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L^2—HYPOELLIPTICITY FOR A CLASS OF OPERATORS OF MIXED TYPE

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Abstract

The present paper gives the notion of L^2–hypoellipticity for differential operator and proves that a class of differential operators of mixed type are of L^2–hypoellipticity on their degenerated curves.

Consider a second order operator

$$Lu = x_1 D^2_{x_1} u + x_2 D^2_{x_2} u + a D_{x_1} u + b D_{x_2} u + cu, \tag{1}$$

where a, b, c are smooth functions. Evidently, (1) is of mixed type in any neighbourhood of $x=0$, and its Hamiltonian vector fields is

$$H_L = 2x_1\xi_1 \frac{\partial}{\partial x_1} + 2x_2\xi_2 \frac{\partial}{\partial x_2} - \xi_1^2 \frac{\partial}{\partial \xi_1} - \xi_2^2 \frac{\partial}{\partial \xi_2}. \tag{2}$$

It is easily seen that the set of radial points, Σ, consists of three parts $\Sigma_1 \cup \Sigma_2 \cup \Sigma_0$, where

$$\begin{aligned} \Sigma_1 &= \{(0,\ x_2,\ \xi_1,\ 0) \mid \xi_1 \neq 0,\ x_2 \in R^1\}, \\ \Sigma_2 &= \{(x_1,\ 0,\ 0,\ \xi_2) \mid \xi_2 \neq 0,\ x_1 \in R^1\}, \\ \Sigma_0 &= \{(0,\ 0,\ \xi_1,\ \xi_2) \mid \xi_1 = \xi_2 \neq 0\}. \end{aligned} \tag{3}$$

We say that the operator $P(x, D_x)$ is of L^2–hypoellipticity at point x_0 if for any given neighbourhood of x_0, $O(x_0)$, there exists another neighborhood of x_0 $O'(x_0) \subset O(x_0)$ such that $u \in C^\infty(O'(x_0))$ when $u \in L^2(O(x_0))$ and $Pu \in C^\infty(O(x_0))$. L^2–hypoellipticity for (1) in $R^2 \setminus \{0\}$ has been discussed in [4]. If $n_1 \in \Sigma_1$ ($n_2 \in \Sigma_2$) we have proved in [4] the following lemma.

Lemma 1. *Let* $\operatorname{Re} \sqrt{-1}\, a(x_1, 0) > 3/2$ ($\operatorname{Re} \sqrt{-1}\, b(0, x_2) > 3/2$) *and let* $u \in L^2_{\text{loc}}(R^2)$, $n_1(n_2) \overline{\in} WF(Lu)$. *Then* $n_1(n_2) \overline{\in} WF(u)$.

Next we shall study the hypoellipticity at $(0, 0, 1, 1) = n_0$.

Lemma 2. *There exist conical neighbourhoods of* n_0, $n_0' = (0, 0, 1, 0)$, $\mathcal{N}(n_0) \subset T^*(R^2_x)$ *and* $\mathcal{N}(n_0') \subset T^*(R^2_y)$, *and an elliptic Fourier operator* F *defined on* $\mathcal{N}(n_0) \times \mathcal{N}(n_0') \subset T^*(R^2_x) \times T^*(R^2_y)$, *and an elliptic pseudodifferential operator* $A_{-1} \in OPS^{-1}(\mathcal{N}(n_0))$, *such that*

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$$FA_{-1}LF^{-1}=y_1D_{y_1}+2y_2D_{y_2}+Q(y, Dy),\ \text{mod } OPS^{-\infty}(\mathscr{N}(n_0')). \tag{4}$$

Here the principal symbol of Q satisfies

$$\operatorname{Im} q_0(0, 0, 1, 0)=\operatorname{Im}(a(0, 0)+b(0, 0)+\sqrt{-1}). \tag{5}$$

Proof Find a generating function $\varphi=S(y, \zeta)\xi_1$, $\zeta=\xi_2/\xi_1$ such that S satisfies the following partial differential equation

$$(S-S_\zeta\zeta)+S_\zeta\zeta^2=y_1S_{y_1}+2y_2S_{y_2}. \tag{6}$$

Obviously, $S=y_1+y_2(1-1/\zeta)$ is a solution o, (6). Therefore

$$\det(\varphi_{y\xi})\neq 0. \text{ as } y=0 \text{f } \xi_1=\xi_2=1,$$

for $\varphi=(y_1+y_2(1-\xi_1/\xi_2))\xi_1$. It follows that φ may be regarded as a generating function for a homogeneous symplectic transformation $\mathscr{X}$ of $T^*(R_x^2)\supset\mathscr{N}(n_0)$ onto $\mathscr{N}((0, 0, 1, 0))\subset T^*(R_y^2)$, which satisfies

$$\begin{aligned} x_1&=\varphi_{\xi_1}=S-\zeta S_\zeta, & x_2&=\varphi_{\xi_2}=S_\zeta,\\ \eta_1&=\varphi_{y_1}=S_{y_1}\xi_1, & \eta_2&=\varphi_{y_2}=S_{y_2}\xi_1. \end{aligned} \tag{7}$$

Combining (6) with (7), we have

$$\mathscr{X}(x_1\xi_1+x_2\xi_2^2/\xi_1)=y_1\eta_1+2y_2\eta_2. \tag{8}$$

On the other hand, in $\mathscr{N}(n_0)$

$$D_{x_1}^{-1}L=x_1D_{x_1}+x_2D_{x_2}^2\cdot D_{x_1}^{-1}+(a+\sqrt{-1})+bD_{x_2}D_{x_1}^{-1}+R_{-1}. \tag{9}$$

Here and later, R_t denotes an operator $\in OPS^t$. From the theory of [1], it follows that there exists a Fourier integral operator F which is elliptic at $n_0\times(0, 0, 1, 0)$ such that (4) is valid for another neighbourhood $\mathscr{N}((0, 0, 1, 0))$. (5) is the immediate consequence of Lemma 4, the proof of which is postponed until the end of the present paper. In fact

$$\text{subprincipal symbol of } F(D_{x_1}^{-1}L_2)F^{-1}=q_0+\frac{3}{2}\sqrt{-1}$$

and

$$\text{subprincipal symbol of } D_{x_1}^{-1}L_2=a+\sqrt{-1}+b\,\xi_2/\xi_1+\frac{3}{2}\sqrt{-1},$$

which implies (5). The proof is completed.

Lemma 3. *Let* $v\in L^2$ *and let* $L_1v=y_1D_{y_1}V+2Y_2D_{y_2}V+Q(y, D_y)V=f\in L^2$. *If*

$$\operatorname{Re}\sqrt{-1}\,q_0(0, 0, \eta)>\frac{3}{2},\ \textit{for all}\ \eta\in S^1, \tag{10}$$

then there is a constant $\rho>0$ *such that for any* $\varphi\in C_c^\infty(B_\rho)$ $(B_\rho=\{y\mid |y|\leqslant\rho\})$ *with*

$$\varphi(y)\geqslant 0,\ y_1\varphi_{y_1}+2y_2\varphi_{y_2}\leqslant 0, \tag{11}$$

the inequality

$$C_\varphi\|\varphi v\|^2\leqslant\operatorname{Re}(\sqrt{-1}\,\varphi L_1v, \varphi v)+\|v\|_{-1}^2, \tag{12}$$

where C_φ *depends on only* φ *and* L_1, *is valid.*

Proof For any $\varphi\in C_c^\infty(R^2)$, we have

$$L_1(\varphi v)-(y_1D_{y_1}\varphi+2y_2D_y\varphi)v-[Q, \varphi]v=\varphi L_1v.$$

Therefore

$$(\sqrt{-1}\,\varphi L_1 v,\ \varphi v).$$
$$=\left(\left(\sqrt{-1}\,Q-\frac{3}{2}\right)\varphi v,\ \varphi v\right)-(\sqrt{-1}\,[Q,\ \varphi]v,\ \varphi v)+(-(y_1\varphi_{y_1}+2y_2\varphi_{y_2})v,\ \varphi v. \tag{13}$$

Noting Condition (10) and using the techniques of [3, Th2.2] and Garding's inequality, we can obtain for any $\varphi\in C_c^\infty(B_\rho)$

$$\operatorname{Re}\left(\left(\sqrt{-1}\,Q-\frac{3}{2}\right)\varphi v,\ \varphi v\right)\geqslant\frac{1}{2}\min_{|\eta|=1}\left(\operatorname{Re}\sqrt{-1}\,q_0(0,\ 0,\ \eta)-\frac{3}{2}\right)\|\varphi v\|^2$$
$$-C_{L_1}\rho\|\varphi v\|^2-C_\varphi\|v\|_{-1}^2.$$

Here C_{L_1} depends on only Q and is independent of φ. Because condition (11) guarantees that the last term of (13) nonnegative and $[Q,\ \varphi]$ is an operator of order-1, it is easy to conclude (12) when ρ is small enough.

Theorem. *Suppcse that* $\operatorname{Re}\sqrt{-1}\,a(0,\ 0)>3/2$, $\operatorname{Re}\sqrt{-1}\,b(0,\ 0)>3/2$. *Then* (1) *is L^2-hypoelliptic at $x=0$.*

Proof If the conditions in this theorem are fulfilled, then one can find a constant $d>0$ such that $\operatorname{Re}\sqrt{-1}\,a(x_1,\ 0)>3/2$, and $\operatorname{Re}\sqrt{-1}\,b(0,\ x_2)>3/2$ when $|x_1|\leqslant d$, $|x_2|\leqslant d$. Hence one can find a disk O_1 in which any characteristic curve is bound to approach to $(x_1,\ 0)$ (or $(0,\ x_2)$) with $|x_1|\leqslant d$ (or $|x_2|\leqslant d$). By hypoellipticity of elliptic operators, it is easily seen that $(x,\ \xi)\in WF(u)$ when $x_1\xi_1^2+x_2\xi_2^2\neq0$ and $Lu\in C^\infty(O_1)$. On the other hand, from Lemma 1 it follows that $(x_1,\ 0,\ 0,\ \xi_2)$ and $(0,\ x_2\ \xi_1,\ 0)$ are not in $WF(u)$ and P is L^2-hypoelliptic at $(x_1,\ 0)$ (or $(0,\ x_2)$) with $x_1\neq0$ (or $x_2\neq0$). In fact, we come to the conclusion that the set $S_1=T^*(O_1)\setminus\{(x_1,\ x_2,\ \xi_1,\ \xi_2)\mid x_1+x_2=0 \text{ and } \xi_1=\xi_2\}$ does not meet $WF(u)$. Let $(x,\ \xi)\in S_1$ with $x\neq0$. Then the bicharacteristic γ passing through $(x,\ \xi)$ are bound to enter a conical neighbourhood of $(x_1^*,\ 0,\ 0,\ \xi_2^*)$ or $(0,\ x_2^*,\ \xi_1^*,\ 0)$ for some $(x_1^*,\ \xi_2^*)$ or $(x_2^*,\ \xi_1^*)$, which does not meet $WF(u)$. By Hörmander's theorem on the propagation of singularities we have $\gamma\cap WF(u)=\emptyset$. Let $(0,\ \xi)=(0,\ 0,\ \xi_1,\ \xi_2)\in S_1$ with $\xi_1>\xi_2>0$. The bicharacteristic $\gamma(s)$ passing through $(0,\ \xi)$ is of the form

$$x_1(s)=0,\quad x_2(s)=0,$$
$$\xi_1(s)=\frac{1}{s+1/\xi_1},\quad \xi_2(s)=\frac{1}{s+1/\xi_2}. \tag{14}$$

It is easy to see that $\gamma(s)$ is bound to enter any conical neighbourhood of $(0,\ 0,\ 1,\ 0)$. The remainder of proof is the same as before. Analogously, we can get the results expected if $(0,\ 0,\ \xi_1,\ \xi_2)\in S_1$ with $0<\xi_1<\xi_2$ or other cases.

Now we proceed to study the set $T^*(O_1)\setminus S_1$. Let F be the Fourier integral operator mentioned in Lemma 2. Set $v=F^{-1}u$, $FD_x^{-1}L_2\ F^{-1}v=f_1$. Then the intersection of $WF(f_1)$ and a conical neighbourhood of $(0,\ 0,\ 1,\ 0)$, $\mathcal{N}_y$, is empty.

$$L_1v=y_1D_{y_1}+2y_2D_{y_2}+Qv=f_1,\ \text{in}\ \mathcal{N}_y, \tag{15}$$

where Q satisfies (5). Construct a homogeneous function $\psi(\eta)$ of order zero, in η, with $\psi(\eta)=1$ as $|\eta_2/\eta_1|\leqslant\delta/2$ and $\psi(\eta)=0$ as $|\eta_2/\eta_1|>\delta$. Let $g(y)\in C_c^\infty(R^2)$ with $g(y)=1$ near $y=O$ and supp $g(y)\times$supp $\psi(\eta)\subset N_y$. Denote by ψ_s^ε the pseudodifferential operator with symbol $\psi(\eta)(1+\eta_1^2)^{s/2}(1+\varepsilon^2\eta_1^2)^{-1/2}$, where $0<\varepsilon\leqslant 1$. Obviously, ψ_s^ε are uniformly bounded in $OPS^s(R^2)$ with respect to ε.

If $v\in H_{\text{loc}}^{s-1}(R^2)$, then $W=\psi_s^\varepsilon(gv)$ makes sense and satisfies

$$L_1W-(-\sqrt{-1}\,H_{L_1}(\psi_s^\varepsilon))(gv)$$
$$=\psi_s^\varepsilon(gf_1)+R_{s-1}^\varepsilon v+\psi_s^\varepsilon((y_1D_{y_1}g+2y_2D_{y_2}g)v). \tag{16}$$

Here $(H_{L_1}(\psi_s^\varepsilon))$ stands for the pseudodifferential operator with symbol $H_{L_1}(\psi_s^\varepsilon)$, R_{s-1}^ε are uniformly bounded in OPS^{s-1} with respect to ε, and the first and third terms are smooth at origin. Computing yields

$$(H_{L_1}(\psi_s^\varepsilon))=-(s-1)\psi_s^\varepsilon+\dot{\psi}_s^\varepsilon+\tilde{R}_{i-1}^\varepsilon,$$

where

$$\sigma(\dot{\psi}_s^\varepsilon)=-(\psi_{\eta_1}(\eta)\eta_1+2\psi_{\eta_2}(\eta)\eta_2)(1+\eta_1^2)^{s/2}(1+\varepsilon^2\eta_1^2)^{-1/2}$$

and $\tilde{R}_{s-1}^\varepsilon$ is ano ther pseudodifferential operator uniformly bounded in $OPS^{s-1}(R^2)$ with respect to ε. Rewrite (16) in the form

$$L_1W+\sqrt{-1}\,(1-s)W$$
$$=\psi_s^\varepsilon(gf_1)+\dot{\psi}_s^\varepsilon(gv)+\tilde{R}_{s-1}^\varepsilon(gv)+R_{s-1}^\varepsilon v+\psi_s^\varepsilon((y_1g_{y_1}+2y_2g_{y_2})v)/\sqrt{-1}. \tag{17}$$

Choose function $\varphi_\rho=\varphi(\rho^{-1}(y_1^2+y_2^2/2))$ with $\varphi(t)=\exp[-1/(1-2t)]$ if $0\leqslant t\leqslant 1/2$ and $\varphi(t)=0$ if $t>1/2$. We can check that this function φ_ρ satisfies (11). Besides, we always assume that ρ is so small that $g=1$ in the support of φ_ρ. By applying Lemma 3 to (17) it 's easy to see that

$$(C_{\varphi_\rho}+(s-1))\|\varphi_\rho W\|^2\leqslant\|W\|_{-1}^2+\mathrm{Re}\,((\sqrt{-1}\,\varphi_\rho(L_1W)+(s-1)W\varphi_\rho),\ \varphi_\rho W). \tag{18}$$

Note that the $\dot{\psi}_s^\varepsilon(gv)$ and $\varphi_\rho\psi_s^\varepsilon((y_1g_{y_1}+2Y_2g_{y_2})v)$ are C^∞. By induction and (18), it follows that

$$(C_{\varphi_\rho}+s-1)\|\varphi_\rho W\|^2\leqslant C_s'+C_s''\|\varphi_\rho W\|,$$

which implies

$$(s-1+C_{\varphi_\rho}/2)\|\varphi_\rho W\|^2\leqslant C_s' \quad \text{if } s\geqslant 1 \tag{19}$$

for another constant C_s'. Letting $\varepsilon\downarrow 0$ in (19) we get $v\in H_s(0,\ 0,\ 1,\ 0)$, which implies $u\in H_s(0,\ 0,\ 1,\ 1)$. Here $H_s(y,\ \eta)$ ($H_s(x,\ \xi)$) denotes the Sobolev space in the microlocal sense. Hence there exists a conical neighbourhood of $(0,\ 0,\ 1,\ 1)$, $\mathcal{N}_x$, such that $u\in H_s(x,\ \xi)$ for any point $(x,\ \xi)\in\mathcal{N}_x$. Form the same reason as before, it follows that $u\in H_s(x,\ \xi)$, when $x_1+x_2=0$, $\xi_1=\xi_2$, $x\in O_1$. Now we have proved $u\in H_s(O_1)$. Repeating the discussion above, one can get $u\in H_{s+n}(O_1)$ for any $n\in\mathbb{Z}$. The proof is completed.

Remark. The condition Re $\sqrt{-1}\,a(0,0)>3/2$ (Re$\sqrt{-1}\,b(0,0)>3/2$)cannot be dropped. For example, $H(x_1)\in L_{\text{loc}}^2$ is the solution to $x_1D_{x_1}^2u+x_2D_{x_2}^2u-\sqrt{-1}D_{x_1}u-3\sqrt{-1}\,D_{x_2}u=0$, but $H(x_1)$is not in $C^\infty(\{0\})$.

Remark. It is easily seen that operator (1) has H^1-hypoellipticity if $\mathrm{Re}\sqrt{-1}\,a(0, 0)>1/2$, $\mathrm{Re}\sqrt{-1}\,b(0, 0)>1/2$. By the same argument as in the proof of the theorem we can know under what condition (1) is of H^m-hypoellipticity for $m<0$.

What we shall do next is involved with the invariance of subprincipal symbol. As is well known, under microlocally equivalent transformation subprincipal symbol is invariant on multiple characteristics. Under the present circumstance, $n_0\in\Sigma_1\cup\Sigma_2\cup\Sigma_0$ is only radial, and not multiple characteristics. In order to avoid this difficulty, we also need the following lemma.

Lemma 4. *Let $\mathscr{X}$ be a homogeneous canonical transformation of $T^*(R^n_x)\supset\Gamma(n_0)$ onto $\Gamma(n_1)\subset T^*(R^n_y)$. Assume that $P\in OPS^m(R^n)$ has real principal symbol and n_0 is in the characteristic set, $\Sigma(p)$. Then one can find an elliptic Fourier integral operator F associated with $\mathscr{X}$ such that*

$$\mathrm{Im}(\mathrm{sub}\,\sigma(FPF^{-1}))=\mathrm{Im}(\mathrm{sub}\,\sigma(P)\circ\mathscr{X}^{-1}) \text{ on } \mathscr{X}(\Sigma(p))\cap\Gamma(n_1). \tag{20}$$

Proof From the definition of subprincipal symbol and the hypothesis that P has real principal symbol, it follows that

$$\mathrm{Im}\,(\mathrm{sub}\,\sigma(p))=-\sqrt{-1}\ (\text{the principal symbol of } (P-P^*)/2).$$

Let us study a Fourier integral operator

$$Fu(x)=\int e^{\sqrt{-1}\,(s(x,\eta)-y\eta)}\,a(x, \eta)u(y)dx\,d\eta \tag{21}$$

associated with $\mathscr{X}$, where $s(x, \eta)$ is the generating function and $a\in S^0$ does not vanish at the conical neighbourhood discussed. Then by the result in [2] we know $\sigma_m(FPF^{-1})=p_m\circ\mathscr{X}^{-1}$. Therefore, it is easy to see that

$$2\,\mathrm{Im}(\mathrm{sub}\,\sigma(FPF^{-1})=-\sqrt{-1}\ \{\text{the principal symbol of } [(F-(F^{-1})^*)PF^{-1}+(F^{-1})^*(P-P^*)F^{-1}-(F^{-1})^*P^*(F^*-F^{-1})]\}. \tag{22}$$

If

$$(F-(F^{-1})^*)u(x)=\int e^{\sqrt{-1}\,(s(x,\eta)-y\eta)}\,q(x, \eta)u(y)dy\,d\eta \quad \text{with } q(x, \eta)\in S^{-1}, \tag{23}$$

then $\sigma_{m-1}((F-(F^{-1})^*)PF^{-1})=(p_m\circ\mathscr{X}^{-1})h_1(y, \eta)$, $\sigma_{m-1}((F^{-1})^*P^*(F^*-F^{-1}))=(p_m\circ\mathscr{X}^{-1})\,h_2(y, \eta)$ for some smooth functions h_i. So (22) implies at once (20), for $p_m=0$ on $\Sigma(p)$.

Now it remains only to find a suitable amplitude $a(x, \eta)$ in (21) to guarantee the validity of (23). Assume that

$$F^{-1}v(y)=\int e^{\sqrt{-1}\,(y\eta-s(x,\eta))}\,b(x, \eta)v(x)dx\,d\eta.$$

By the standard calculus of Fourier integral operators, we obtain $b\in S^0$ and the principal symbol of b,

$$b_0(x, \eta)=|\det s_{x\eta}(x, \eta)|/a_0(x, \eta).$$

Obviously, (23) is valid if $a_0(x, \eta)=|\det s_{x\eta}(x, \eta)|^{1/2}$. The proof is completed.

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