F and denfity function f. Denote by $F_n(\cdot)$ the empirical distribution function of X_i, and

$$a=\sup\{x\colon F(x)=0\},\quad b=\inf\{x\colon F(x)=1\}.$$

Assume $-\infty<a<b<\infty$, and we have the following lemma.

Lemma 3. *If f has second derivative and for $\varepsilon>0$ there is a constant M such that*

$$\sup_{a\le x\le b}f(x)\le M,\qquad \sup_{a+\varepsilon\le x\le b-\varepsilon}|f''(x)|\le M,$$

then for n large enough

$$\sup_{a+\varepsilon\le x\le b-\varepsilon}|\hat{f}_n(x)-f(x)|=O\left(\left(\frac{\log n}{n}\right)^{\frac{2}{5}}\right)\quad \text{a. s.}$$

Proof Observing

$$\sup_{a+h_n/2\leqslant x\leqslant b-h_n/2}\left|\frac{\sqrt{n}\,[(F_n(x+h_n/2)-F(x+h_n/2))-(F_n(x-h_n/2)-F(x-h_n/2))]}{h_n}-\frac{B_n(F(x+h_n/2))-B_n(F(x-h_n/2))}{h_n}\right|$$

$$\leqslant\sup_{a+h_n/2\leqslant x\leqslant b-h_n/2}\left|\frac{\sqrt{n}\,(F_n(x+h_n/2)-F(x+h_n/2))-B_n(F(x+h_n/2))}{h_n}\right|$$

$$+\sup_{a+h_n/2\leqslant x\leqslant b-h_n/2}\left|\frac{\sqrt{n}\,(F_n(x-h_n/2)-F(x-h_n/2))-B_n(F(x-h_n/2))}{h_n}\right|$$

and assuming that $\Sigma h_n^{-1}e^{-v_n^2/3}<\infty$, for n large enough we obtain

$$\sup_{a+\varepsilon\leqslant x\leqslant b-\varepsilon}\left|\sqrt{n}\left[\frac{F_n(x+h_n/2)-F_n(x-h_n/2)}{h_n}-\frac{F(x+h_n/2)-F(x-h_n/2)}{h_n}\right]\right|$$

$$\leqslant\sup_{a+\varepsilon\leqslant x\leqslant b-\varepsilon}\left|\frac{B_n(F(x+h_n/2))-B_n(F(x-h_n/2))}{h_n}\right|+O(n^{-1/2}(\log n)h_n^{-1})\quad\text{a. s.}\tag{5}$$

Furthermore we have

$$\frac{F(x+h_n/2)-F(x-h_n/2)}{h_n}=f(x)+\frac{h_n^2}{48}[f''(x^*)+f''(x^{**})]\tag{6}$$

by Taylor expansion, where $x\leqslant x^*\leqslant x+h_n/2$, $x-\frac{h_n}{2}\leqslant x^{**}\leqslant x$. Hence

$$\sup_{a+\varepsilon\leqslant x\leqslant b-\varepsilon}\left|\hat{f}_n(x)-f(x)-\frac{h_n^2}{48}[f''(x^*)+f''(x^{**})]\right|$$

$$\leqslant\frac{1}{\sqrt{n}}\sup_{a+\varepsilon\leqslant x\leqslant b-\varepsilon}\left|\frac{B_n(F(x+h_n/2))-B_n(F(x-h_n/2))}{h_n}\right|+O\left(\frac{\log n}{nh_n}\right)\quad\text{a. s.,}$$

or

$$\sup_{a+\varepsilon\leqslant x\leqslant b-\varepsilon}|\hat{f}_n(x)-f(x)|\leqslant\frac{1}{\sqrt{n}}\sup_{a+\varepsilon\leqslant x\leqslant b-\varepsilon}\left|\frac{B_n(F(x+h_n/2))-B_n(F(x-h_n/2))}{h_n}\right|$$

$$+O\left(\max\left(\frac{\log n}{nh_n},\ h_n^2\right)\right)\quad\text{a. s.}$$

Notice that

$$\sup_{a+\varepsilon\leqslant x\leqslant b-\varepsilon}\left|\frac{B_n(F(x+h_n/2))-B_n(F(x-h_n/2))}{h_n}\right|$$

$$=\sup_{F(a+\varepsilon-h_n/2)\leqslant t\leqslant F(b-\varepsilon-h_n/2)}\left|\frac{B_n(t+f(u^*)h_n)-B_n(t)}{h_n}\right|,$$

where $u\leqslant u^*\leqslant u+h_n$. For n large enough, it is less than

$$\sup_{0\leqslant t\leqslant 1-Mh_n}\ \sup_{0\leqslant s\leqslant Mh_n}\left|\frac{B_n(t+s)-B_n(t)}{h_n}\right|\leqslant\frac{V_n(Mh_n)^{1/2}}{h_n}\quad\text{a. s.}$$

Therefore

$$\sup_{a+\varepsilon\leqslant x\leqslant b-\varepsilon}|\hat{f}_n(x)-f(x)|=O\left(\max\left(\frac{\log n}{nh_n},\ h_n^2,\ \frac{V_n}{\sqrt{nh_n}}\right)\right).$$

Now we choose $h_n=n^{-1/5}(\log n)^{1/5}$, $V_n=\sqrt{6\log n}$. It is easy to verify

$$\Sigma h_n^{-1}e^{-v_n^2/3}=\Sigma n^{1/5}(\log n)^{-1/5}e^{-2\log n}<\infty.$$

The statement follows from

$$\max\left(\frac{\log n}{nh_n},\ h_n^2,\ \frac{V_n}{\sqrt{nh_n}}\right)=\sqrt{6}\,n^{-2/5}(\log n)^{2/5}.$$

Corollary. *If X_i are i. i. d. random variables uniformly distributed on* (0, 1], *then*

$$\sup_{h_n/2\le x\le 1-h_n/2}|\hat f_n(x)-1|=O\left(\frac{\sqrt{\log n}}{\sqrt{nh_n}}\right)$$

provided $h_n^{-1}\le\frac{n}{\log n}$. *In particular,* $h_n^{-1}=\log\log n$,

$$O\left(\frac{\sqrt{\log n}}{\sqrt{nh_n}}\right)=O\left(\frac{\sqrt{\log n\cdot\log\log n}}{\sqrt{n}}\right).$$

Proof Choosing $V_n=\sqrt{3\log[n(\log n)(\log\log n)^{1+\beta}h_n^{-1}]}$, $\beta>0$, we have $h_n^{-1}e^{-v_n^2/3}=[n(\log n)(\log\log n)^{1+\beta}]^{-1}$ and $\frac{V_n}{\sqrt{nh_n}}=O\left(\frac{\sqrt{\log n}}{\sqrt{nh_n}}\right)$. The statement follows from $f(x)=1$ and $f''(x)=0$.

§ 3. Strong Uniform Consistency of $f_n^*(x)$

Now we let X_1, X_2, $\cdots$ X_n be a sequence of i. i. d. random variables with continuous distribution function $F(x)$. The X_i are censored on the right by Y_i $(1\le_r\le n)$, which is a sequence of i. i. d. random variables with continuous distribution function $G(y)$ and also independent of the X_i sequence. One only observes

$$Z_i=\min(X_i,\ Y_i),\quad \delta_i=I_{(X_i\le Y_i)}.$$

Let $S(x)=1-F(x)$ be survival function and $\hat S(x)$ be the Kaplan-Meier estimator[3], i. e.,

$$\hat S(x)=\begin{cases}\prod\limits_{Z_{(i)}<x}1-\left(\frac{1}{n-i+1}\right)^{\delta_{(i)}}, & x<\max Z_i,\\ 0, & x\ge\max Z_i,\end{cases}$$

where $Z_{(i)}$ are the order statistics of Z_i and $\delta_{(i)}$ are the corresponding censoring indicator functions.

$$\hat F(x)=1-\hat S(x).$$

Define

$$H(x)=P(Z\le x)=1-(1-F(x)(1-G(x)),$$

$$H^1(x)=P(Z\le x,\ \delta=1)=\int_{-\infty}^{x}(1-G(s))dF(s).$$

Let $T_n<T=\inf\{x:\ H(x)=1\}$, such that $1-F(T_n)\ge\left(2(1+\delta)\frac{\log n}{n}\right)^{1/2}(\delta>0)$, $b_n=(1-H(T_n))^{-1}$,

$$U_n(x)=\int_{-\infty}^{x} B_n^0(s)(1-H(s))^{-2}dH^1(s)+B_n^1(x)(1-H(x))^{-1} -\int_{-\infty}^{x} B_n^1(s)(1-H(s))^{-2}dH(s), \tag{7}$$

where $B_n^0(x)$, $B_n^1(x)$ are Gaussian processes with

$$EB_n^0(x)=EB_n^1(x)=0,$$
$$EB_n^0(x)B_n^0(s)=\min(H(x),\ H(s))-H(x)H(s),$$
$$EB_n^1(x)B_n^1(s)=\min(H^1(x),\ H^1(s))-H^1(x)H^1(s),$$
$$EB_n^1(x)B_n^0(s)=\min(H^1(x),\ H^1(s))-H^1(x)H(s).$$

Furthermore[4]

$$\{B_n^0(x),\ -\infty<x<\infty\}\overset{\mathscr{D}}{=}\{B(H(u)),\ -\infty<u<\infty\},$$
$$\{B_n^1(x),\ -\infty<x<\infty\}\overset{\mathscr{D}}{=}\{B(H^1(u)),\ -\infty<u<\infty\}, \tag{8}$$

where $B(\cdot)$ is a Brownian bridge.

Lemma 4[4]. *Under above assumptions*

$$P\{\sup_{-\infty<x\le T_n}|n^{1/2}(S(x)-\hat{S}(x))-S(x)U_n(x)|>r(n)\}\le Qn^{-(1+\delta)}$$

where Q is some constant and

$$r(n)=O(\max(n^{-1/3}b_n^2(\log n)^{3/2},\ n^{-1/2}b_n^4\log n,\ n^{-3/2}b_n^6(\log n)^2)).$$

In particular, $T_n\equiv T^*<T$ is a constant and we can find $\varepsilon>0$ such that $T^*+\varepsilon<T$, for $\delta>0$. We have

$$\sup_{-\infty<x\le T^*+\varepsilon}|n^{1/2}(S(x)-\hat{S}(x))-S(x)U_n(x)|=O(n^{-1/3}(\log n)^{3/2})\quad \text{a. s.}$$

Now by definition (2),

$$f_n^*(x)=\frac{\hat{S}(x-h_n/2)-\hat{S}(x+h_n/2)}{h_n}.$$

We observe

$$\sup_{-\infty<x\le T^*+\varepsilon-h_n/2}\left|\frac{\sqrt{n}\,[(\hat{F}(x+h_n/2)-\hat{F}(x-h_n/2))-(F(x+h_n/2)-F(x-h_n/2))]}{h_n}\right.$$
$$\left.-\frac{[S(x-h_n/2)U_n(x-h_n/2)-S(x+h_n/2)U_n(x+h_n/2)]}{h_n}\right|$$
$$\le\sup_{-\infty<x\le T^*+\varepsilon-h_n/2}\frac{1}{h_n}|\sqrt{n}\,(\hat{S}(x-h_n/2)-S(x-h_n/2))-S(x-h_n/2)U_n(x-h_n/2)|$$
$$+\sup_{-\infty<x\le T^*+\varepsilon-h_n/2}\frac{1}{h_n}|\sqrt{n}\,(\hat{S}(x+h_n/2)-S(x+h_n/2))-S(x+h_n/2)U_n(x+h_n/2)|$$
$$=O(n^{-1/3}(\log n)^{3/2}h_n^{-1}).$$

Hence

$$\sup_{-\infty<x\le T^*+\varepsilon-h_n/2}\left|\sqrt{n}\left[f_n^*(x)-\frac{(F(x+h_n/2)-F(x-h_n/2))}{h_n}\right]\right|$$
$$\le\sup_{-\infty<x\le T^*+\varepsilon-h_n/2}\left|\frac{S(x-h_n/2)U_n(x-h_n/2)-S(x+h_n/2)U_n(x+h_n/2)}{h_n}\right|$$
$$+O(n^{-1/3}(\log n)^{3/2}h_n^{-1})$$

$$\leqslant \sup_{-\infty<x\leqslant T^*+\varepsilon-h_n/2}\left|\frac{U_n(x-h_n/2)-U_n(x+h_n/2)}{h_n}\right|$$

$$+\sup_{-\infty<x\leqslant T^*+\varepsilon}|U_n(x)|\cdot\sup_{-\infty<x\leqslant T^*+\varepsilon-h_n/2}\left|\frac{S(x-h_n/2)-S(x+h_n/2)}{h_n}\right|$$

$$+O(n^{-1/3}(\log n)^{3/2}h_n^{-1}).$$

We denote C_i various constants below.

Lemma 5. *Under the conditions of lemma 4*

$$\sup_{-\infty<x\leqslant T^*+\varepsilon}|U_n(x)|=O(\sqrt{\log n})\quad \text{a. s.}$$

Proof Since

$$|U_n(x)|\leqslant\sup_{s\leqslant x}|B_n^0(s)|\int_{-\infty}^{x}\frac{dH^1(s)}{(1-H(s))^2}+\frac{|B_n^1(x)|}{1-H(x)}+\sup_{s\leqslant x}|B_n^1(s)|\cdot\int_{-\infty}^{x}\frac{dH(s)}{(1-H(s))^2}$$

and

$$\sup_{-\infty<x\leqslant T^*+\varepsilon}|U_n(x)|\leqslant C_1\sup_{-\infty<x\leqslant T^*+\varepsilon}|B_n^0(x)|+C_2\sup_{-\infty<x\leqslant T^*+\varepsilon}|B_n^1(x)|,$$

we can use the inequlity[4]

$$P\{\sup_{-\infty<x<\infty}|B_n^i(x)|>Z\}\leqslant 2\exp(-2Z^2),\ Z>0,\ i=0,\ 1.$$

Let $Z=(\log n)^{1/2}$. Thus

$$P\{\sup_{-\infty<x\leqslant T^*+\varepsilon}|B_n^i(x)|>\sqrt{\log n}\}\leqslant\frac{2}{n^2},$$

by Borel–Cantelli Lemma, the desired conclusion follows.

Lemma 6. *Suppose that F and G have density functions f and g respectively such that* $\sup_{-\infty<x\leqslant T^*+\varepsilon/2}f(x)\leqslant M$, $\sup_{-\infty<x\leqslant T^*+\varepsilon/2}g(x)\leqslant M$ *for some constant M. Then*

$$\sup_{-\infty<x\leqslant T^*+\varepsilon-h_n/2}\left|\frac{U_n(x+h_n/2)-U_n(x-h_n/2)}{h_n}\right|=O(\max(\sqrt{\log n},\ h_n^{-1/2}V_n))\quad \text{a. s.}$$

provided $\sum h_n^{-1}e^{-v_n^2/2}<\infty$.

Proof By the definition of U_n,

$$|U_n(x+h_n/2)-U_n(x-h_n/2)|$$

$$\leqslant\sup_{x-h_n/2\leqslant s\leqslant x+h_n/2}|B_n^0(s)|\int_{x-h_n/2}^{x+h_n/2}\frac{dH^1(s)}{(1-H(s))^2}+\left|\frac{B_n^1(x+h_n/2)-B_n^1(x-h_n/2)}{1-H(x+h_n/2)}\right|$$

$$+|B_n^1(x-h_n/2)|\cdot\left|\frac{1}{1-H(x-h_n/2)}-\frac{1}{1-H(x+h_n/2)}\right|$$

$$+\sup_{x-h_n/2\leqslant s\leqslant x+h_n/2}|B_n^1(s)|\int_{x-h_n/2}^{x+h_n/2}(1-H(s))^{-2}dH(s).$$

Since

$$\begin{aligned}&H^1(x+h_n/2)-H^1(x-h_n/2)\\&\leqslant H(x+h_n/2)-H(x-h_n/2)\\&=(1-F(x-h_n/2))(G(x+h_n/2)-G(x-h_n/2))\\&\quad+(1-G(x+h_n/2))(F(x+h_n/2)-F(x-h_n/2))\\&\leqslant 2Mh_n,\end{aligned}$$

it is clear that

$$\sup_{-\infty<x\leq T^*+\varepsilon-h_n/2} |U_n(x+h_n/2)-U_n(x-h_n/2)|$$
$$\leq C_3 \sup_{-\infty<x\leq T^*+\varepsilon} |B_n^0(x)| \cdot |H^1(x+h_n/2)-H^1(x-h_n/2)|$$
$$+C_4 \sup_{-\infty<x\leq T^*+\varepsilon-h_n/2} |B_n^1(x+h_n/2)-B_n^1(x-h_n/2)|$$
$$+C_5 \sup_{-\infty<x\leq T^*+\varepsilon} |B_n^1(x)| \cdot \sup_{-\infty<x\leq T^*+\varepsilon-h_n/2} |H(x+h_n/2)-H(x-h_n/2)|$$
$$\leq 2MC_3h_n \sup_{-\infty<x\leq T^*+\varepsilon} |B_n^0(x)| + C_4 \sup_{-\infty<x\leq T^*+\varepsilon-h_n/2} |B_n^1(x+h_n/2)-B_n^1(x-h_n/2)|$$
$$+2MC_5h_n \sup_{-\infty<x\leq T^*+\varepsilon} |B_n^1(x)|$$
$$=C_4 \sup_{-\infty<x\leq T^*+\varepsilon} |B_n^1(x+h_n/2)-B_n^1(x-h_n/2)| + h_nO(\sqrt{\log n}) \quad \text{a. s.}$$

by the proof of Lemma 5. Therefore it is enough to show

$$\sup_{-\infty<x\leq T^*+\varepsilon-h_n/2} |B_n^1(x+h_n/2)-B_n^1(x-h_n/2)| = O(h_n^{1/2}V_n).$$

It is clear by (8), for n large enough, that

$$P(\sup_{-\infty<x\leq T^*+\varepsilon-h_n/2} |B_n^1(x+h_n/2)-B_n^1(x-h_n/2)| > \sqrt{2M}\, h_n^{1/2}V_n)$$
$$\leq P(\sup_{0\leq t'\leq 1-2Mh_n} \sup_{0\leq s'\leq 2Mh_n} |B(t'+s')-B(s')| > \sqrt{2M}\, h_n^{1/2}V_n)$$
$$=P(\sup_{0\leq t'\leq 1-h_n'} \sup_{0\leq s'\leq h_n'} |B(t'+s')-B(s')| > (h_n')^{1/2}V_n).$$

So

$$\sup_{-\infty<t\leq T^*+\varepsilon-h_n/2} \left| \frac{B_b^1(t+h_n/2)-B_n^1(t-h_n/2)}{h_n} \right| \leq \sqrt{2M}\, h_n^{-1/2}V_n \quad \text{a. s.}$$

Since $\Sigma h_n^{-1}e^{-v_n^2/3}<\infty$, it completes the proof.

Combining Lemma 5, Lemma 6 and (9), we obtain

$$\sup_{-\infty<x\leq T^*+\varepsilon-h_n/2} \left| f_n^*(x) - \frac{F(x+h_n/2)-F(x-h_n/2)}{h_n} \right|$$
$$=O\left(\max\left(\frac{\sqrt{\log n}}{\sqrt{n}}, \frac{V_n}{\sqrt{nh_n}}, \frac{1}{\sqrt{n}} n^{-1/3}(\log n)^{3/2}h_n^{-1}\right)\right). \tag{10}$$

Theorem. *Suppose that F and G have density functions f and g respectively such that* $\sup_{-\infty<x\leq T^*+\varepsilon} f(x)\leq M$, $\sup_{-\infty<x\leq T^*+\varepsilon} g(x)\leq M$ *for some constant M. Furthermore assume that $f''(x)$ exists and* $\sup_{-\infty<x\leq T^*+\varepsilon} |f''(x)|\leq M_1$ *for some constant M_1. Then*

$$\sup_{-\infty<x\leq T^*} |f_n^*(x)-f(x)| = O\left(\left(\frac{\log n}{n}\right)^{2/5}\right) \quad \text{a. s.}$$

Proof Assuming $\Sigma h_n^{-1}e^{-v_n^2/3}<\infty$, by (6) and

$$\sup_{-\infty<x\leq T^*+\varepsilon-h_n/2} \left| f_n^*(x) - \frac{F(x+h_n/2)-F(x-h_n/2)}{h_n} \right|$$
$$\leq \sup_{-\infty<x\leq T^*+\varepsilon-h_n/2} |f_n^*(x)-f(x)| + O(h_n^2) \quad \text{a. s.}$$

we have

$$\sup_{-\infty<x\leq T^*} |f_n^*(x)-f(x)| = O\left(\max\left(\sqrt{\frac{\log n}{n}}, \frac{V_n}{\sqrt{nh_n}}, \frac{n^{-1/3}(\log n)^{3/2}}{\sqrt{n}\,h_n}, h_n^2\right)\right)$$

for large n. Let $h_n = n^{-1/5}(\log n)^{1/5}$, $V_n = \sqrt{6\log n}$. Then $\Sigma h_n^{-1} e^{-v_n^2/3} < \infty$ and

$$\max\left(\sqrt{\frac{\log n}{n}}, \frac{U_n}{\sqrt{nh_n}}, \frac{n^{-1/3}(\log n)^{3/2}}{\sqrt{n}\, h_n}, h_n^2\right) = \sqrt{6}\, n^{-2/5}(\log n)^{2/5}.$$

This proves the theorem.

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