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# MULTIPLIERS OF SEGAL ALGEBRAS $A_p(G)$

OUYANG GUANGHONG (欧阳光中)*

**Abstract**

Let $G$ be a locally compact but non-compact abelian group. It is proved that $M(A_p(G), L_1(G))=M(G)$ and $M(A_p(G), L_1(G)\cap C_0(G))=M(L_1(G), L_1(G)\cap C_0(G))$. If $G$ is discrete, then $M(A_p(G), L_1(G))=A_p(G)$, $M(A_p(G), L_1(G)\cap C_0(G))=A_p(G)$.

Let $G$ be a locally compact but non-compact abelian group with Haar measure $\lambda$, and $\hat{G}$ the character group of $G$. The space of all complex-valued integrable functions with respect to $\lambda$ is denoted by $L_1(G)$, which is a Banach algebra, where the multiplication is convolution. Let

$$A_p(G)=\{f\in L_1(G): \hat{f}\in L_p(\hat{G})\} \quad 1\leqslant p<\infty.$$

The norm of an element in space $A_p(G)$ is defined by

$$\|f\|_{A_p}=\|f\|_1+\|\hat{f}\|_p, \quad \forall f\in A_p(G).$$

Then $A_p(G)$ is a Segal algebra.

Suppose that $(S_1(G), \|\ \|_{S_1})$ and $(S_2(G), \|\ \|_{S_2})$ are two Segal algebras and $T$ is a bounded linear operator from $S_1(G)$ to $S_2(G)$, if $T$ commutes with every translation operator $\tau_x(x\in G)$, that is $T\,\tau_x=\tau_x T$, then $T$ is called a mutiplier from $S_1(G)$ to $S_2(G)$. We denote the collection of all multipliers by $M(S_1(G), S_2(G))$, which is a Banach algebra.

Figa-Talamanca and Gaudry[1] proved that the following are equivalent:

(1) $T\in M(A_p(G), A_p(G))$.

(2) There exists a unique measure $\mu\in M(G)$ such that

$$Tf=\mu*f,\ \forall f\in A_p(G),$$

where $M(G)$ is the Banach algebra of all bounded regular complex valued Borel measures on $G$.

Moreover, the correspondence between $T$ and $\mu$ defines an isometric algebra isomorphism from $M(A_p(G), A_p(G))$ onto $M(G)$.

Ouyang[3] has studied the character of $M(L_1(G), A_p(G))$. Let

$$M_{A_p}(G)=\{\mu\in M(G): \hat{\mu}\in L_p(\hat{G})\},$$

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* Department of Mathematics, Fudan University, Shanghai, China

$$\|\mu\|_{M_{A_p}}=\overline{\lim_n}\|\alpha_n*\mu\|_{A_p},$$

where $\{\alpha_n\}$ is a bounded approximate identity of $L_1(G)$ such that $\|\alpha_n\|_1=1$ and $\hat{\alpha}_n$ has the compact support for every $n$. Then the following are equivalent:

(1) $T\in M(L_1(G),\ A_p(G))$.

(2) There exists a unique measure $\mu\in M_{A_p}(G)$ such that

$$Tf=\mu*f,\ \forall f\in L_1(G).$$

Moreover, the correspondence between $T$ and $\mu$ defines an isometric algebra isomorphism from $M(L_1(G),\ A_p(G))$ onto $M_{A_p}(G)$.

Now, in this paper, we will consider $M(A_p(G),\ L_1(G))$ and $M(A_p(G),\ L_1(G)\cap C_0(G))$.

Let $C_c(\hat{G})$ be the space of all continuous complex valued functions with compact support in $\hat{G}$ and let

$$P(L_1(G))=\{f\in L_1(G):\hat{f}\in C_c(\hat{G})\}.$$

We have known that $P(L_1(G))\subset S(G)$, $S(G)$ is any segal algebra on $\hat{G}$. The space $P(L_1(G))$ is dense in $S(G)$.[4]

Let $f$ be a function belonging to $P(L_1(G))$, $\tau_x$ $(x\in G)$ a translation operator. For $x_0,\ x_1,\ \cdots,\ x_n,\ \{x_n\}\subset G$, $x_0=e$ the identity of group $G$, we define

$$f_n=\frac{\tau_{x_0}f+\tau_{x_1}f+\cdots+\tau_{x_n}f}{n+1}. \tag{1}$$

From

$$\hat{f}(\gamma)=\frac{1+(x_1,\ \gamma)+\cdots+(x_n,\ \gamma)}{n+1}\hat{f}(\gamma)$$

and $\hat{f}(\gamma)\in C_c(\hat{G})$, we have $\hat{f}_n\in C_c(\hat{G})$ and $f_n\in P(L_1(G))$.

For every $\varepsilon>0$, there exists $g\in C_c(G)$ ($C_c(G)$ is the space of all continuous complex valued functions on $G$ which have compact support) such that

$$\|f-g\|_1<\varepsilon.$$

Let

$$g_n=\frac{\tau_{x_0}g+\tau_{x_1}g+\cdots+\tau_{x_n}g}{n+1}. \tag{2}$$

By the homogeneous structures of $L_1(G)$, it is easy to see that

$$\begin{aligned}\|f_n-g_n\|_1&\leqslant\frac{1}{n+1}(\|f-g\|_1+\|\tau_{x_1}(f-g)\|_1+\cdots+\|\tau_{x_n}(f-g)\|_1)\\&\leqslant\frac{1}{n+1}(n+1)\|f-g\|_1<\varepsilon.\end{aligned} \tag{3}$$

We first establish the following lemma.

**Lemma.** *There exists a sequence $x_0,\ x_1,\ \cdots,\ x_n,\cdots$ in $G$ such that*

(a) $\|g_n\|_2\to 0\ (n\to\infty)$, (4)

(b) $\|f_n\|_2\to 0\ (n\to\infty)$, (5)

(c) $\|\hat{f}_n\|_p\to 0\ (n\to\infty)$, $1\leqslant p<\infty$, (6) *where* $\{f_n\}$ *and* $\{g_n\}$ *are defined by* (1) *and* (2) *respectively.*

*Proof* Suppose supp $g=K$, a compact set in $G$. There exists a sequence $x_0$, $x_1, \cdots, x_n, \cdots$ in $G$ such that

$$(Kx_i)\cap(Kx_j)=\varnothing,\quad i\neq j.$$

Since $G$ is a non-compact locally compact abelian group.

Furthermore we have

$$\operatorname{supp}\tau_{x_i}g\cap\operatorname{supp}\tau_{x_j}g=\varnothing,\quad i\neq j.$$

We now prove the proposition (a):

$$\|g_n\|_2^2=\frac{1}{(n+1)^2}\int_G(\tau_{x_0}g(x)+\tau_{x_1}g(x)+\cdots+\tau_{x_n}g(x))^2d\lambda(x)$$
$$=\frac{1}{(n+1)^2}\int_G\left\{\sum_{i=0}^n[\tau_{x_i}g(x)]^2+2\sum_{0\le i<j\le n}\tau_{x_i}g(x)\cdot\tau_{x_j}g(x)\right\}d\lambda(x),$$

but

$$\int_G[\tau_{x_i}g(x)]^2d\lambda(x)=\|g\|_2^2,$$
$$\int_G\tau_{x_i}g(x)\cdot\tau_{x_j}g(x)d\lambda(x)=0,\quad i\neq j.$$

Therefore

$$\|g_n\|_2^2=\frac{1}{(n+1)^2}(n+1)\|g\|_2^2\to0\quad(n\to\infty).$$

In order to prove the proposition (b), we write

$$\|f_n\|_2^2=\frac{1}{(n+1)^2}\int_G[\tau_{x_0}f(x)+\tau_{x_1}f(x)+\cdots+\tau_{x_n}f(x)]^2d\lambda(x)$$
$$=\frac{1}{(n+1)^2}\int_G\left[\sum_{i=0}^n\tau_{x_i}g(x)+\sum_{i=0}^n\tau_{x_i}(f-g)(x)\right]^2d\lambda(x)$$
$$=\frac{1}{(n+1)^2}\int_G\left\{\left[\sum_{i=0}^n\tau_{x_i}g(x)\right]^2+\left[\sum_{i=0}^n\tau_{x_i}(f-g)(x)\right]^2\right.$$
$$\left.+\sum_{i,j=0}^n\tau_{x_i}g(x)\cdot\tau_{x_j}(f-g)(x)\right\}d\lambda(x)$$
$$=\|g_n\|_2^2+\|f_n-g_n\|_2^2+\frac{1}{(n+1)^2}\sum_{i,j=0}^n\int_G\tau_{x_i}g(x)\cdot\tau_{x_j}(f-g)(x)d\lambda(x)$$
$$=I_n^{(1)}+I_n^{(2)}+I_n^{(3)}.$$

$I_n^{(1)}\to0$ $(n\to\infty)$ since (4). Estimate $I_n^{(2)}$ as follows: $f-g$ is a bounded function on $G$ because $g\in C_c(G)$ and $f\in C_0(G)$ ($C_0(G)$ denotes the space of all continuous complex valued functions on $G$ vanishing at infinity). Suppose $|f(x)-g(x)|\le B$, $\forall x\in G$, where $B$ is a constant. Hence

$$\|f_n-g_n\|_2^2\le B\int_G|f_n(x)-g_n(x)|dx\to0\quad(n\to\infty)$$

since $\|f_n-g_n\|_1\to0$ $(n\to\infty)$.

Last, we estimate $I_n^{(3)}$. There exists a constant $A$ such that $|g(x)|\le A$, $\forall x\in G$ since $g\in C_c(G)$. Hence $|\tau_{x_i}g(x)|\le A$, $\forall x\in G$, $i=0, 1, 2, \cdots$.

$$\left|\int_G \tau_{x_i} g(x)\cdot\tau_{x_j}(f-g)(x)d\lambda(x)\right| \leqslant A\int_G |\tau_{x_j}(f-g)(x)|d\lambda(x)$$
$$\leqslant A\|f-g\|_1 < A\varepsilon.$$

Thus $I_n^{(3)}\to 0(n\to\infty)$ and (b) is true.

As for (c), $f_n\in P(L_1(G))$ and $\|f_n\|_2\to 0$ $(n\to\infty)$ implies that $\|\hat{f}_n\|_2\to 0$ $(n\to\infty)$, then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$, $\hat{f}_{n_k}(\gamma)\to 0$ $(k\to\infty)$ almost everywhere on $\hat{G}$. But $\hat{f}_{n_k}$ are continuous functions, it shows that $\hat{f}_{n_k}(\gamma)\to 0$ $(k\to\infty)$ at every point on $\hat{G}$.

By the facts

$$\hat{f}_{n_k}(\gamma)=\frac{1+(x_1, \gamma)+\cdots+(x_n, \gamma)}{n+1}\hat{f}(\gamma),\ \forall\gamma\in\hat{G},$$
$$|(x_i, \gamma)|\leqslant 1,\ \forall\gamma\in\hat{G},\ i=1, 2, \cdots, n.$$

Then, clearly,

$$|\hat{f}_{n_k}(\gamma)|\leqslant|\hat{f}(\gamma)|,\ \forall\gamma\in\hat{G}.$$

But $\hat{f}_{n_k}\in C_c(\hat{G})$ and $\hat{f}\in C_c(\hat{G})$. Lebesgue dominated covergence theorem gives

$$\|\hat{f}_{n_k}\|_p\to 0\ (k\to\infty).$$

If we choose the sequence $x_0, x_1, \cdots, x_n, \cdots$ as $x_0, x_{n_1}, x_{n_2}, \cdots, x_{n_k}, \cdots$, the lemma is proved.

**Theorem 1.** *The following are equivalent:*

(1) $T\in M(A_p(G), L_1(G))$.

(2) *There exists a unique measure* $\mu\in M(G)$ *such that*

$$Tf=\mu * f,\quad \forall f\in A_p(G).$$

*Moreover, the correspondence between* $T$ *and* $\mu$ *defines an isometric algebra isomorphism from* $M(A_p(G), L_1(G))$ *onto* $M(G)$.

*Proof* We first prove that (1) implies (2). Suppose $T\in M(A_p(G), L_1(G))$. For every $f\in P(L_1(G))$, $Tf\in L_1(G)$. Then for every $\varepsilon>0$, there exists $s\in G$, such that

$$\|Tf+\tau_s(Tf)\|_1\geqslant 2\|f\|_1-\varepsilon. \tag{7}$$

It shows that there exists a sequence $x_0, x_1, \cdots, x_n, \cdots, \{x_n\}\subset G$, $x_0=e$, such that

$$\|\tau_{x_0}(Tf)+\tau_{x_1}(Tf)+\cdots+\tau_{x_n}(Tf)\|_1\geqslant(n+1)\|Tf\|_1-n\varepsilon. \tag{8}$$

Futhermore, we can require that the sequence $\{x_n\}$ satisfy the requirement of the lemma at all.

Recall that $T$ is a multiplier and the definition of $f_n$, then

$$Tf_n=T\left(\frac{\tau_{x_0}f+\tau_{x_1}f+\cdots+\tau_n f}{n+1}\right)=\frac{\tau_{x_0}(Tf)+\tau_{x_1}(Tf)+\cdots+\tau_{x_n}(Tf)}{n+1}.$$

Substituting it into (8) and noticing that $T$ is a bounded linear operator from $A_p(G)$ to $L_1(G)$, we have

$$\|Tf\|_1\leqslant\|Tf_n\|_1+\varepsilon\leqslant\|T\|\,\|f_n\|_{A_p}+\varepsilon\leqslant\|T\|\,(\|f_n\|_1+\|\hat{f}_n\|_p)+\varepsilon. \tag{9}$$

By the lemma $\|\hat{f}_n\|_p\to 0(n\to\infty)$, $\varepsilon>0$ is an arbitrary number and $L_1(G)$ is a homogeneous Banach algebra, $\|f_n\|_1=\|f\|_1$, then

$$\|Tf\|_1 \leqslant \|T\| \|f\|_1, \quad \forall f \in P(L_1(G)).$$

This shows that $T$ defines a bounded linear operator from $P(L_1(G))$ which possesses $L_1(G)$ — norm to $L_1(G)$ and $T$ commutes with translation. Since $P(L_1(G))$ is dense in $L_1(G)$, $T$ determines a unique bounded linear operator $T$ (we also use the same notation $T$) from $L_1(G)$ to $L_1(G)$, whose norm remains the same. Moreover, it is easy to see that $T$ is a multiplier from $L_1(G)$ to $L_1(G)$.

According to a wellknown theorem on the multiplier of $L_1(G)$, there exists a unique measure $\mu \in M(G)$ such that

$$Tf = \mu * f, \quad \forall f \in L_1(G),$$

and this correspondence between $T$ and $\mu$ defines an isometric algebra isomorphism from $M(L_1(G), L_1(G))$ onto $M(G)$.

Conversely, suppose that $\mu \in M(G)$. We define an operator $T$:

$$Tf = \mu * f, \quad \forall f \in A_p(G).$$

Obviously, $\mu * f \in L_1(G)$ since $L_1(G)$ is an idea in $M(G)$. By the properties of convolution, it is easy to prove that $T$ is a bounded linear operator from $A_p(G)$ to $L_1(G)$ and commutes with any translation operator, that is $T \in M(A_p(G), L_1(G))$.

We now consider the character of $M(A_p(G), L_1(G) \cap C_0(G))$, where $L_1(G) \cap C_0(G)$ is a Segal algebra with the norm

$$\|f\|_{L_1 \cap C_0} = \|f\|_1 + \|f\|_\infty, \quad \forall f \in L_1(G) \cap C_0(G).$$

**Theorem 2.** *$M(A_p(G), L_1(G) \cap C_0(G)) = M(L_1(G), L_1(G) \cap C_0(G))$, where "$=$" means isometric and algebra isomorphism.*

*Proof* Suppose $T \in M(A_p, L_1(G) \cap C_0(G))$. For every $f \in C_0(G)$ and each $\varepsilon > 0$, there exists $s \in G$ such that

$$\|f + \tau_s f\|_\infty \geqslant 2\|f\|_\infty - \varepsilon,$$

so that there exists a sequence $x_0, x_1, \cdots, x_n, \cdots, \{x_n\} \subset G$, $x_0 = e$, such that

$$\|\tau_{x_0} f + \tau_{x_1} f + \cdots + \tau_{x_n} f\|_\infty \geqslant (n+1)\|f\|_\infty - n\varepsilon. \tag{10}$$

Let $f$ be an arbitrary function in $P(L_1(G))$. We have $Tf \in L_1(G) \cap C_0(G)$. Recall that $T$ is a multiplier and $f_n$ is defined by (1). Then

$$\begin{aligned}\|Tf_n\|_{L_1 \cap C_0} &= \|Tf_n\|_1 + \|Tf_n\|_\infty \\ &= \|Tf\|_1 + \frac{1}{n+1}\|\tau_{x_0}(Tf) + \tau_{x_1}(Tf) + \cdots + \tau_{x_n}(Tf)\|_\infty \\ &\geqslant \|Tf\|_1 + \|Tf\|_\infty - \varepsilon = \|Tf\|_{L_1 \cap C_0} - \varepsilon\end{aligned}$$

by (10). Hence we get the following inequality

$$\begin{aligned}\|Tf\|_{L_1 \cap C_0} &\leqslant \|Tf_n\|_{L_1 \cap C_0} + \varepsilon \leqslant \|T\| \|f_n\|_{A_p} + \varepsilon \\ &= \|T\|(\|f_n\|_1 + \|\hat{f}_n\|_p) + \varepsilon.\end{aligned}$$

But $\|f_n\|_1 = \|f\|$, $\|\hat{f}_n\|_p \to 0 \ (n \to \infty)$, $\varepsilon > 0$ is an arbitrary number, then

$$\|Tf\|_{L_1 \cap C_0} \leqslant \|T\| \|f\|_1, \quad \forall f \in P(L_1(G)).$$

$P(L_1(G))$ is dense in $L_1(G)$ that $T$ determines a unique multiplier from

$L_1(G)$ to $L_1(G)\cap C_0(G)$ and remains the same norm.

Conversely, suppose $T\in M(L_1(G), L_1(G)\cap C_0(G))$. It is easy to see that $T$ is a multiplier from $A_p(G)$ to $L_1(G)\cap C_0(G)$ if $T$ is restricted on $A_p(G)$.

Theorems 3 and 4 will consider the case when $G$ is discrete.

**Theorem 3.** *Suppose $G$ is a discrete group. Then the following are equivalent:*

(1) $T\in M(A_p(G), L_1(G))$.

(2) *There exists a unique function $g_T\in A_p(G)$ such that*

$$Tf=g_T*f,\ \forall f\in A_p(G).$$

*Moreover, the correspondence between $T$ and $g_T$ defines an algebra isomorphism from $M(A_p(G), L_1(G))$ onto $A_p(G)$, and the norms are equivalent.*

*Proof* We have $A_p(G)=L_1(G)=M(G)$ if $G$ is discrete, where '=' means the eqnality of the two sets in both sides and the preservation of algebra operations. Moreover, $L_1(G)$-norm is the same one as $M(G)$-norm if $G$ is discrete.

Take $g_T=T\delta$, where $\delta$ is an identity of $A_p(G)=L_1(G)$. Obviously

$$Tf=T(\delta*f)=T\delta*f,\quad \forall f\in A_p(G),$$

and $\|T\|=\|T\delta\|_M=\|T\delta\|_1$ by Theorem 1. It remains to prove that $L_1(G)$-norm and $A_p(G)$-norm are equivalent.

Since $\|u*v\|_1\leqslant\|u\|_{A_p}\|v\|_1$ for every $u\in A_p(G)$, $v\in L_1(G)$, $L_1(G)$ is a Banach $A_p(G)$-module. According to Module factorization theorem[2], for every $f\in L_1(G)$ and each $\varepsilon>0$, there exist $g\in A_p(G)$, $h\in L_1(G)$ such that

$$f=g*h,\ \|g\|_{A_p}\leqslant K,\ \|f-g\|_1<\varepsilon, \tag{11}$$

where $K$ is a constant number.

On the other side, $\|g*h\|_{A_p}\leqslant\|g\|_{A_p}\|h\|_1$. Substituting it into (10), we have

$$\|f\|_{A_p}=\|g*h\|_{A_p}\leqslant K(\|f\|_1+\varepsilon).$$

It is obtaind that

$$\|f\|_1\leqslant\|f\|_{A_p}\leqslant K\|f\|_1,\quad \forall f\in L_1(G),$$

since $\varepsilon>0$ is arbitrary. Then, $L_1(G)$-norm and $A_p(G)$-norm are equivalent when $G$ is discrete.

**Theorem 4.** *Suppose $G$ is a discrete group. Then the following are equivalent.*

(1) $T\in M(A_p(G), L_1(G)\cap C_0(G))$.

(2) *There exists a unique function $g_T\in A_p(G)$ such that*

$$Tf=g_T*f,\quad \forall f\in A_p(G).$$

*Proof* It is similar to Theorem 3. We also have

$$A_p(G)=L_1(G)=M(G)=L_1(G)\cap C_0(G),$$

and $L_1(G)\cap C_0(G)$-norm is equivalent to $L_1(G)$-norm when $G$ is discrete.

## References

[1] Larsen, R., An Introduction to the theory of Multipliers, Springer-Verlag (1971).
[2] Hewitt, E. & Ross, K. A., Abstract harmonic analysis, Springer-Verleg, II (1970).
[3] Ouyang, G. Z., Multiplier operators from $L_1(G)$ to a Segal algebra, *Chinese Annals of Math.*, **5A**: 2(1984), 247—252.
[4] Wang, H. C., Homogeneous Banach algebras, Marcell Dekker Inc (1977).