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THE TOPOLOGICAL AND DYNAMICAL PROPERTIES OF NONORIENTABLE SURFACES

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Abstract

Four topological and dynamical properties of nonorientable surfaces are proved. The first is that for every continuous flow defined on any nonorientable closed surface, there exist periodic or singular closed orbits. In the case of the projective plane, it confirms a conjecture of professor Ye Yian-qian in his lecture notes "dynamical systems on surfaces". Secondly, the author gives an exact upper bound of the number of closed curves on nonorientable surfaces, which do not intersect each other and the complement of their sum is still connected. The third is concerned with the upper and lower bound of the number of the periodic or singular closed orbits with certain properties. The last is related to the connectedness of the complement of a lifting curve on two-fold covering space. The first property may be considered as a generalization of Kneser theorem from Klein bottle to general nonorientable surfaces and the second as a generalization of [4] Theorem 9.3.6 from orientable surfaces to nonorientable surfaces.

In this paper, we show four topological and dynamical properties of non-orientable surfaces.

First, we give a definition of singular closed orbits (simply by SCO).

Definition. *Let L be a connected set of singular points, generalized foci and parabolic orbits. Then L is called an SCO, if and only if*

1) *When two parabolic orbits belong tothe same parabolic sector, they must tend to (or come from) different saddles or different para-bolic sectors of a saddle;*

2) *When we regard each generalized focus as a single point, L is a simple closed curve;*

3) *When the number of singular points is finite, any two generalized foci (if exist) in L cannot be connected only by one parabolic orbit which belongs to L (here we consider a generalized focus as a critical point with only one parabolic sector).*

In Theorems 1 and 2, we need the following hypothesis.

(H) The set of the accumulations of singular points is countable.

Now, we prove a general property of dynamic system on the real projective

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plane PR^2.

For simplicity, we denote by PCO the periodic orbit, and let n (may be $+\infty$) be the number of the singular points, n_1 (n_2), k be the number of the resources (sinks) and the singular points with index $+1$ respectively, supposing the latter possesses k_1(k_2) repelling (attracting) parabolic sectors and separatrices. We denote $m=\mathrm{Max}\{k_1+n_1,\ k_2+n_2\}$.

Theorem 1. *Suppose f is a continuous flow defined on PR^2, and the condition(H) is satisfied. Then there exists at least one PCO or SCO. In particular,*

1) if either there is only one singular point and it is elementary or there is a closed transversal to which the distribution of the singular points is concentrated, then there is a one-sided PCO;

2) if $n<+\infty$, $k=0$ ($k>0$), then there are at least two (m) PCO's or SCO's, among which there is a one-sided PCO or SCO.

In order to prove the theorem, we need some lemmas.

By virtue of [1] Lemma 3, we have the following lemma.

Lemma 1. *If $n<+\infty$, then*

$$M(M')\geqslant\begin{cases}0 & \text{for } k>0,\\ 1 & \text{for } k=0,\end{cases}\qquad N(N')\geqslant\begin{cases}n-1 & \text{for } k>0,\\ n & \text{for } k=0,\end{cases}\tag{1}$$

where $M(M')$ and $N(N')$ are the numbers of the separatrices of class B and class A going out (into) saddles respectively.

Now, let $T(T')$ be the number of the separatirces going out (into) singular points, and regard each parabolic sector as a single separatrix. Then if we notice that in the proof of [1] Lemma 3, the number of the separatrices going out (into) the singular points with index $+1$ is considered as 0, we immediately have the following lemma.

Lemma 2. *If $k>0$, then*

$$T\geqslant M+N+k_1\geqslant n+k_1-1,\qquad T'\geqslant M'+N'+k_2\geqslant n+k_2-1.\tag{2}$$

Lemma 3.[1] *For any positive integers E, K, if $E+K-1$ orbits joint E singular points, and each end of each orbit tends to one of these E points, then they construct at least K SCO's.*

For the sake of conciseness, below, we may as well assume $m=k_1+n_1\geqslant k_2+n_2$.

Proof of Theorem 1 Case 1) has been proved in the paper [2].

Now we prove case 2).

Let s be the number of the PCO's. We consider the tendency of the T separatrices. Because they cannot go into n_1 resources, their ω-limit sets consist of no more than $n-n_1$ singular points and s number PCO of. Then by

$$T \geqslant \begin{cases} n+1=(n+s)+(2-s)-1 \text{ for } k=0, \\ n+k_1-1=(n-n_1+s)+(n_1+k_1-s)-1 \text{ for } k>0, \end{cases} \tag{3}$$

and regard each PCO as a point in the meaning of Lemma 3, the first part of case 2) follows immediately.

We assume there is no one-sided PCO or SCO. Because each two-sided PCO or SCO bounds a disk ([6] Lemma (4) (iii)), and the number of the SCO's and the families of homotopic PCO's is finite, we can contract these disks into a single singular point repeatedly. At last, there will be no PCO or SCO, but it is contrary to what we have proved, so the second part of case 2) is proved.

Next we consider the case $n=+\infty$, and suppose there is no PCO or SCO.

Under this assumption, it is easy to see that the following statements are true.

a) The boundary of each elliptic sector of any saddle S contains no singular point besides S itself (otherwise the boundary will be an SCO by the definition).

b) An elliptic sector cannot be situated between two parabolic sectors (otherwise its boundary must possess singular points besides S).

c) An elliptic sector can only be situated between either two hyperbolic sectors or one hyperbolic sector and one parabolic sector (by b)).

d) Each elliptic sector can be contracted into the saddle itself and unchange its index (by the index formula and c), so we may as well consider that there is no elliptic sector.

e) Any parabolic orbit (not a separatrix) cannot go into a parabolic sector (otherwise by the strong ω-stability of trajectories, this kind of parabolic orbits will fill in a region and its boundary will be an SCO).

f) Because the sum of the indexes of singular points is $+1$ and there is no elliptic sector, there exists at least one parabolic sector (here we regard a sink or source as a critical point with only one parabolic sector). We consider a parabolic orbit r (suppose r belongs to a repelling sector) of the singular point S. By e), r only can go into a saddle S_1 and r is a separatrix of S_1. By the same reason of e), r must be the separatrix of two hyperbolic sectors of S_1. Then there exists other separatrix L_1, which goes out S_1, and only can go into some saddle S_2 such that L_1 is a separatrix of S_2 and at least one side of L_1 is a hyperbolic sector of S_2. Thus we have another separatrix L_2 going out S_2, $\cdots$

g) Let L_r be the curve made up of the orbit segments: r, L_1, L_2, $\cdots$, S^2 be the two-fold covering space of PR^2 and L_r^* be a lift of L_r in S^2. It is easy to see that L_r^* can only tend to some point P_r^* and its image P_r (under the covering mapping) must be an accumulation of singular points (otherwise L_r can be still extended forward).

Now if we notice that the set of the accumulations of singular points is countable, and the set $\{L_r | r$ is a parabolic orbit of $S\}$ is a continuum, we can conclude that there are infinitely many SCO's. But it is contrary to our assumption of no SCO, thus the theorem is also true in case $n=+\infty$.

Then we show the first general property of nonorientable surfaces.

Let M be a closed nonorientable surface with genus g, and f be a continuous flow defined on M.

Theorem 2. *If n, the number of the singular points, is finite, and there are k_1 (k_2) repelling (attracting) parabolic sectors and separatrices possessed by the singular points with index $+1$, n_1(n_2) sources (sinks), then C, the number of the PCO's and SCO's of f, satisfies the inequality*

$$C \geqslant g+m-R-1, \tag{4}$$

where $m=\max_{i=1,2}\{n_i+k_i\}$, $R \leqslant [(g-1)/2]$ is the number of different closures of nontrivial recurrent orbits; if $n=+\infty$, but the condition (H) is satisfied and there is at least one parabolic sector (or a sink or source), then there is at least one PCO or SCO of f.*

Proof By Theorem 1, we need only to consider the case $g \geqslant 2$.

The proof is similar to that of Theorem 1, we only need to make some remarks as follows.

i) The formula $(1)_2$ now becomes

$$N(N') \geqslant n-(2-g)=n+g-2.$$

ii) When $n<+\infty$, we consider each closure of nontrivial recurrent orbits as a single "point" (in the meaning of Lemma 3). Then the formulae $(2)_1$ and $(3)_2$ can be rewritten as

$$T \geqslant M+N+k_1 \geqslant n+k_1+g-2,$$

and

$$T \geqslant (n-n_1+R+s)+(n_1+k_1-R-s+g-1)-1.$$

Then by Lemma 3 ($E=n-n_1+R+s$, $K=g+m-s-R-1$) and

$$C \geqslant s+K=g+m-R-1,$$

the formula (4) follows.

iii) If $n=+\infty$, the continuous curve L_r, now, may go into a closure of nontrivial recurrent orbits besides the possibility to go into an accumulation.

By [7], $R \leqslant [(g-1)/2]$, so the proof of Theorem 1 is still valid here.

Remark 1. Since we have used some new technique in the proof of Theorem 2, compared with [3], we can give up here the finiteness assumption for the number of singular points, and increase the lower bound for the number of PCO's

* The notation [ ] denotes the integral part.

and SCO's.

Now we show the second general property of nonorientable surfaces.

Theorem 3. *Suppose M is a nonorientable surface with genus g. Then there exist $m=s+n$ closed curves, which do not intersect each other, such that the complement of their sum on M is still connected, where s is the number of one-sided curves and n that of the two-sided curves, if and only if (s, n) satisfies*

$$s+2n\leqslant\begin{cases} g-1 & \text{if } g-s \text{ is odd,} \\ g & \text{if } g-s \text{ is even.} \end{cases} \tag{5}$$

Proof We will use induction.

When $g=1$, $M=PR^2$. If there exist $m=s+n$ non-intersected closed curves such that their complement is connected, then we must have $(s, n)=(1, 0)$, because the complement of any two-sided curve on PR^2 is disconnected ([6]) and any two one-sided curves must have non-null intersection. On the other hand, only $(s, n)=(1, 0)$ can satisfy (5).

When $g=2$, $M=K^2$, i. e., the Klein bottle. It is easy to see that only $(s, n)=(2, 0)$ or $(0, 1)$ can satisfy (5), besides $(s, n)=(1, 0)$. In the first case we may choose two one-sided curves of type $k^{\pm1}h^n$ for some integer n ([5] or [2]). In the second case we can choose a two-sided curve of type $h^{\pm1}$. Obviously, any other m non-intersected closed curves must cut K^2 into different pieces.

Supose the theorem holds when $g=k-1$ and $g=k$, we want to show it is still true when $g=k+1$.

Now assume there exist $m=s+n$ non-intersected closedcurves such that their complement is connected.

If $s>0$, then let L be one of the one-sided curves, and g_1 be the genus of $M_1=M-L$. From

$$2-g=\chi(M)=\chi(M_1)=\begin{cases} 2-2g_1-1 & \text{if } M_1 \text{ orientable,} \\ 2-g_1-1 & \text{if } M_1 \text{ nonorientable,} \end{cases}$$

we have

$$g_1=\begin{cases} (g-1)/2 & \text{if } M_1 \text{ orientable,} \\ g-1 & \text{if } M_1 \text{ nonorientable.} \end{cases}$$

If M_1 is orientable, then g is odd and by [4] Theorem 9.3.6

$$n\leqslant(g-1)/2,\ s=1.$$

It follows that

$$s+2n\leqslant g.$$

If M_1 is nonorientable, then by $s=s_1+1$ and inductive assumption we have

$$s+2n=1+s_1+2n\leqslant\begin{cases} g_1=g-1 & \text{if } g-s \text{ is odd,} \\ g_1+1=g & \text{if } g-s \text{ is even.} \end{cases}$$

If $s=0$, then L is a two-sided curve. We can prove $(s, n)=(0, n)$ must satisfy (5) in the same way, if we notice that in this case

$$g_1=\begin{cases}(g-2)/2 & \text{if } M_1 \text{ orientable } (g \text{ must be even}),\\ g-2 & \text{if } M_1 \text{ nonorientable.}\end{cases} \tag{6}$$

On the other hand, if we have been given a pair of numbers (s, n) which satisfies (5), we have to show there are $m=s+n$ non-intersected closed curves on M such that their complement is connected.

It is very easy if we regard M as a sphere with g cross-caps, i. e. if M^* is a sphere with g boundaries $l_1, l_2, \cdots, l_g$, H is a mapping which identifies the diagonal points of l_i $(i=1, 2, \cdots, g)$, $L_i=H(l_i)$, then $M=H(M^*)$. Evidently, we may choose $L_1, L_2, \cdots, L_s$ as s one-sided curves. If $s<g$ and $n>0$, we may choose n pairs of one-sided curves: $(L_{s+1}, L_{s+2}), \cdots, (L_{s+2n-1}, L_{s+2n})$. To each pair of curves (L_{s+i}, L_{s+i+1}), we first cut out a small arc from each curve, then use two pieces of arcs contained in sphere to joint them into a simple closed curve (see Figure 1.1), which must be two-sided and its complement is connected (see Figure 1.2).

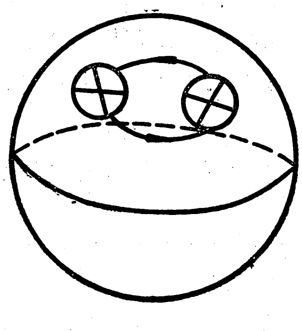

Figure 1.1

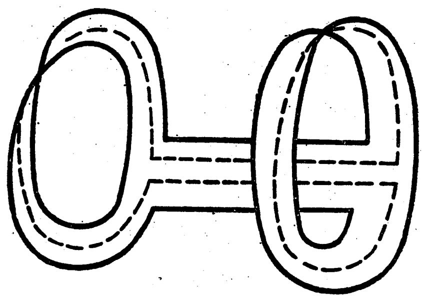

Figure 1.2

Thus we have finished the proof of the theorem.

As an application of Theorems 1–3, we prove the third generalproperty of nonorientable surfaces.

Theorem 4. *Suppose M is a nonorientable surface with genus g, f is a continuous flow on M with $m=s+n$ non-null-homotopic and non-intersected PCO's and SCO's which are not homotopic to each other, where s (n) is the number of one-sided (two-sided) PCO's and SCO's. Then*

i) *$s\leqslant g$ and there exists a flow f such that $s=g$;*

ii) *$s\geqslant 1$ if g is odd and the number of singular points is finite;*

iii) *$g-s$ is even if the number of singular points is finite; moreover, $n+r\leqslant \max\{1, (3g+s-6)/2\}$ if $g>1$ and M is compact and without boundary, where r is the number of the different closures of nontrivial recurrent orbits. $2r+s\leqslant g$**.*

** It follows that $n+2r\leqslant 2g-3$ and $s+n+4r\leqslant \max\{1, 3g-3\}$.

Proof The first part of conclusion i) is a simply corollary to Theorem 3, since the complement of any one-sided curve is connected.

If we construct a continuous flow on the south semi-sphere (with g cross-caps) as Figure 2.1 and on the north semi-sphere as Figure 2.2, then it is easy to see that the second part of conclusion i) is also true.

We use induction to prove ii).

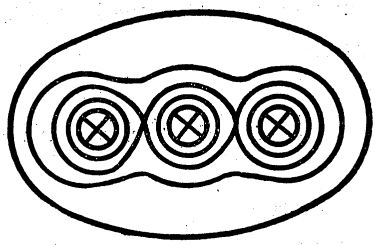

Figure 2.1

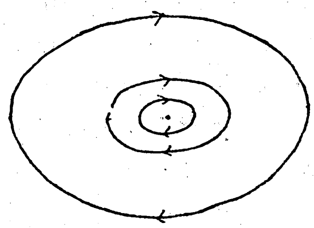

Figure 2.2

By Theorem 1, the conclusion follows when $g=1$.

Suppose it keeps true when $g\leqslant 2k-1$, for $k\geqslant 1$.

Now we consider the case $g=2k+1$. Assume there is no one-sided PCO and SCO. Then by Theorem 2, there exists a two-sided PCO or SCO L. By the same inference in the proof of the second part of Theorem 1, we may assume L is non-null-homotopic.

If $M-L$ is connected, then by formula (6), $M-L$ must be nonorientable, and $g_{M-L}=g-2$. By induction hypothesis, we lead to a contradiction. So the second conclusion follows in this case.

If $M-L$ is disconnected, denote it by $M-L=M_1\cup M_2$ and let g_i be the genus of M_i, for $i=1$, 2. From the relation

$$\chi(M)=\chi(M-L)=\chi(M_1)+\chi(M_2),$$

we have

$$2-g=\begin{cases}(2-g_1-1)+(2-g_2-1) & \text{if both are nonorientable,}\\(2-g_1-1)+(2-2g_2-1) & \text{if only one is orientable,}\\(2-2g_1-1)+(2-2g_2-1) & \text{if both are orientable.}\end{cases}$$

In the first case, $g_1+g_2=g$. It is easy to see that either g_1 or g_2 is odd. Then by the induction hypothesis, conclusion ii) follows.

In the second case, $g=g_1+2g_2$, and g_1 is odd. So the situation is the same as above.

The third case is impossible, because the left hand is odd, whereas the right hand is even.

Before proving the third conclusion, we introduce a lemma.

Lemma 4. *Suppose M is an orientable closed surface with genus g, f is a* *continuous flow on M. Then*

$$n+r\leqslant\begin{cases}1 & \text{when } g=1,^{**}\\ 3g-3 & \text{when } g>1.\end{cases}$$

This lemma can be proved by induction similarly, if we notice $r\leqslant g$ ([8]) and the property A: the largest number of the non-null-homotopic two-sided PCO's or SCO's which do not intersect and are not homotopic to each other and not homotopic to any boundary on a compact surface with one hole (two holes) is larger by one (two) than that of the same surface without hole.

The proof of the third part of Theorem 4:

Lemma 5. *If M is a nonorientable closed surface with genus g where g is even and $s=0$, then*

$$n+r\leqslant\begin{cases}1 & \text{if } g=2,\\ (3g-6)/2 & \text{if } g>2.\end{cases}$$

Proof This is an immediate consequence of lemma 4, if we consider the two-fold covering surface T_{g-1} of M.

We can also give an intuitive proof as follows. Consider M as a sphere with g cross-caps $L_1, \cdots, L_g$. Because $s=0$, by Theorem 2 and Theorem 4 (ii), there must exist $m(\leqslant g/2)$ two-sided PCO's, SCO's or CNR's (i. e., the closures of nontrivial recurrent orbits) connecting $L_1, \cdots, L_g$, such that their complement M_1 is a sphere with $2m$ $(\leqslant g)$ holes (in the case of CNR, we replace CNR by a closed transversal). Now it is easy to see that there are at most $2m-3$ non-null-homotopic two-sided PCO's or SCO's on M_1 which are not homotopic to (and do not inversect) boundaries and each other. Thus the lemma follows.

Now we consider the case $s>0$. If $s=g$, we consider M as a sphere with g cross-caps $L_1, \cdots, L_g$, and each L_i is a PCO or SCO. In this case, $M-\Sigma L_i$ is a sphere with g holes, it is easy to see that around these holes there may be at most $2g-3$ non-null-homotopic two-sided PCO's or SCO's which are not homotopic to (and do not intersect) each other and $r=0$.

If $s<g$, let $L_1, \cdots, L_s$ be these one-sided PCO's or SCO's and $M_1=M-\Sigma L_i$. By the conclusion (ii) we have just proved, $g-s$ must be even.

First suppose M_1 is orientable. We can regard M as a torus with genus $(g-s)/2$ and with s cross-caps $L_1, \cdots, L_s$. In the same manner as above, we can see that there are at most $2s-1$ non-null-homotopic two-sided PCO's, SCO's (they are not homotopic to and do not intersect each other) or CNR's $C_1, \cdots, C_{2s-1}$ around $L_1, \cdots, L_s$. Then we consider the surface M_2, the connected component of $M-\Sigma C_i$ with genus $(g-s)/2$, and notice the property A. The conclusion (iii) follows from Lemma 4 in this case.

** Han Mao-an helps me to obtain this correct number and proves it in detail.

Now suppose M_1 is nonorientable. We can also easily see that there are at most $2s-1$ non-null-homotopic PCO's, SCO's (not homotopic to and intersect each other) or CNR's $C_1, \cdots, C_{2s-1}$ around $L_1, \cdots, L_s$. Let M_2 be the compactized surface of $M_1 - \Sigma C_i$. Then the conclusion (iii) follows from Lemma 5 if we notice that the genus of M_2 is $g-s$ and make use of the proprety A.

Now we prove the inequality $s+2r \leqslant g$. When $r=0$, it follows from (i). When $r>0$, assume L is a closed transversal through a point of some nontrivial recurrent orbit. Then $M-L$ is a connected and nonorientable surface with genus $g-2$. By using induction, weobtain $s+2r \leqslant g$ immediately.

Thus we have completed the proof of Theorem 4.

Let M be a nonorientable surface with genus g, L be a Jordan curve on M, and L^* be a lift of L on the normal two-fold covering space M^* of M. The last general property of nonorientable surfaces we are going to prove is the following theorem.

Theorem 5. *If the number g is even (odd) and L is one-sided (two-sided and $M-L$ is connected), then M^*-L^* is connected.*

Proof We regard M as a sphere with g cross-caps. Let $M'(L')$ be a copy of $M(L)$. Then M^* may be considered as the connected sum of M and M' through gluing their g pair of cross-caps (here regard each cross-cap as a circle), and L^* may be considered as the connected sum of L and L'.

We prove the first part of the theorem.

Because g is even, by $\chi(M-L)=\chi(M)=2-g$ and

$$\chi(M-L)=\begin{cases} 2-g_{M-L}-1 & \text{if } M-L \text{ is nonorientable,} \\ 2-2g_{M-L}-1 & \text{if } M-L \text{ is orientable,} \end{cases}$$

it is easy to see that $M-L\ (M'-L')$ is a connected nonorientable surface with $g-1$ cross-caps and a boundary $L(L')$.

By virtue of the orientability of M^*-L^* and

$$M^*-L^*=(M-L)\cup(M'-L'),$$

it follows that M^*-L^* is, topologically, the connected sum of $M-L$ and $M'-L'$ through gluing $g-1$ pair of cross-caps and connecting L, L' into L^*.

Obviously, M^*-L^* is connected.

The second part can be proved in a similar way if we notice that $M-L$ is still nonorientable by the formula

$$2-g=\begin{cases} 2-g_1-2 & \text{if } M-L \text{ is nonorientable,} \\ 2-2g_1-2 & \text{if } M-L \text{ is orientable.} \end{cases}$$

Remark 2. Theorem 5 is used to prove the existence of PCO of a continuous flow defined on nonorientable surface in my recent work (see[9]).

By the proof of Theorem 5, we have the following corollary.

Corollary. *If $M-L$ is connected and nonorientable, then M^*-L^* is connected.*

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