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THE ROTATION NUMBER AND DIRECTION OF A CONTINUOUS FLOW ON THE TORUS

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Abstract

In this paper the rotation direction r_f and rotation number ρ_f for any continuous flow on the torus are defined by applying Weil's theorem proved by Markley. The following results are obtained:

(i) $r_f=0$ iff all orbits of f are proper and each limit set of f is homotopic to zero on T^2;

(ii) if $r_f\neq 0$, then ρ_f is irrational iff f has at least one non-trivial P stable orbit, ρ_f is rational iff f has at least one non-zero-homotopic closed or singular closed orbit.

Then a method of computing the rotation number of certain flows is given.

§1. The Rotation Number of Flows

Let T^2 be the torus, R^2 a real plane, and $(R^2. p)$ a covering space of T^2. We will use the following results.

Weil's theorem[1]. *Let* α: $[0, \infty)\to T^2$ *be a simple curve on the torus. Let* $\tilde{\alpha}$: $[0, \infty)\to R^2$ *be any lift of* α *to* R^2. *If* $|\tilde{\alpha}(t)|\to\infty$ *as* $t\to\infty$, *then* $\lim\limits_{t\to\infty}\tilde{\alpha}(t)/|\tilde{\alpha}(t)|$ *exists, where* $\tilde{\alpha}(t)=(\tilde{\alpha}_1(t), \tilde{\alpha}_2(t))$, $|\tilde{\alpha}(t)|=((\tilde{\alpha}_1(t))^2+(\tilde{\alpha}_2(t))^2)^{1/2}$.

Corollary[1]. *Let* α, β: $[0, \infty)\to T^2$ *be simple curves which do not intersect. Let* $\tilde{\alpha}$, $\tilde{\beta}$: $[0, \infty)\to R^2$ *be lifts of* α *and* β *respectively. If* $|\tilde{\alpha}(t)|\to\infty$ *and* $|\beta(t)|\to\infty$ *as* $t\to\infty$, *then*

$$\lim_{t\to\infty}\tilde{\alpha}(t)/|\tilde{\alpha}(t)|=\pm\lim_{t\to\infty}\tilde{\beta}(t)/|\tilde{\beta}(t)|.$$

Now let f be a continuous flow on T^2. From Lemma 1[2] there is a unique continuous flow $\tilde{f}$ (called the lift of f) with the property that $p\circ\tilde{f}=f\circ p$. Let $\tilde{f}(q, t)$ be a lift of an orbit $f(q, t)$ of f. It is an orbit of $\tilde{f}$. If there is a point $q\in T^2$ such that $|\tilde{f}(q, t)|\to\infty$ as $t\to+\infty$, then by Weil's theorem $\lim\limits_{t\to\infty}\tilde{f}(q, t)/|\tilde{f}(q, t)|=r_q^+$ exists.

Put

$$Q^+=\{q\in T^2: |\tilde{f}(q, t)|\to\infty \text{ as } t\to+\infty\}.$$

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For any $q_1, q_2 \in Q^+$, $r_{q_1}^+ = \pm r_{q_2}^+$ from the corollary above. If there is a point $q \in T^2$ such that $|\tilde{f}(q, t)| \to \infty$ as $t \to -\infty$, $\lim_{t\to-\infty} \tilde{f}(q, t)/|\tilde{f}(q, t)| = r_q^-$ exists by the same theorem.

Put

$$Q^- = \{q \in T^2: |\tilde{f}(q, t)| \to \infty \text{ as } t \to -\infty\}.$$

Then $r_{q_1}^- = \pm r_{q_2}^-$ for any $q_1, q_2 \in Q^-$.

If $Q^+ \cup Q^- \neq \varnothing$, then $r_{q_1}^+ = \pm r_{q_2}^-$ for any $q_1, q_2 \in Q^+ \cup Q^-$ by the same corollary.

From the facts above we can introduce the following definition.

Definition. *Let $Q = Q^+ \cup Q^-$. If $Q \neq \varnothing$, we have a pair of numbers $r_f = (\rho_1, \rho_2)$ with $\rho_1 \geqslant 0$ uniquely determined by f, called the rotation direction of f. In this case $\rho_f = \rho_2/\rho_1$ is called the rotation number of f.*

If $Q = \varnothing$, we define the rotation direction of f as zero, i. e., $r_f = (0, 0)$. In this case the rotation number of f is not defined.

Remark 1. The sign of ρ_f does not depend on the sign of $\pm r_f$, and it may be positive, negative, or infinite.

It is easy to see that $|\tilde{f}(q, t)| \to \infty$ as $t \to +\infty$ or $-\infty$ if q is strictly P^+ or P^- stable. The same is true for non-zero-homotopic periodic orbits.

Lemma 1. *Let f be a continuous flow on T^2. Suppose that $q \in T^2$ is a $P^+(P^-)$ stable point such that $\lim \tilde{f}_2(q, t)/\tilde{f}_1(q, t) = \rho$ exists as $t \to +\infty(-\infty)$, where $\tilde{f}(q, t) = (\tilde{f}_1(q, t), \tilde{f}_2(q, t))$ is a lift of $f(q, t)$ to R^2. Then there is a constant $C > 0$ such that*

$$|\tilde{f}_2(q, t) - \rho\tilde{f}_1(q, t)| \leqslant C$$

for all $t \geqslant 0 (\leqslant 0)$.

Proof Put

$$(\varphi(t), \psi(t)) = (\tilde{f}_1(q, t), \tilde{f}_2(q, t)).$$

Without loss of generality, we can suppose q is P^+ stable, and $\varphi(0) = \psi(0) = 0$, $\varphi(t_N) = N$, where $t_N \to +\infty$ as $N \to \infty$. And from Lemma 2[3] (which is still valid here from its proof) we have the following estimates for every positive integer $N \geqslant 1$:

$$|\psi(t_N)/N - \rho| \leqslant 2/N, \quad |\varphi(t) - N| \leqslant 1,$$

$$|\psi(t_N) - \rho N| \leqslant 2, \quad |\psi(t) - \psi(t_N)| \leqslant M_1 \text{ for } t_N \leqslant t \leqslant t_{N+1},$$

where $M_1 > 0$ is a constant.

Thus, $\varphi(t) = N + \theta$, $\psi(t) = \psi(t_N) + \xi$, $|\theta| \leqslant 1$, $|\xi| \leqslant M_1$. Notice that $\psi(t) - \rho\varphi(t) = \psi(t_N) - \rho N + \xi - \rho\theta$. We obtain $|\psi(t) - \rho\varphi(t)| \leqslant 3 + \rho + M_1$, and the proof is completed.

Theorem 1. *Let f be a continuous flow on T^2. Let $\tilde{f}(q, t)$ be a lift of the orbit $f(q, t)$ through point $q \in T^2$.*

(1) *If $|\tilde{f}(q, t)| \to \infty$ as $t \to +\infty(-\infty)$, then $\lim \tilde{f}_2(q, t)/\tilde{f}_1(q, t)$ exists as $t \to +\infty(-\infty)$, and the limit value is the rotation number ρ_f of f.*

(2) *ρ_f is an irrational (rational) number if $f(q, t)$ is strictly P^+ or P^- stable (periodic).*

Proof We deal with the case where $t\to+\infty$. Similarly for $t\to-\infty$. From Weil's theorem, $\lim_{t\to+\infty}\tilde{f}(q, t)/|\tilde{f}(q, t)|$ exists. It follows that $\lim_{t\to+\infty}\tilde{f}_i(q, t)/|\tilde{f}(q, t)| = \rho_i$ exists for $i=1, 2$. Thus

$$\lim \tilde{f}_2(q, t)/\tilde{f}_1(q, t) = \rho_2/\rho_1 = \rho.$$

Suppose $f(q, t)$ is strictly P^+ or P^- stable. Then by [4] we can suppose it is strictly P stable since ρ is independent of q. Thus we have from Lemma 1

$$|\tilde{f}_2(q, t)-\rho\tilde{f}_1(q, t)| \leqslant C \text{ for all real } t$$

for some constant $C>0$. Namely, the curve $\tilde{f}(q, t)$ lies between the parallel lines $y=\rho x\pm C$. Hence, ρ is an irrational number from Theorem 3.3[1].

Finally, it is evident that ρ is a rational number if $f(q, t)$ is a non-zero-homotopic periodic orbit.

Theorem 2. *Let f be a continuous flow on T^2.*

(1) *The rotation direction r_f is zero iff all orbits of f are proper and the limit set of every orbit is homotopic to zero on T^2.*

(2) *Suppose $r_f\neq 0$. Then the rotation number ρ_f is irrational iff f has strictly P stable orbits; ρ_f is rational iff f has at least one non-zero-homotopic closed or (usual) singular closed orbit on T^2.*

Proof (1) We first prove the necessity. Because $r_f=0$, f has no strictly P^+ or P^- stable orbits, i.e., its all orbits are proper. From Theorem 2[5], for any $q\in T^2$, $\Omega(q)$ and $A(q)$ have one of the following properties:

(i) a single singular point;

(ii) a periodic orbit;

(iii) an invariant curve which consists of singular points and proper orbits connecting them.

By Theorem 1 we need only to prove $\Omega(q)$ and $A(q)$ are homotopic to zero for case (iii). If not, say $\Omega(q)$ is not homotopic to zero, it is clear that $\Omega(q)=L$ must be a singular closed orbit. Thus, without loss of generality, we can suppose L is simply closed. Then we have that T^2-$\Omega(q)$ is a cylinder with two boundary circles L_1 and L_2. There must be an $L_i (i=1$ or $2)$ such that $\Omega(q)=L_i$ on the cylinder from the Poincare-Bendixson theory. Now suppose L is of type (n, m). Let $\tilde{L}$ be a universal lift of L. It is easy to know that there is a lift $\tilde{f}(q, t)$ of $f(q, t)$ such that for any neighbourhood U of $\tilde{L}$ in R^2 there is a $T>0$ such that $\tilde{f}(q, t)\in U$ for all $t\geqslant T$. Thus, $\lim_{t\to+\infty} d(\tilde{f}(q, t), \tilde{L})=0$, where d denotes the usual distance on R^2. We also note that $\Omega(q)=L$. It follows that $|\tilde{f}(q, t)|\to\infty$ as $t\to+\infty$, and $\lim \tilde{f}_2(q, t)/\tilde{f}_1(q, t)= m/n$, a contradiction.

Conversely, let $f(q, t)$ be a proper orbit of f and $\Omega(q)$, $A(q)$ are homotopic to zero. Then there exist two disks D_1, D_2 on T^2, and a positive T such that $f(q, t)\in D_1(D_2)$ for all $t\geqslant T(\leqslant -T)$. Thus any lift of $f(q, t)$ is bounded in R^2.

(2) Let $r_f\neq 0$. Firstly, if f has strictly P stable orbits, ρ_f is irrational from Theorem 1. If ρ_f is irrational, but f has no strictly P stable orbits, it follows from Theorem 2[5] that for any $q\in T^2$ the limit set $\Omega(q)$ is a single singular point, a periodic orbit, or an invariant curve. The same is true for $A(q)$. By $r_f\neq 0$ and the conclusion above, there must exists a $q\in T^2$ such that $\Omega(q)$ or $A(q)$ is a non-zero-homotopic invariant closed curve. For the sake of definition we can suppose $\Omega(q)$ is such a closed curve. If it contains no singular points, then it is periodic, and ρ_f is rational by Theorem 1, a contradiction. If it contains at least one singular point, then in the same way as in (1) we can prove ρ_f is also rational. Thus, we have just proved that f must have strictly P stable orbits if ρ_f is an irrational number.

Secondly, if f has a closed or singular closed orbit L which is not homotopic to zero, then T^2-L is a cylinder. Thus, f has no strictly P stable orbits, and ρ_f is rational from the argument above. Conversely, if ρ_f is rational, then all orbits of f are proper. It follows that there exists a $q\in T^2$ such that one of the limit sets $\Omega(q)$ and $A(q)$ is a non-zero-homotopic closed or singular closed orbit of f. The proof is complete.

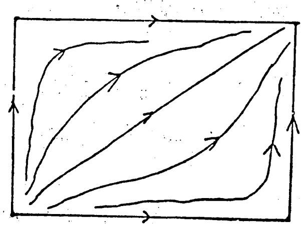

Fig. 1.

Remark 2. From Theorem 2 and its proof, geometrically that $r_f=0$ means that any lift of every non-singular orbit of f is a bounded simple curve of R^2.

Example 1. Consider the flow f on T^2 obtained by an identification of the opposite sides of the following square with given orbit structure (Fig. 1). By Theorem 2, $r_f=0$ and ρ_f is not defined.

In fact, f has two singular closed orbits which are of types (1, 0) and (0, 1) respectvely, all other singular closed orbits are of type (1, 1). In this case it is natural that ρ_f is not definite.

If f is a continuous flow defined on the Klein bottle K^2, and $\tilde{f}$ is its lift to the torus T^2, then by section 4.2 of [6] the rotation number of $\tilde{f}$ is either 0 or ∞ as long as it is well defined. In this case the flow $\tilde{f}$ (therefore f) has at least one non-zero-homotopic closed or singular closed orbit.

§2. Computation of the Rotation Number

In this part we calculate the rotation number for certain flows on T^2 by using the first integral and linear transformation.

Consider the following torus equations

$$\frac{dx}{dt}=X(x, y), \quad \frac{dy}{dt}=Y(x, y) \tag{1}$$

with the condition as follows.

H_1: X, Y are functions of class C^1 and are periodic of period one in both variables. $T^2=R^2/Z^2$.

For the sake of simplity, denote by (1)′ the lift system of the system (1). Now we suppose that (1)′ (not (1)) possesses a first integral $H(x, y)\in C^1$ Then from Lemma 1 and the proof of Theorem 2 we have the following theorem.

Theorem 3. *Suppose the equation $H(x, y)=c$ for some c has an unbounded branch L on which there are no singular points of the system (1)′. Then*

(i) *there are constants C_1, C_2 and ρ such that L lies between the lines $y=\rho x+C_1$ and $y=\rho x+C_2$ (or $x=C_1$ and $x=C_2$),*

(ii) *the ρ (or ∞) is the rotation number of the system (1).*

In paper [3] the author considered the system (1) with an integral invariant, i. e., it satisfies the following conditions, besides H_1:

H_2: the system (1) possesses an integral invariant $U(x, y)$ of class C^1 defined on T^2:

H_3: the singular points of (1) are isolated.

Under the conditions above the system (1)′ has a first integral of the form

$$H(x, y)=ax+by+h(x, y),$$

where

$$a=-\int_0^1 U(x, 0)X(x, 0)dx, \quad b=\int_0^1 U(0, y)Y(0, y)dy,$$

and $h\in C^1$ is periodic of period 1 in x and y.

Now we consider the following slightly more general system

$$\frac{dx}{dt}=C(x, y)X(x, y), \quad \frac{dy}{dt}=C(x, y)Y(x, y), \tag{2}$$

where X, Y satisfy the conditions H_1, H_2, H_3, and $C\in C^1$ is periodic of period 1 in x and y (maybe vanishes at some points). Obviously, the global orbit structures of (1) and (2) are equivalent up to movable singular points. Thus, from Theorems 2 and 3, and the results of [3] we have immediately the following theorem.

Theorem 4. *Suppose the system (2) has only isolated singular points.*

(i) *If $(a, b)\neq(0, 0)$, then the system (2) has a nontrivial P stable orbit which is dense in an open set of T^2 iff $\rho=-a/b$ is irrational; it has a group of non-zero-homotopic periodic orbits, and its all orbits, except singular points and separatrices connecting them, are closed iff ρ is rational.*

(ii) *If the system (2) has no nontrivial P stable orbits and all periodic orbits are homotopic to zero, then the rotation direction of (2) is zero, and $a=b=0$.*

However, the following example shows that the system (2) may have non-zero-homotopic periodic orbits when $a=b=0$.

Example 2. Consider the following system

$$\frac{dx}{dt}=\sin y,\ \frac{dy}{dt}=b\sin x. \tag{3}$$

Then

(i) $\rho=0(\infty)$ if $0<|b|<1(|b|>1)$, in this case the system (3) has a group of periodic orbits of type (1, 0) ((0, 1));

(ii) the rotation direction of (3) is zero if $|b|=1$, in this case all periodic orbits are homotopic to zero.

In fact, we have $H(x, y)=b\cos x-\cos y$. If $|b|\neq 1$ there is no singular point of (3) satisfying $H(x, y)=0$. And for $|b|>1$ the equation $H(x, y)=0$ has two branches:

$$y=\operatorname{Arc}\cos(b\cos x) \text{ for } 0<x<\pi \text{ and for } \pi<x<2\pi \text{ respectively,}$$

whose projections to T^2 are closed curves of type (0, 1).

Example 3. Consider

$$\frac{dx}{dt}=B\cos x+a+\cos y+\sin y,\ \frac{dy}{dt}=B\sin x. \tag{4}$$

We have

(i) If $|B|>1+|a|$, then $\rho=\infty$, in this case the system (4) has exactly two periodic orbits of type (0, 1).

(ii) If $|a|>1+|B|$, or $|a|+|B|<1$, then $\rho=0$, and (4) has a group of perioido orbits of type (1, 0).

(iii) If $a=0$, $|B|=1$, then $r=0$, in this case the orbit structure of the system (4) is as shown in the sketch below.

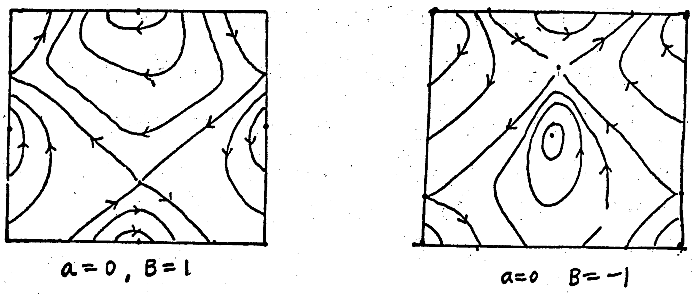

Fig. 2

In fact, the lift (4)′ of (4) has an integral factor e^y, and a first integral of the form

$$H(x, y)=e^y(a+\sin y+B\cos x).$$

If $|B|>1+|a|$, then the equation $H(x, y)=0$ has exactly two branches:

$$y=\operatorname{Arc}\sin(-a-B\cos x) \text{ for } 0<x<\pi, \text{ and } \pi<x<2\pi \text{ respectively.}$$

Thus $\rho=\infty$. Notice that $(x_0, 0)$ is a periodic point iff $H(x_0, 0)=H(x_0, 2\pi)$, which gives $\cos x_0=-a/B$. This shows that the system (4) has exactly two limit cycles.

The other cases are easy to discuss.

The example above shows that the system (1) may have limit cycles (i. e., isolated closed orbits) although its lift system has a first integral.

Remark 3. More generally than above, it is easy to see that the following systems

$$\frac{dx}{dt}=F_1(y)\cos x+F_2(y),\quad \frac{dy}{dt}=F_0(y)\sin x$$

and

$$\frac{dx}{dt}=-F_1(y)\sin x+F_2(y),\quad \frac{dy}{dt}=F_0(y)\cos x,$$

where $F_i\in C^1$ is periodic of period 1 in y, $i=1, 2, 3$, both have an integral factor of the form $F_0^{-1}\exp\left(\int F_1/F_0\,dy\right)$. Hence, we can calculate their rotation numbers by using their first integrals.

In the rest we offer another method to calculate the rotation number for some systems.

Let T be a linear transformtion of the form

$$T:\ \begin{array}{l} u=ax+by \\ v=cx+dy, \end{array}\quad |ad-bc|\neq 0$$

which carries the system (1) to the following

$$\frac{du}{dt}=X_1(u, v),\quad \frac{dv}{dt}=Y_1(u, v) \tag{5}$$

such that the functions X_1, Y_1 are periodic of period 1 in both variables. Thus (5) is also a torus system. Now suppose the rotation number ρ of (1) is well defined. Then there is a point $q\in T^2$ such that the orbit $\tilde{f}(q, t)$ of (1)′ (which is a lift of the orbit $f(q, t)$ of (1)) is unbounded. Let $\tilde{q}$ be the lift of q to R^2 with $\tilde{q}=\tilde{f}(q, 0)$, and $\tilde{q}_1=T\tilde{q}$. Let $\tilde{f}_1(q, t)=T\tilde{f}(q, t)$. Then $\tilde{f}_1(q, t)$ is the orbit of the system (5) through $\tilde{q}_1$, and is unbounded since T is non-singular. This means that the rotation number ρ_1 of (5) is also well defined. From the form of T we have the following relation

$$\rho_1=\frac{c+d\rho}{a+b\rho}. \tag{6}$$

We summarize the above as follows.

Theorem 5. *Let the torus system* (1) *be changed into the torus system* (5) *through a non-singular linear transform*

$$T:\ u=ax+by,\ v=cx+dy.$$

Then the rotation number of (1) *is well defined iff that of* (5) *is well defined, in this case they are related by formula* (6).

Corollary 1. *If T is such that $(X, Y)=\pm(X_1, Y_1)C$, where $C(x, y)>0$, then*

(i) $\rho=\infty$ *or* $\rho=c/(a-d)$ *if* $b=0$;

(ii) ρ *satisfies the quadratic equation* $b\rho^2+(a-d)\rho-c=0$ *if* $b\neq0$.

Corollary 2. *Let the rotation number ρ of* (1) *be well defined.*

(i) *If the system* (1) *is invariant under the variable change $x\to y$, $y\to x$, $t\to\pm t$, then $\rho=\pm1$, in this case it has at least one non-zero-homotopic closed or singular closed orbit of type* (1, 1).

(ii) *If* (1) *is invariant under $x\to x$, $y\to -y$, $t\to\pm t$, or $x\to -x$, $y\to y$, $t\to\pm t$, then $\rho=0$ or ∞, in this case it has at least one closed or singular closed orbit of type* (1, 0) *or* (0, 1).

Example 4. From Corollary 2, the following system

$$\frac{dx}{dt}=P(\sin^2 x,\ \sin y,\ \cos x,\ \cos y),$$

$$\frac{dy}{dt}=g(\sin x)Q(\sin^2 x,\ \sin y,\ \cos x,\ \cos y),$$

where P, Q, $g\in C^1$ and g is odd, has the rotation number $\rho=0$ or ∞ as long as it is well defined. In particular, $\rho=0$ if $P>0$. From Example 3 the system above may have limit cycles.

Example 5. Consider the system

$$\frac{dx}{dt}=\sin(y+nx)+b,\ \frac{dy}{dt}=a,\ a\neq0,\ n\geqslant1.$$

It is changed into

$$\frac{du}{dt}=a+nb+n\sin u,\ \frac{dv}{dt}=a$$

under the variable change $u=y+nx$, $v=y$. Obviously, for $|a+nb|\leqslant n$, $\rho_1=\infty$, and so $\rho=-n$.

For $|a+nb|>n$, the new system has the following first integral

$$H(u, v)=Au+h(u)-v,$$

where $h\in C^1$ is periodic of period 2π, $A=\dfrac{a(a+nb)}{|a+nb|((a+nb)^2-n^2)^{1/2}}$, in this case $\rho=nA/(1-A)$. Thus for any integer $n\geqslant1$, the stated system has always exactly two periodic orbits of type $(1, n)$ if $|a+nb|<n$, and has a unique semi-stable limit cycle if $|a+nb|=n$. And its all orbits are periodic iff $nA/(1-A)$ is rational if $|a+nb|>n$.

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