Chin. Ann. of Math.
9B (2) 1988

THE STABILITY OF LARGE SCALE SYSTEMS WITH INFINITE DELAY

ZHANG YI (章 毅)*

Abstract

In this paper, the author uses the method of making Liapunov functional to study the stability of large scale systems with infinite delay. Some simple criteria for stability are obtained.

With fast development of science and technology, appeared large scale systems with complex structure in many subjects [1-4]. Nowadays, we are still at the beginning of studying the theory of large scale systems and are short of methods. As the problems of the stability of large scale systems with delay are difficult and complicated, little achievements bave been made.

In this paper, we shall study the stability of large scale systems with infinite delay, some stability criteria are obtained. We make the course simple and clear and prevent it from the complex calculation often done in the study of large scale systems, and offer some other ways to the study of stability of large scale systems with delay.

Consider the following system

$$\dot{x}_i(t)=A_{ii}(t)x_i(t)+\sum_{\substack{j=1\\ j\neq i}}A_{ij}(t)x_j(t)+\sum_{j=1}^{r}B_{ij}(t)x_j(t-\tau(t))\quad (i=1,\cdots,r). \tag{1}$$

Assume the isolated subsystems as follows

$$\dot{x}_i(t)=A_{ii}(t)x_i(t)\quad (i=1,\cdots,r), \tag{2}$$

where $x_i=\mathrm{col}(x_1^{(i)},\cdots\cdots,x_{n_i}^{(i)})$ $(i=1,\cdots,r)$, $\sum_{i=1}^{r}n_i=n$, $x^T=(x_1^T,\cdots,x_r^T)$ and $A_{ij}(t)$, $B_{ij}(t)$ $(i、j=1,\cdots,r)$ are $n_i\times n_j$ real continuous function matrices. The delay $\tau(t)$ is a nonnegative and continuous function.

The initial condition is

$$x_i(t)=\varphi_i(t),\ -\infty<t\leqslant t_0\ (i=1,\cdots,r), \tag{3}$$

where $\varphi_i(t)$ $(i=1,\cdots,r)$ are bounded and continuous on $(-\infty,t_0]$.

In this paper, we always assume that solution of system (1) and (3) exists and

Manuscript received December 14, 1985. Revised October 13, 1986.

* Department of Mathematics, Chengdu Institue of Telecommunications Engineering, Chengdu, Sichuan, China.

is unique.

Suppose $\|A_{ij}(t)\| \leqslant a_{ij} (i \neq j,\ i, j=1, \cdots\cdots, r)$, $\|B_{ij}(t)\| \leqslant b_{ij}\ (i, j=1, \cdots, r)$, where $a_{ij} \geqslant 0$, $b_{ij} \geqslant 0$ are constants. Define $\|\varphi_i\| = \sup_{-\infty < t \leqslant t_0} \|\varphi_i(t)\|\ (i=1, \cdots, r)$.

To the isolated subsystems (2), we assume that $Y_i(t, s)$, $(t, s \geqslant t_0)$ $(i=1, \cdots, r)$, satisfy

$$\begin{cases} \dfrac{\partial Y_i(s, t)}{\partial t} = A_{ii}(t) Y_i(s, t), \\ Y_i(s, s) = I_i \text{ (identity matrix)}. \end{cases} \tag{4}$$

Definition 1. *If there exist constants $a_{ii} < 0$ $(i=1, \cdots, r)$ such that*

$$\|Y_i(s, t)\| \leqslant \exp(a_{ii}(t-s)) \quad (i=1, \cdots, r),$$

then the isolated subsystems are called to have property A.

Definition 2. *If there exist nonpositive functions $r_i(t)$ $(i=1, \cdots, r)$ such that*

$$\|Y_i(s, t)\| \leqslant \exp\left(\int_s^t r_i(\theta) d\theta\right) \quad (i=1, \cdots, r),$$

then the isolated subsystems are called to have property B.

Theorem 1. *If the system* (1) *satisfies the following conditions:*

(i) *the isolated subsystems have property A;*

(ii) $\sup_{t \geqslant t_0} \dot{\tau}(t) < 1$;

(iii) $a_{ii} + k b_{ii} < 0$ $(i=1, \cdots, r)$ *and* $\operatorname{Re}\lambda (a_{ij} + k b_{ij})_{r \times r} < 0$, *where* $k = (1 - \sup_{t \geqslant t_0} \dot{\tau}(t))^{-1}$,

then the zero solution of (1) *is asymptotically stable.*

Proof From condition (iii), it follows that $(a_{ij} + k b_{ij})_{r \times r}$ is a stable Metzler matrix. According to [5], there exist constants $\alpha_i > 0$ $(i=1, \cdots, r)$ such that

$$\sum_{i=1}^{r} \alpha_i (a_{ij} + k b_{ij}) < 0 \quad (j=1, \cdots, r). \tag{5}$$

Applying variation of parameters to system (1) gives

$$x_i(t) = Y_i(t_0, t)\varphi_i(t_0) + \int_{t_0}^{t} Y_i(s, t) \Big[\sum_{\substack{j=1 \\ j \neq i}}^{r} A_{ji}(s)\, x_j(s) + \sum_{j=1}^{r} B_{ij}(s) x_j(s - \tau(s)) \Big] ds, \ t \geqslant t_0.$$

From condition (i), we have

$$\begin{aligned} \|x_i(t)\| \leqslant & \|Y(t_0, t)\| \cdot \|\varphi_i(t_0)\| + \int_{t_0}^{t} \exp(a_{ii}(t-s)) \Big[\sum_{\substack{j=1 \\ j \neq i}}^{r} \|A_{ij}(s)\| \cdot \|x_j(s)\| \\ & + \sum_{j=1}^{r} \|B_{ij}(s)\| \cdot \|x_j(s - \tau(s))\| \Big] ds \\ \leqslant & \|\varphi_i\| \exp(a_{ii}(t - t_0)) + \int_{t_0}^{t} \exp(a_{ii}(t-s)) \Big[\sum_{\substack{j=1 \\ j \neq i}}^{r} a_{ij} \|x_j(s)\| \\ & + \sum_{j=1}^{r} b_{ij} \|x_j(s - \tau(s))\| \Big] ds. \end{aligned}$$

Let

$$P_i(t)=\begin{cases}\|\varphi_i\|, & -\infty<t\leqslant t_0,\\ \|\varphi_i\|\exp(a_{ii}(t-t_0))+\int_{t_0}^{t}\exp(a_{ii}(t-s))\Big[\sum_{\substack{j=1\\ j\neq i}}^{r}a_{ij}\|x_j(s)\|+\sum_{j=1}^{r}b_{ij}\|x_j(s-\tau(s))\|\Big]ds, & t\geqslant t_0.\end{cases}$$

Then $\|x_i(t)\|\leqslant P_i(t)$, $-\infty<t<+\infty$ $(i=1,\cdots,r)$. Moreover

$$\begin{aligned}\dot{P}_i(t)&=a_{ii}P_i(t)+\sum_{\substack{j=1\\ j\neq i}}^{r}a_{ij}\|x_j(t)\|+\sum_{j=1}^{r}b_{ij}\|x_j(t-\tau(t))\|\\ &\leqslant a_{ij}P_i(t)+\sum_{\substack{j=1\\ j\neq i}}^{r}a_{ij}P_j(t)+\sum_{j=1}^{r}b_{ij}P_j(t-\tau(t))\\ &=\sum_{j=1}^{r}a_{ij}P_j(t)+\sum_{j=1}^{r}b_{ij}P_j(t-\tau(t)).\end{aligned}$$

Let

$$V(t)=\sum_{i=1}^{r}\Big\{\alpha_i\Big[P_i(t)+k\int_{t-\tau(t)}^{t}\Big(\sum_{j=1}^{r}b_{ij}P_j(s)\Big)ds\Big]\Big\}.$$

Then

$$\begin{aligned}\dot{V}(t)&=\sum_{i=1}^{r}\Big\{\alpha_i\Big[\dot{P}_i(t)+k\sum_{j=1}^{r}b_{ij}(P_j(t)-(1-\dot{\tau}(t))P_j(t-\tau(t)))\Big]\Big\}\\ &\leqslant\sum_{i=1}^{r}\Big\{\alpha_i\Big[\sum_{j=1}^{r}(a_{ij}+kb_{ij})P_j(t)+\sum_{j=1}^{r}b_{ij}(1-k(1-\dot{\tau}(t)))P_j(t-\tau(t))\Big]\Big\}\\ &\leqslant\sum_{j=1}^{r}\Big[\sum_{i=1}^{r}\alpha_i(a_{ij}+kb_{ij})\Big]P_j(t)\\ &\leqslant-\beta\sum_{j=1}^{r}P_j(t),\end{aligned}\tag{6}$$

where

$$\beta=-\max_{1\leqslant j\leqslant r}\Big[\sum_{i=1}^{r}\alpha_i(a_{ij}+kb_{ij})\Big]>0.$$

It follows, by an integration of (6), that

$$V(t)\leqslant V(t_0)-\beta\int_{t_0}^{t}\Big(\sum_{j=1}^{r}P_j(s)\Big)ds,$$

that is

$$V(t)+\beta\int_{t_0}^{t}\Big(\sum_{j=1}^{r}P_j(s)\Big)ds\leqslant V(t_0).\tag{7}$$

Notice that

$$V(t)\geqslant\sum_{i=1}^{r}\alpha_iP_i(t)\geqslant\bar{\alpha}\sum_{i=1}^{r}P_i(t)\geqslant\bar{\alpha}\sum_{i=1}^{r}\|x_i(t)\|,$$

$$V(t_0)=\sum_{i=1}^{r}\Big\{\alpha_i\Big[P_i(t_0)+k\int_{t_0-\tau(t_0)}^{t_0}\Big(\sum_{j=1}^{r}b_{ij}P_j(s)\Big)ds\Big]\Big\}\leqslant(\bar{\bar{\alpha}}+krb\tau(t_0))\sum_{i=1}^{r}\|\varphi_i\|,$$

where

$$\bar{\alpha}=\min_{1\leqslant i\leqslant r}(\alpha_i),\quad \bar{\bar{\alpha}}=\max_{1\leqslant i\leqslant r}(\alpha_i),\quad b=\max_{1\leqslant i,j\leqslant r}(b_{ij}).$$

Then

$$\bar{\alpha}\sum_{i=1}^{r}\|x_i(t)\|\leqslant V(t)\leqslant V(t_0)\leqslant(\bar{\bar{\alpha}}+krb\tau(t_0))\sum_{i=1}^{r}\|\varphi_i\|.$$

Hence, the zero solution of (1) is stable.

Since $V(t_0)\geqslant V(t)\geqslant \bar{a}\sum_{i=1}^{r}P_i(t)$, $\sum_{i=1}^{r}P_i(t)$ is bounded. From

$$\left(\sum_{i=1}^{r}P_i(t)\right)'\leqslant\sum_{i=1}^{r}\sum_{j=1}^{r}[a_{ij}P_j(t)+b_{ij}P_j(t-\tau(t))],$$

it follows that $\left(\sum_{i=1}^{r}P_i(t)\right)'$ is also bounded. Thus $\sum_{i=1}^{r}P_i(t)$ is uniformly continuous on $[t_0, +\infty)$.

From (7), we get to know that $\int_{t_0}^{+\infty}\left(\sum_{i=1}^{r}P_i(s)\right)ds$ is convergent. Then

$$\sum_{i=1}^{r}P_i(t)\to 0\ (t\to+\infty)$$

and

$$\sum_{i=1}^{r}\|x_i(t)\|\to 0\ (t\to+\infty).$$

Thus, the zero solution of (1) is asymptotically stable and the proof is completed.

If we let norm be $\|x\|=\sqrt{\sum|x_i|^2}$, we can get the following results.

Corollary 1. *If the system* (1) *satisfies the following conditions:*

(i) $\mathrm{Re}\lambda[A_{ii}(t)+A_{ii}^T(t)]\leqslant 2a_{ii}<0$, $(i=1, \cdots, r)$, $t\geqslant t_0$, a_{ii} *are constants;*

(ii) $\sup_{t\geqslant t_0}\dot{\tau}(t)<1$;

(iii) $a_{ii}+kb_{ii}<0$ $(i=1,\cdots, r)$ *and* $\mathrm{Re}\lambda(a_{ij}+kb_{ij})_{r\times r}<0$ *where* $k=(1-\sup_{t\geqslant t_0}\dot{\tau}(t))^{-1}$,

then the zero solution of (1) *is asymptotically stable.*

According to [6], under condition (i), we have

$$\|Y_i(s, t)\|\leqslant\exp(a_{ii}(t-s))\qquad(i=1, \cdots, r).$$

Therefore the corollary holds.

Corollary 2. *Consider the system*

$$\begin{cases}\dot{x}_i(t)=\sum_{j=1}^{n}a_{ij}(t)x_j(t)+\sum_{j=1}^{n}b_{ij}(t)x_j(t-\tau(t)),\\ x_i(t)=\varphi_i(t),\ -\infty<t\leqslant t_0,\end{cases}\tag{8}$$

where $\varphi_i(t)$ $(i=1, \cdots, n)$ *are bounded and continuous on* $(-\infty, t_0]$. a_{ij}, b_{ij} $(i, j=1, \cdots, n)$ *are constants. If the system satisfies the following conditions:*

(i) $a_{ii}<0$ $(i=1, \cdots, n)$;

(ii) $\sup_{t\geqslant t_0}\dot{\tau}(t)<1$;

(iii) $\mathrm{Re}\lambda(D)<0$, *where*

$$D=\begin{pmatrix}a_{11}+k|b_{11}| & |a_{12}|+k|b_{12}|\cdots|a_{1n}|+k|b_{1n}|\\ \cdots\cdots & \cdots\cdots\\ |a_{n1}|+k|b_{n1}| & |a_{n2}|+k|b_{n2}|\cdots a_{nn}+k|b_{nn}|\end{pmatrix},\ k=(1-\sup_{t\geqslant t_0}\dot{\tau}(t))^{-1};$$

then zero solution of (8) *is asymptotically stable.*

Theorem 2. *If the system* (1) *satisfies the following conditions:*

(i) *the isolated subsystems have property B;*

(ii) $\sup_{t\geqslant t_0}\dot{\tau}(t)<1$;

(iii) $\|A_{ij}(t)\|$ $(i\neq j;\ i,\ j=1,\ \cdots,\ r)$ *are bounded;* $\|B_{ij}(t)\|$ $(i,\ j=1,\ \cdots,\ r)$ *are nonincreasing functions; and*

$$r_j(t)+\sum_{\substack{i=1\\ i\neq i}}^{r}\|A_{ij}(t)\|+k\sum_{i=1}^{r}\|B_{ij}(t)\|\leqslant-\beta<0,\ t\geqslant t_0\ (j=1,\ \cdots,\ r),$$

where $k=(1-\sup_{t\geqslant t_0}\dot{\tau}(t))^{-1}$, β *is a constant;*

then the zero solution of (1) *is asymptotically stable.*

Proof By variation of parameters, we have

$$\|x_i(t)\|\leqslant\|\varphi_i\|\exp\left(\int_{t_0}^{t}r_i(\theta)d\theta\right)+\int_{t_0}^{t}\exp\left(\int_{s}^{t}r_i(\theta)d\theta\right)$$
$$\times\Big[\sum_{\substack{j=1\\ j\neq i}}^{r}\|A_{ij}(s)\|\cdot\|x_j(s)\|+\sum_{j=1}^{r}\|B_{ij}(s)\|\cdot\|x_j(s-\tau(s))\|\Big]ds,\ t\geqslant t_0.$$

Let

$$P_i(t)=\begin{cases}\|\varphi_i\|, & -\infty<t\leqslant t_0,\\ \|\varphi_i\|\exp\left(\int_{t_0}^{t}r(_i\theta)d\theta\right)+\int_{t_0}^{t}\exp\left(\int_{s}^{t}r_i(\theta)d\theta\right) & \\ \quad\times\Big[\sum_{\substack{j=1\\ j\neq i}}^{r}\|A_{ij}(s)\|\cdot\|x_j(s)\|+\sum_{j=1}^{r}\|B_{ij}(s)\|\cdot\|x_j(s-\tau(s))\|\Big]ds, & t\geqslant t_0.\end{cases}$$

Let

$$V(t)=\sum_{i=1}^{r}\Big[P_i(t)+k\int_{t-\tau(t)}^{t}\Big(\sum_{j=1}^{r}\|B_{ij}(s)\|P_j(s)\Big)ds\Big].$$

Then, in a way smilar to the proof of Theorem 1, this theorem can be proved.

Using methods similar to the above, we can study the following nonlinear large scale system with infinite delay

$$\dot{x}_i(t)=A_{ii}(t)x_i(t)+f_i[t,\ x(t),\ x(t-\tau(t))]\ (i=1,\ \cdots,\ r),\tag{9}$$

where $\tau(t)\geqslant0$ and $f_i(t,\ 0,\ 0)\equiv0$. The isolated subsystems are still (2).

Suppose that

$$\|f_i[t,\ x(t),\ x(t-\tau(t))]\|\leqslant\sum_{j=1}^{r}\bar{a}_{ij}\|x_j(t)\|+\sum_{j=1}^{r}b_{ij}\|x_j(t-\tau(t))\|,$$

where $\bar{a}_{ij}\geqslant0$, $b_{ij}\geqslant0$ $(i,\ j=1,\ \cdots,\ r)$ are constants.

Theorem 3. *If the system* (9) *satisfies the following conditions:*

(i) *the isolated subsystems have property* A;

(ii) $\sup_{t\geqslant t_0}\dot{\tau}(t)<1$;

(iii) $a_{ii}^{(1)}+kb_{ii}<0$ $(i=1,\ \cdots,\ r)$ *and* $\mathrm{Re}\lambda(a_{ij}^{(1)}+kb_{ij})<0$,

where

$$a_{ij}^{(1)}=\begin{cases}a_{ii}+\bar{a}_{ii}, & i=j,\\ \bar{a}_{ij}, & i\neq j,\end{cases}\quad k=(1-\sup_{t\geqslant t_0}\dot{\tau}(t))^{-1};$$

then the zero solution of (9) *is asymptotically stable.*

In this paper, we have considered the systems with infinite delay, but if the delay are bounded, all the theorems of above still holds.

References

[1] Qin Yuanxun, Liu Yongcheng & Wang Lian, *The Stability of Dynamical Systems with Finite Delay*, Science Press, 1963 (in Chinese).

[2] Liao Xiaoxin, *Xuxu Tongbao*, **27**: 10 (1982), 577—580.

[3] Burton, T. A., *Funkcialaj Ekvacioj*, **22** (1979), 67—76.

[4] Halanay, A., Differential Equations: Stability, Oscillations, Time Lags, Academic Press, New York and London, 1965.

[5] Siljak, D. D., Large Scale Dynamic Systemsl Stability and Structure, Elsevier North-Holland Inc, New York, 1978.

[6] Лозинский, С. М., Изв высших Учебных заведени й Матаматика, **5** (1958), 52—96.