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ON COLLECTIONWISE SUBNORMAL SPACES

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Abstract

Collectionwise subnormality is characterized by some covering properties. It follows that collectionwise subnormality is preserved under closed mappings.

In paper [1], J. C. Smith gave some characteristic properties of collectionwise normal spaces by covering properties. Naturally, we shall ask whether collectionwise subnormal spaces can be characterized by covering properties. In this paper, we give an affirmative answer. From this result we prove that collectionwise subnormality is preserved under closed mappings,

We denote the union of family $\mathscr{U}$ by $\mathscr{U}^*$.

A space X is called weak $\bar{\theta}$-refinable if every open cover of X has a refinement $\bigcup_{n=1}^{\infty}\mathscr{G}_n$ satisfying (1) $\mathscr{G}_n$ is an open collection for each n, (2) each $x\in X$, has finite positive order with respect to some $\mathscr{G}_n$, (3) the open cover $\{\mathscr{G}_n^*\}_{n=1}^{\infty}$ is point finite. The open cover $\bigcup_{n=1}^{\infty}\mathscr{G}_n$ is called weak $\bar{\theta}$-cover.

A weak $\bar{\theta}$-cover $\bigcup_{n=1}^{\infty}\mathscr{G}_n$ is called boundly weak $\bar{\theta}$-cover[3] if there exists an integer N such that

$$\bigcup_{n=1}^{\infty}\{x\in X:\ 0<\operatorname{ord}(x,\ \mathscr{G}_n)\leqslant N\}=X.$$

Lemma 1. (Long Bing[3]) *For a space X, every boundly weak $\bar{\theta}$-cover has a refinement $\mathscr{F}=\bigcup_{n=1}^{\infty}\mathscr{F}_n$ such that $\mathscr{F}_0=\{\emptyset\}$, for each $n\geqslant 1$, $\mathscr{F}_n$ is a discrete closed family in $X\setminus\bigcup_{i=0}^{n-1}\mathscr{F}_i^*$ and $\mathscr{F}$ is a cover of X.*

A space X is collectionwise subnormal (J. Chaber [4]) if for each discrete closed family $\mathscr{F}$ of X, there exists a countable closed cover $\mathscr{E}$ of X such that for each $E\in\mathscr{E}$, $\mathscr{F}|_E$ has a pairwise disjoint open in E expansion.

J. Chaber proved in [4] that the above definition is equivalent to the statement that for each discrete closed family $\mathscr{F}$ of X, there exists a countable family $\{\mathscr{V}_n\}_{n=1}^{\infty}$ of open expansions of $\mathscr{F}$ such that for each $x\in X$, there exists an n

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with $\mathrm{ord}(x, \mathscr{V}_n) \leqslant 1$. In the following, we regard this characteristic property as the definition of collectionwise subnormal spaces.

Lemma 2. *Let X be a collectionwise subnormal space. If $\mathscr{F}=\{F_r: r \in \Gamma\}$ is a discrete closed family which refines open cover $\mathscr{V}$, then there exists a σ-discrete closed family $\mathscr{F}'$ and G_δ-set A such that*

(1) *$\mathscr{F}'$ refines $\mathscr{V}$,*

(2) *$\mathscr{F}^* \subset A \subset \mathscr{F}'^*$.*

Proof Since $\mathscr{F}$ refines $\mathscr{V}$, we see that for each $r \in \Gamma$ there exists a $V_r \in \mathscr{V}$ such that $F_r \subset V_r$. By the property of collectionwise subnormal spaces, there exists a countable family $\{\mathscr{V}_n\}_{n=1}^{\infty}$ of open expansion of $\mathscr{F}$ such that for each $x \in X$ there is an n with $\mathrm{ord}(x, \mathscr{V}_n) \leqslant 1$. Clearly, we may assume that each $\mathscr{V}_n$ refines $\mathscr{V}$, $\mathscr{V}_n = \{V_{nr}: r \in \Gamma\}$.

Let $S_n = X \setminus \mathscr{V}_n^*$. Then S_n and $\mathscr{F}^*$ are disjoint closed subsets. Note that X is subnormal (i. e., each disjoint closed subset pair has a disjoint G_δ-expansion). Hence, we have G_δ-sets $\bigcap_{i=1}^{\infty} Q_{ni}$ and $\bigcap_{i=1}^{\infty} W_{ni}$, where each Q_{ni} or W_{ni} is open in X, such that $\mathscr{F}^* \subset \bigcap_{i=1}^{\infty} Q_{ni}$, $S_n \subset \bigcap_{i=1}^{\infty} W_{ni}$, and $\left(\bigcap_{i=1}^{\infty} Q_{ni}\right) \cap \left(\bigcap_{i=1}^{\infty} W_{ni}\right) = \emptyset$. Let

$$\mathscr{G}_{ni} = \{W_{ni} \cap V: V \in \mathscr{V}\} \cup \{V_{nr}: r \in \Gamma\},$$

$$\mathscr{F}'_{ni} = \{X \setminus \cup \{\mathscr{G}_{ni} \setminus \{g\}\}: g \in \mathscr{G}_{ni}\}, \text{ and}$$

$$\mathscr{F}' = \bigcup_{n,i=1}^{\infty} \mathscr{F}'_{ni}.$$

Clearly, $\mathscr{G}_{ni}$ is an open cover of X. It follows that each $\mathscr{F}'_{ni}$ is a discrete closed family and $\mathscr{F}'$ is a σ-discrete closed family.

Let $A = \bigcap_{n,i=1}^{\infty} Q_{ni}$. Then A is a G_δ-set and $\mathscr{F}^* \subset A$. We show that $A \subset \mathscr{F}'^*$. For each $x \in A$, by the property of $\{\mathscr{V}_n\}_{n=1}^{\infty}$, there exists an n_0 such that $\mathrm{ord}(x, \mathscr{V}_{n_0}) \leqslant 1$. Since $x \notin \bigcap_{i=1}^{\infty} W_{n_0 i}$, there exists an i_0 such that $x \notin W_{n_0 i_0}$. Since $\mathscr{G}_{n_0 i_0}$ is a cover and $x \notin W_{n_0 i_0}$, x must be in some $V_{n_0 r}$, but $\mathrm{ord}(x, \mathscr{V}_{n_0}) \leqslant 1$. Hence, there is one and only one member $V_{n_0 r_0}$ of $\mathscr{V}_{n_0}$ such that $x \in V_{n_0 r_0}$. Therefore, $x \in X \setminus \cup \{\mathscr{G}_{n_0 i_0} \setminus \{V_{n_0 r_0}\}\} \subset \mathscr{F}'^*_{n_0 i_0} \subset \mathscr{F}'^*$. This ompletes the proof.

The main result in this paper is the following theorem.

Theorem 1. *For every space X, the following conditions are equivalent:*

(1) *X is collectionwise subnormal.*

(2) *Every boundly weak $\bar{\theta}$-cover of X has a σ-discrete closed refinement.*

(3) *Every boundly weak $\bar{\theta}$-cover of X has a σ-locally finite closed refinement.*

(4) *Every boundly weak $\bar{\theta}$-cover of X has a σ-closure preserving closed refinement.*

(5) *Every boundly weak $\bar{\theta}$-cover of X has a σ-cushioned refinement.*

Proof It is clear that (2)⇒(3)⇒(4)⇒(5). We show that (1)⇒(2) and (5) ⇒(1).

(1)⇒(2). Let $\mathscr{G}$ be a boundly weak $\bar{\theta}$-cover of X. By Lemma 1, there exists a refinement $\mathscr{B}=\bigcup_{n=0}^{\infty}\mathscr{B}_n$ such that $\mathscr{B}_0=\{\emptyset\}$, for each $n\geqslant 1$, $\mathscr{B}_n$ is a discrete closed family in $X\setminus\bigcup_{i=1}^{n-1}\mathscr{B}_i^*$ and $\mathscr{B}$ is a cover. We denote the set of all positive intergers by N. For any i_0, $i_1\in N$, we set $\mathscr{F}_{i_0i_1}=\{\emptyset\}$, G_δ-set $A_{i_0}=\emptyset$ and open set $G_{i_0i_1}=\emptyset$. Suppose that there exist discrete closed families $\mathscr{F}_{i_0\cdots i_k}$, $(i_0, \cdots, i_k)\in N^{k+1}$, $1\leqslant k\leqslant n$, and G_δ-sets $A_{i_0\cdots i_k}=\bigcap_{j=1}^{\infty}G_{i_0\cdots i_k,j}$ $(i_0, \cdots, i_k)\in N^{k+1}$, $1\leqslant k\leqslant n-1$, $j\in N$, such that

(i) each $\mathscr{F}_{i_0\cdots i_k}$ refines $\mathscr{G}$,

(ii) for each tuple $(i_0, \cdots, i_k)\in N^{k+1}$, $k<n$,

$$\bigcup_{j=1}^{\infty}\mathscr{F}^*_{i_0\cdots i_kj}\supset A_{i_0\cdots i_k}\supset\mathscr{B}_k^*\setminus\bigcup_{j=1}^{k}G_{i_0\cdots i_j},$$

For $k=n+1$, we construct discrete closed families $\mathscr{F}_{i_0\cdots i_{n+1}}$ and G_δ-sets

$$A_{i_0\cdots i_n}=\bigcap_{j=1}^{\infty}G_{i_0\cdots i_n,j}.$$

At first, we note that for each tuple $(i_1, \cdots, i_m)$ with $m\leqslant n$ we have $\bigcup_{i=1}^{m}G_{i_0\cdots i_j}\supset\bigcup_{i=0}^{m-1}\mathscr{B}_i^*$. In fact, it is clearly true for $m=1$. If it is also true for $m=s<n$, then

$$\bigcup_{j=1}^{s+1}G_{i_0\cdots i_j}=\bigcup_{j=1}^{s}G_{i_0\cdots i_j}\cup G_{i_0\cdots i_{s+1}}\supset\bigcup_{j=1}^{s}G_{i_0\cdots i_j}\cup A_{i_0\cdots i_s}$$

$$\supset\bigcup_{j=1}^{s}G_{i_0\cdots i_j}\supset\left(\mathscr{B}_s^*\setminus\bigcup_{j=1}^{s}G_{i_0\cdots i_j}\right)\supset\bigcup_{j=1}^{s}G_{i_0\cdots i_j}\cup\mathscr{B}_s^*\supset\bigcup_{i=0}^{s}\mathscr{B}_i^*.$$

This means that $\bigcup_{j=1}^{n}G_{i_0\cdots i_j}\supset\bigcup_{i=0}^{n-1}\mathscr{B}_i^*$. Since $\mathscr{B}_n$ is a discrete closed family in $X\setminus\bigcup_{i=0}^{n-1}\mathscr{B}_i^*$, $\mathscr{B}_n'=\left\{B\setminus\bigcup_{j=1}^{n}G_{i_0\cdots i_j}: B\in\mathscr{B}_n\right\}$ is a discrete closed family in X. By Lemma 2, there exists a σ-discrete closed family $\bigcup_{j=1}^{\infty}\mathscr{F}_{i_0\cdots i_nj}$ and a G_δ-set $A_{i_0\cdots i_n}$ such that $\bigcup_{j=1}^{\infty}\mathscr{F}_{i_0\cdots i_nj}$ refines $\mathscr{G}$ and $\bigcup_{j=1}^{\infty}\mathscr{F}^*_{i_0\cdots i_nj}\supset A_{i_0\cdots i_n}\supset\mathscr{B}_n'^*$. Since $\mathscr{B}_n'^*=\mathscr{B}_n^*\setminus\bigcup_{j=1}^{n}G_{i_0\cdots i_j}$, these $\mathscr{F}_{i_0\cdots i_{n+1}}$ and $A_{i_0\cdots i_n}$ satisfy (i) and (ii). By induction, we see that for each tuple $(i_0, \cdots, i_n)$ there exists a discrete closed family $\mathscr{F}_{i_0\cdots i_n}$ and a G_δ-set $A_{i_0\cdots i_{n-1}}$, and they satisfy (i) and (ii).

We set $\mathscr{F}=\bigcup\{\mathscr{F}_{i_0\cdots i_n}: (i_0, \cdots, i_n)\in N^{n+1}, n=1, 2, 3\cdots\}$. Then it is clear that $\mathscr{F}$ is a σ-discrete closed family which refines $\mathscr{G}$. Now we prove $\mathscr{F}$ is a cover. For any $x\in X$, there is an n_0 with $x\in\mathscr{B}^*_{n_0}$. If $x\in A_{i_0\cdots i_k}$ with some $(i_0, \cdots, i_k)\in N^{k+1}$ where $k<n_0$, then, by (ii), $x\in A_{i_0\cdots i_k}\subset\bigcup_{j=1}^{\infty}\mathscr{F}^*_{i_0\cdots i_kj}\subset\mathscr{F}^*$. If

$$x \notin \bigcup \left\{ A_{i_0 \cdots i_k} \mid (i_0, \cdots, i_k) \in \bigcup_{j=0}^{n_0-1} N^{j+1} \right\},$$

then there exists a tuple $(i_0, \cdots, i_{n_0})$ with $x \notin \bigcup_{j=1}^{n_0} G_{i_0 \cdots i_j}$ and hence $x \in \mathscr{B}^* \setminus \bigcup_{j=1}^{n_0} G_{i_0 \cdots i_j}$. By (ii), $x \in A_{i_0 \cdots i_{n_0}} \subset \bigcup_{j=0}^{\infty} \mathscr{F}^*_{i_0 \cdots i_{n_0} j} \subset \mathscr{F}^*$. It follows that $\mathscr{F}$ is a cover.The proof of (1) $\Rightarrow$ (2) is complete.

(5)$\Rightarrow$(1). Let $\mathscr{F} = \{F_r: r \in \Gamma\}$ be a discrete closed family. For each $r \in \Gamma$, put $G_r = X \setminus \bigcup_{r' \neq r} F_{r'}$ and $G = X \setminus \mathscr{F}^*$. It is clear that $\mathscr{G} = \{G\} \cup \{G_r: r \in \Gamma\}$ is a boundly weak $\bar{\theta}$-cover of X. By (5), there exists a σ-cushioned refinement $\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$. Let $f_n: \mathscr{H} \to \mathscr{G}$ be the cushioned mapping (i.e., if $\mathscr{H}'_n \subset \mathscr{H}_n$, then $\overline{\bigcup \mathscr{H}'} \subset \bigcup \{f_n(h): h \in \mathscr{H}'\}$).

For each $r \in \Gamma$, let $V_{nr} = X \setminus \bigcup \{h \in \mathscr{H}_n: f_n(g) \neq G_r\}$. Then $V_{nr} \supset X \setminus \bigcup_{r' \neq r} G_{r'} \supset F_r$, this means that $\mathscr{V}_n = \{V_{nr}: r \in \Gamma\}$ is an open expansion of $\mathscr{F}$. Hence, $\{\mathscr{V}_n\}_{n=1}^{\infty}$ is the required one. In fact, for each $x \in X$, there is an n_0 such that $x \in \mathscr{H}^*_{n_0}$. Then if r, $r' \in \Gamma$ and $r \neq r'$, we have

$$V_{n_0 r} \cap V_{n_0 r'} \subset (X \setminus \bigcup \{h \in \mathscr{H}_{n_0}: f_{n_0}(h) \neq G_r\})$$
$$\cap (X \setminus \bigcup \{h \in \mathscr{H}_{n_0}: f_{n_0}(h) \neq G_{r'}\}) = X \setminus \mathscr{H}^*_{n_0}.$$

It follows that $x \notin V_{n_0 r} \cap V_{n_0 r'}$, this means that ord $(x, \mathscr{V}_{n_0}) \leqslant 1$. Therefore, X is collectionwise subnormal. The proof of Theorem 1 is complete.

Corollary 1. *Let X be a collectionwise subnormal space and f be a closed mapping from X onto Y. Then Y is collectionwise subnormal.*

Proof Let $\mathscr{G}$ be any boundly weak $\bar{\theta}$-cover of Y. Then we can easily see that $f^{-1}(\mathscr{G}) = \{f(g): g \in \mathscr{G}\}$ is also a boundly weak $\bar{\theta}$-cover of X. By Theorem 1 there exists a σ-closure preserving closed refinement $\mathscr{F}$ of $f^{-1}(\mathscr{G})$. Then $f(\mathscr{F}) = \{f(F): F \in \mathscr{F}\}$ is a σ-closure preserving closed refinement of $\mathscr{G}$. By Theorem 1, Y is collectionwise subnormal.

Recall that a space X is strong quasi-paracompact (Liu Ying-ming[5]) if every open cover of X has a refinement $\mathscr{F} = \bigcup_{n=0}^{\infty} \mathscr{F}_n$ such that $\mathscr{F}_0 = \{\varnothing\}$ for each $n \geqslant 1$, $\mathscr{F}_n$ is a discrete closed family in $X \setminus \bigcup_{i=0}^{n-1} \mathscr{F}^*_i$ and $\mathscr{F}$ is a cover of X. Long Bing proved in [3] that a space X is strong quasi-paracompact if every open cover of X has a boundly weak $\bar{\theta}$-refienment. We hence have the following corollary.

Corollary 2. *A space X is subparacompact iff it is strong quasi-paracompact and collectionwise subnormal.*

The author proved in [6] that the class of strong quasi-paracompact spaces lies between the class of θ-refinable spaces and the class of weak $\bar{\theta}$-refinable spaces. Therefore Corollary 2 slightly improves a result of J. Chaber in [4].

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