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UNIFORM ASYMPTOTIC STABILITY OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract

With the help of a Liapunov functional with semi-negative definite derivative, Barbashin-Krasovskii's theorem is extended to nonautonomous functional differential equations, a reducing dimension approach is presented for the uniform asymptotic stability of high dimension systems, and some sufficient conditions of uniform asymptotic stability are obtained.

For $x\in R^n$, $r\geqslant 0$, $|x|$ denotes the Euclidean norm of x, $R^n_H=\{x\in R^n;\ |x|<H\}$ for $H>0$, C_n denotes the space of continuous functions mapping $[-r,\ 0]$ into R^n with the super-norm $\|\cdot\|$, $C^H_n=\{\varphi\in C_n;\ \|\varphi\|<H\}$. If $x(u)$ is a continuous n-vector function defined on $-r\leqslant u<A$ $(A>0)$, then for $t\in[0,\ A)$, x_t denotes the restriction of x to the interval $[t-r,\ t]$ so that x_t is an element of C_n defined by $x_t(s)=x(t+s)$, for $-r\leqslant s\leqslant 0$.

Consider the system

$$\dot{x}(t)=X(t,\ x_t) \tag{1}$$

where X: $R^+\times C^H_n\to R^n$ is continuous with $X(t,\ 0)=0$. We denote by $x(t_0,\ \varphi)$ a solution of (1) with initial condition $\varphi\in C^H_n$, where $x_{t_0}(t_0,\ \varphi)=\varphi$ and by $x(t;\ t_0,\ \varphi)$ the value of $x(t_0,\ \varphi)$ at t. For simplicity, we assume that for any $(t_0,\ \varphi)\in R^+\times C^H_n$, $x(t_0,\ \varphi)$ exists uniquely. For a continuous functional V: $R^+\times C^H_n\to R$, the derivative of V along the solutions of (1) is defined as

$$\dot{V}_{(1)}(t,\ \varphi)=\lim_{h\to 0^+}\sup\frac{1}{h}\{V(t+h,\ x_{t+h}(t,\ \varphi))-V(t,\ \varphi)\}.$$

A wedge W is a strictly increasing continuous function defined on R^+ with $W(0)=0$. Throughout this paper, an integrally positive function λ is a nonnegative measurable function, defined on R^+ such that $\int_J\lambda(t)\,dt=+\infty$ for every $J=\bigcup_{m=1}^{\infty}[a_m,\ b_m]$ with $a_m<b_m<a_{m+1}$ and $b_m-a_m\geqslant\delta$ for all $m=1,\ 2,\ \cdots$ and for a constant $\delta>0$.

The following lemma is proved in [5].

Lemma 1. *If a measurable function* λ: $R^+\to R^+$ *is integrally positive, then for*

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every $\alpha>0$ *and* $\delta>0$ *there exists a positive integer* $K(\alpha, \delta)$ *such that for every set* $J=\bigcup_{m=1}^{K}[a_m, b_m]$ *with* $0\leqslant a_m<a_m+\delta\leqslant b_m\leqslant a_{m+1}$ *for* $1\leqslant m\leqslant K-1$, *we have* $\int_J\lambda(t)\,dt\geqslant\alpha$.

Theorem 1. *Suppose that there exist continuous functionals* V, P: $R^+\times C_n^H\to R^+$ *and wedges* $W_i(i=1, 2, 3)$ *such that*

(i) $W_1(|\varphi(0)|)\leqslant V(t, \varphi)\leqslant W_2(\|\varphi\|)$,

(ii) *for any* $\sigma>0$ *there exists* $\mu(\sigma)>0$ *such that for any* $(t, \varphi)\in R^+\times C_n^H$ *with* $P(t, \varphi)\geqslant\sigma$, *we have*

$$\dot V_{(1)}(t, \varphi)\leqslant-\lambda(t)P(t, \varphi)-\mu(\sigma)|W_3(P(t, \varphi))_{(1)}|,$$

(iii) *for any* $\varepsilon>0$ *there exist* $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$ *and* $T_1(\varepsilon)>0$ *such that for any* $t_0\geqslant0$ *and any solution* $x(t)$ *of* (1) *defined on* $[t_0-r, t_0+T_1(\varepsilon)]$ *with* $\|x_t\|<H$, $P(t, x_t)<\delta_1(\varepsilon)$ *for* $t\in[t_0, t_0+T_1(\varepsilon)]$ *and* $\int_{t_0}^{t_0+T_1(\varepsilon)}\lambda(s)P(s, x_s)\,ds<\delta_2(\varepsilon)$, *there exists* $\tau\in[t_0, t_0+T_1(\varepsilon)]$ *with* $\|x_\tau\|<\varepsilon$ *(or* $V(\tau, x_\tau)<\varepsilon$*)*.

Then the zero solution of (1) *is uniformly asymptotically stable.*

Proof. For any $\varepsilon>0$, choose $\delta(\varepsilon)>0$ so that $W_2(\delta(\varepsilon))<W_1(\varepsilon)$. It is easy to prove that $[t_0\geqslant0,\ \|\varphi\|<\delta(\varepsilon)]$ implies $|x(t;\, t_0, \varphi)|<\varepsilon$ for all $t\geqslant t_0$. So the zero solution of (1) is uniformly stable.

Let $\delta_0=\delta(H)$ and $\|\varphi\|<\delta_0$, $x(t)=x(t;\, t_0, \varphi)$, $V(t)=V(t, x_t)$. By (iii) there exist $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$ and $T_1(\varepsilon)>0$ such that for any $\bar t\geqslant t_0$, if $\int_{\bar t}^{\bar t+T_1(\varepsilon)}\lambda(s)P(s, x_s)\,ds<\delta_2(\varepsilon)$ and if $P(t, x_t)<\delta_1(\varepsilon)$ for $t\in[\bar t, \bar t+T_1(\varepsilon)]$, then there exists $\tau\in[\bar t, \bar t+T_1(\varepsilon)]$ with $\|x_\tau\|<\delta(\varepsilon)$.

Let $K=K\left(\dfrac{2W_2(H)+1}{\delta_1(\varepsilon)}, 1\right)$. Then we can assert that there exists $t_2\in[\bar t, \bar t+K]$ such that $P(t_2, x_{t_2})<\frac{1}{2}\delta_1(\varepsilon)$. Otherwise, if for all $t\in[\bar t, \bar t+K]$ we have $P(t, x_t)\geqslant\frac{1}{2}\delta_1(\varepsilon)$, then

$$\dot V(t)\leqslant-\lambda(t)P(t, x_t)\leqslant-\frac{1}{2}\delta_1(\varepsilon)\lambda(t)$$

and thus

$$V(\bar t+K)\leqslant V(\bar t)-\int_{\bar t}^{\bar t+K}\frac{1}{2}\delta_1(\varepsilon)\lambda(t)\,dt\leqslant W_2(H)-\frac{1}{2}\delta_1(\varepsilon)\int_{\bar t}^{\bar t+K}\lambda(t)\,dt<0.$$

This is contrary to $V\geqslant0$.

If $\int_{t_2}^{t_2+T_1(\varepsilon)}\lambda(t)P(t, x_t)\,dt<\delta_2(\varepsilon)$ and $P(t, x_t)<\delta_1(\varepsilon)$ for $t\in[t_2, t_2+T_1(\varepsilon)]$, then there exists $\tau\leqslant t_2+T_1(\varepsilon)\leqslant\bar t+K+T_1(\varepsilon)$ with $\|x_\tau\|<\delta(\varepsilon)$ and thus $|x(t)|<\varepsilon$ for $t\geqslant\tau$.

If there exists $t_3\in[t_2, t_2+T_1(\varepsilon)]$ with $P(t_3, x_{t_3})\geqslant\delta_1(\varepsilon)$, then there exist $t_2<t_4<t_5\leqslant t_3$ such that $P(t_4, x_{t_4})=\delta_1(\varepsilon)/2$, $P(t_5, x_{t_5})=\delta_1(\varepsilon)$ and $\frac{1}{2}\delta_1(\varepsilon)<P(t, x_t)<\delta_1(\varepsilon)$

for $t\in(t_4, t_5)$, and therefore

$$\dot V(t)\leqslant-\mu\left(\frac{1}{2}\delta_1(\varepsilon)\right)|W_3(P(t, x_t))| \text{ for } t\in[t_4, t_5].$$

This implies

$$V(t_5)\leqslant V(t_4)-\mu\left(\frac{1}{2}\delta_1(\varepsilon)\right)\left[W_3(\delta_1(\varepsilon))-W_3\left(\frac{1}{2}\delta_1(\varepsilon)\right)\right].$$

So

$$V(\bar t+T_1(\varepsilon)+K)\leqslant V(t_5)\leqslant V(\bar t)-\mu\left(\frac{1}{2}\delta_1(\varepsilon)\right)\left[W_3(\delta_1(\varepsilon))-W_3\left(\frac{1}{2}\delta_1(\varepsilon)\right)\right].$$

If $\int_{t_2}^{t_2+T_1(\varepsilon)}\lambda(t)P(t, x_t)dt\geqslant\delta_2(\varepsilon)$, then

$$V(\bar t+K+T_1(\varepsilon))\leqslant V(t_2+T_1(\varepsilon))\leqslant V(t_2)-\int_{t_2}^{t_2+T_1(\varepsilon)}\lambda(t)P(t, x_t)dt$$
$$\leqslant V(\bar t)-\delta_2(\varepsilon).$$

Let $\bar t=t_0+k[T_1(\varepsilon)+K]$. We have either

(A) $|x(t)|<\varepsilon$ for $t\geqslant t_0+(k+1)[K+T_1(\varepsilon)]$, or

(B) $V(t_0+(k+1)(K+T_1(\varepsilon)))\leqslant V(t_0+k(K+T_1(\varepsilon)))$
$$-\min\left\{\delta_2(\varepsilon),\ \mu\left(\frac{1}{2}\delta_1(\varepsilon)\right)\left[W_3(\delta_1(\varepsilon))-W_3\left(\frac{1}{2}\delta_1(\varepsilon)\right)\right]\right\}.$$

Choose a positive integer N with

$$N>\frac{W_2(H)}{\min\left\{\delta_2(\varepsilon),\ \mu\left(\frac{1}{2}\delta_1(\varepsilon)\right)\left[W_3(\delta_1(\varepsilon))-W_3\left(\frac{1}{2}\delta_1(\varepsilon)\right)\right]\right\}}.$$

Since $V\geqslant0$, (B) holds at most for a finite number of $k=0, 1, \cdots, N$, and therefore $|x(t)|<\varepsilon$ for $t\geqslant t_0+N[K+T_0(\varepsilon)]$. This completes the proof.

Corollary 1. *Suppose that* (i) *and* (iii) *of Theorem* 1 *hold and*

(a) $\dot V_{(1)}(t, \varphi)\leqslant-P(t, \varphi)$,

(b) *there exists a constant* $L\geqslant0$ *and a wedge* W_3 *such that*

$$|W_3(P(t, \varphi))_{(1)}|\leqslant L \text{ for } (t, \varphi)\in R^+\times C_n^H.$$

Then the zero solution of (1) *is uniformly asymptotically stable.*

Proof For any $\sigma>0$ and $(t, \varphi)\in R^+\times C_n^H$ with $P(t, \varphi)\geqslant\sigma$, we have

$$\dot V_{(1)}(t, \varphi)\leqslant-P(t, \varphi)\leqslant-\frac{1}{2}P(t, \varphi)-\frac{1}{2}P(t, \varphi)\cdot|W_3(P(t, \varphi))_{(1)}|/L$$
$$\leqslant-\frac{1}{2}P(t, \varphi)-\frac{\sigma}{2L}|W_3(P(t, \varphi))_{(1)}|.$$

Then by Theorem 1, we can complete the proof.

Remark 1. If $P(t, \varphi)=W(|\varphi(0)|)$ for a wedge W, then (iii) of Theorem 1 holds naturally, and thus Theorem 1 and Corollary 1 reduce to Theorem 8.3.8 and Theorem 8.3.2 of [2], respectively.

Example 1. Consider now the system

$$\begin{cases} \dot{x}_1(t)=x_2(t), \\ \dot{x}_2(t)=-\frac{k}{m}x_1(t)-\frac{b(t)}{m}x_2(t)-\frac{q(t)}{m}x_2(t-r) \end{cases} \tag{2}$$

obtained from the equation

$$m\ddot{z}(t)+b(t)\dot{z}(t)+q(t)\dot{z}(t-r)+kz(t)=0, \tag{3}$$

where m and k are positive constants, $b(t)$ and $q(t)$ are positive bounded continuous functions and $\inf\left\{b(t)-\frac{q^2(t)}{b(t-r)};\ t\geqslant 0\right\}>0$.

Let $V(t, x_{1t}, x_{2t})=\frac{1}{2}kx_1^2(t)+\frac{1}{2}mx_2^2(t)+\frac{1}{2}\int_{t-r}^{t} b(s)x_2^2(s)ds$. Then

$$\begin{aligned} &\frac{d}{dt}V(t, x_{1t}, x_{2t}) \\ &=-\frac{1}{2}b(t)x_2^2(t)-q(t)x_2(t)x_2(t-r)-\frac{1}{2}b(t-r)x_2^2(t-r) \\ &=-\frac{1}{2}b(t-r)\left[x_2(t-r)+\frac{q(t)}{b(t-r)}x_2(t)\right]^2-\frac{1}{2}\left[b(t)-\frac{q^2(t)}{b(t-r)}\right]x_2^2(t) \\ &\leqslant-\frac{1}{2}\left[b(t)-\frac{q^2(t)}{b(t-r)}\right]x_2^2(t). \end{aligned}$$

Let $P(t, x_{1t}, x_{2t})=x_2^2(t)$. For any $\varepsilon>0$ choose $\delta_1(\varepsilon)<\min\left\{\frac{k\varepsilon}{16h^2}, \frac{\varepsilon}{hr+m}\right\}$ and $T_1(\varepsilon)>\frac{4m\sqrt{\delta_1(\varepsilon)}}{\sqrt{k\varepsilon}}+r$, where $h>0$ is an upper bound of $b(t)$ and $q(t)$. For a solution $(x_1(t), x_2(t))$ of (2) on $[t_0-r, t_0+T_1(\varepsilon)]$ with $|x_1(t)|+|x_2(t)|<1$ and $x_2^2(t)<\delta_1(\varepsilon)$ for $t\in[t_0, t_0+T_1(\varepsilon)]$, we have

$$\dot{x}_2(t)\leqslant-\frac{k}{m}x_1(t)+\frac{2h}{m}\sqrt{\delta_1(\varepsilon)} \text{ for } t\in[t_0+r, t_0+T_1(\varepsilon)].$$

If $|x_1(t)|\geqslant\sqrt{\frac{\varepsilon}{k}}$ for all $t\in[t_0+r, t_0+T_1((\varepsilon)]$, without loss of generality we assume $x_1(t)\geqslant\sqrt{\frac{\varepsilon}{k}}$, then

$$\dot{x}_2(t)\leqslant-\frac{\sqrt{k}}{m}\sqrt{\varepsilon}+\frac{2h}{m}\sqrt{\delta_1(\varepsilon)}\leqslant-\frac{\sqrt{k}}{2m}\sqrt{\varepsilon}.$$

This implies $x_2(t_0+T_1(\varepsilon))\leqslant x_2(t_0+r)-\frac{\sqrt{k}}{2m}\sqrt{\varepsilon}\,[T_1(\varepsilon)-r]$, which is contrary to $|x_2(t)|<\sqrt{\delta_1(\varepsilon)}$ for $t\in[t_0, t_0+T_1(\varepsilon)]$.

Therefore there exists $\tau\in[t_0+r, t_0+T_1(\varepsilon)]$ with $|x_1(\tau)|<\sqrt{\frac{\varepsilon}{k}}$ and thus

$$V(\tau, x_{1\tau}, x_{1\tau})\leqslant\frac{1}{2}\varepsilon+\frac{m}{2}\delta_1(\varepsilon)+\frac{1}{2}h\delta_1(\varepsilon)r<\varepsilon.$$

By Corollary 2, the zero solution of (2) is uniformly asymptotically stable.

Remark 2. By the extension of Barbashin–Krasovskii's theorem (Theorem D of [6]), Krasovskii discussed the case where b and q are constants. But his result was available for autonomous systems only.

Remark 3. From the proof of Theorem 1, we know that if the zero solution of (1) is uniformly stable, then the positive definite condition $W_1(|\varphi(0)|)\leqslant V(t, \varphi)$ is not required.

Remark 3 motivates the following generalization of Theorem 1.

Corollary 2. *Suppose that there exist continuous functionals* V, P: $R^+\times C_n^H\to R^+$, *wedges* W_1, W_2, W_3 *and a constant* $\mu>0$ *such that*

(i) $V(t, \varphi)\leqslant W_2(\|\varphi\|)$, $P(t, \varphi)\leqslant W_2(\|\varphi\|)$,

(ii) $W_1(|\varphi(0)|)\leqslant V(t, \varphi)+P(t, \varphi)$,

(iii) $\dot V_{(1)}(t, \varphi)\leqslant-\mu|W_3(P(t, \varphi))_{(1)}|-P(t, \varphi)$,

(iv) (iii) *of Theorem* 1 *holds.*

Then the zero solution of (1) *is uniformly asymptotically stable.*

Proof Noticing Remark 3, it suffices to prove the uniform stability of the zero solution of (1). Let $x(t)$ be a solution of (1) defined for $t\geqslant t_0-r$. Then by (i) and (iii), we get

$$V(t, x_t)\leqslant V(t_0, x_{t_0})\leqslant W_2(\|x_{t_0}\|)$$

and

$$V(t, x_t)\leqslant V(t_0, x_{t_0})-\mu|W_3(P(t, x_t))-W_3(P(t_0, x_{t_0}))|.$$

Therefore

$$\begin{aligned}W_3(P(t, x_t))&\leqslant W_3(P(t_0, x_{t_0}))+\frac{1}{\mu}[V(t_0, x_{t_0})-V(t, x_t)]\\&\leqslant W_3(P(t_0, x_{t_0}))+\frac{1}{\mu}W_2(\|x_{t_0}\|).\end{aligned}$$

Thus from (ii), we get

$$\begin{aligned}W_1(|x(t)|)&\leqslant V(t, x_t)+P(t, x_t)\\&\leqslant W_2(\|x_{t_0}\|)+W_3^{-1}(W_3(W_2(\|x_{t_0}\|))+\frac{1}{\mu}W_2(\|x_{t_0}\|)).\end{aligned}$$

For any $\varepsilon>0$ choose $\delta(\varepsilon)>0$ such that

$$W_2(\delta(\varepsilon))+W_3^{-1}\Big(W_3(W_2(\delta(\varepsilon)))+\frac{1}{\mu}W_2(\delta(\varepsilon))\Big)<W_1(\varepsilon).$$

Then $\|x_{t_0}\|<\delta(\varepsilon)$ implies $|x(t)|<\varepsilon$ for $t\geqslant t_0$. This completes the proof.

In the results above, $\dot V_{(1)}(t, \varphi)$ is only required to be semi-negative definite. This makes it easy to construct Liapunov functionals in many practical problems. For convenience of practical use, we give a series of criteria to ensure (iii) of Theoerm 1.

Lemma 2. *Suppose that there exist a continuous functional* W: $R^+\times C_n^H\to R$ *and wedges* $W_i(i=1, 2, 3)$ *such that*

$$0\leqslant W(t, \varphi)\leqslant W_1(\|\varphi\|),$$

$$\dot W_{(1)}(t, \varphi)\leqslant-W_2(V(t, \varphi))+W_3(P(t, \varphi)).$$

Then (iii) *of Theorem* 1 *holds.*

Proof For any $\varepsilon>0$ choose $\delta_1(\varepsilon)$ and $T_1(\varepsilon)>0$ such that $W_3(\delta_1(\varepsilon))<\frac{1}{2}W_2(\varepsilon)$ and $T_1(\varepsilon)>2W_1(H)/W_2(\varepsilon)$. Let $x(t)$ be a solution of (1) defined on $[t_0-r,\ t_0+T_1(\varepsilon)]$ with $\|x_t\|<H$, $P(t, x_t)<\delta_1(\varepsilon)$. If for all $t\in[t_0,\ t_0+T_1(\varepsilon)]$ we have $V(t, x_t)\geqslant\varepsilon$, then

$$\dot{W}(t, x_t)\leqslant -W_2(V(t, x_t))+W_3(P(t, x_t))\leqslant -\frac{1}{2}W_2(\varepsilon)$$

and thus

$$W(t_0+T_1(\varepsilon),\ x_{t_0}+T_1(\varepsilon))\leqslant W(t_0, x_{t_0})-\frac{1}{2}W_2(\varepsilon)T_1(\varepsilon)<0.$$

This is contrary to $W\geqslant 0$. Therefore there exists $\tau\in[t_0,\ t_0+T_1(\varepsilon)]$ with $V(\tau, x_\tau)<\varepsilon$. This completes the proof.

Lemma 3. *Suppose that there exists a continuous functional* W: $R^+\times C_n^H\to R^+$ *and wedges* $W_i(i=1, \cdots, 4)$ *such that*

(i) $0\leqslant W(t, \varphi)\leqslant W_2(\|\varphi\|)$,

(ii) $W_1(|\varphi(0)|)\leqslant W(t,\varphi)+P(t, \varphi)$,

(iii) $\dot{W}_{(1)}(t, \varphi)\leqslant -W_3(W(t, \varphi))+W_4(P(t, \varphi))$.

Then (iii) *of Theorem* 1 *holds.*

Proof For any $\varepsilon>0$ choose $\delta_1(\varepsilon)$ and $T_1(\varepsilon)>0$ such that $\delta_1(\varepsilon)<\frac{1}{2}W_1(\varepsilon)$, $W_4(\delta_1(\varepsilon))<\frac{1}{2}W_3\left(\frac{1}{4}W_1(\varepsilon)\right)$ and $T_1(\varepsilon)>2_2W(H)/W_3\left(\frac{1}{4}W_1(\varepsilon)\right)+r$. Let $x(t)$ be a solution of (1) defined on $[t_0-r,\ t_0+T_1(\varepsilon)]$ with $\|x_t\|<H$ and $P(t, x_t)<\delta_1(\varepsilon)$ for $t\in[t_0,\ t_0+T_1(\varepsilon)]$ and let $W(t)=W(t, x_t)$. If for all $t\in[t_0\ t_0+T_1(\varepsilon)-r]$, we have $W(t)\geqslant\frac{1}{4}W_1(\varepsilon)$, then

$$\begin{aligned}\dot{W}(t)&\leqslant -W_3(W(t))+W_4(P(t, x_t))\\&\leqslant -W_3\left(\frac{1}{4}W_1(\varepsilon))\right)+W_4(\delta_1(\varepsilon))\\&\leqslant -\frac{1}{2}W_3\left(\frac{1}{4}W_1(\varepsilon)\right).\end{aligned}$$

This implies

$$W(t_0+T_1(\varepsilon)-r)\leqslant W(t_0)-\frac{1}{2}W_3\left(\frac{1}{4}W_1(\varepsilon)\right)[T_1(\varepsilon)-r]<0,$$

Which is contrary to $W\geqslant 0$. So there exists $t_1\in[t_0,\ t_0+T_1(\varepsilon)-r]$ with

$$W(t_1)<\frac{1}{4}W_1(\varepsilon).$$

If there exists $t_2\in[t_1,\ t_0+T_1(\varepsilon)]$ with $W(t_2)\geqslant\frac{1}{2}W_1(\varepsilon)$, then there exist $t_1<t_3<t_4\leqslant t_2$ such that $W(t_3)=\frac{1}{4}W_1(\varepsilon)$, $W(t_4)=\frac{1}{2}W_1(\varepsilon)$ and

$$\frac{1}{4}W_1(\varepsilon)<W(t)<\frac{1}{2}W_1(\varepsilon)$$

for $t\in(t_3, t_4)$. So, for $t\in t_3, t_4]$, we have

$$\dot{W}(t)\leqslant -W_3(W(t))+W_4(P(t, x_t))\leqslant -\frac{1}{2}W_3\Big(\frac{1}{4}W_1(\varepsilon)\Big)<0.$$

This implies $W(t_4)\leqslant W(t_3)$, which is contrary to the choice of t_3 and t_4. So

$$W(t)<\frac{1}{2}W_1(\varepsilon)$$

for $t\in[t_1, t_0+T_1(\varepsilon)]$, and thus

$$W_1(|x(t)|)\leqslant W(t)+P(t, x_t)\leqslant\frac{1}{2}W_1(\varepsilon)+\delta_1(\varepsilon)<W_1(\varepsilon)$$

for $t\in[t_1, t_0+T_1(\varepsilon)]$. This implies $\|x_{t_0+T_1(\varepsilon)}\|<\varepsilon$. The proof is completed.

Lemma 4. *Suppose that there exist a continuous function W: $R^+\times R^n_H\to R^+$ and wedges W_i $(i=1, \cdots 4)$ such that*

(i) $W_1(|\varphi(0)|)\leqslant W(t, \varphi(0))+P(t, \varphi)$ *for* $(t, \varphi)\in R^+\times C^H_n$,

(ii) $W(t, x)\leqslant W_2(|x|)$ *for* $(t, x)\in R^+\times R^n_H$,

(iii) *for any $\alpha>\beta>0$ there exists $\gamma>0$ such that*

$$\dot{W}_{(1)}(t, \varphi(0))\leqslant -W_3(W(t, \varphi(0)))+W_4)P(t, \varphi))$$

whenever $(t, \varphi)\in R^+\times C^H_n$, $\beta\leqslant W(t, \varphi(0))\leqslant\alpha$ *and* $W(t+s, \varphi(s))\leqslant W(t_1\varphi(\sigma))+\gamma$ *for* $s\in[-r, 0]$.

Then (iii) *of Theorem* 1 *holds.*

Proof For any $\varepsilon>0$ choose $\delta_1(\varepsilon)>0$ so that $\delta_1(\varepsilon)<\frac{1}{2}W_1(\varepsilon)$ and $W_4(\delta_1(\varepsilon))<\frac{1}{2}W_3\left(\frac{1}{2}W_1(\varepsilon)\right)$. By (iii), there exists $\gamma(\varepsilon)<\delta_1(\varepsilon)$ such that

$$\dot{W}_{(1)}(t, \varphi(0))\leqslant -W_3(W(t, \varphi(0)))+W_4(P(t, \varphi))$$

whenever $(t, \varphi)\in R^+\times C^H_n$, $\frac{1}{2}W_1(\varepsilon)\leqslant W(t, \varphi(0))\leqslant W_2(H)$ and $W(t+s, \varphi(s))\leqslant W(t, \varphi(0))+\gamma$ for $s\in[-r, 0]$. For this given $\gamma(\varepsilon)>0$ choose a positive integer $N(\varepsilon)>0$ with $\frac{1}{2}W_1(\varepsilon)+N(\varepsilon)\gamma(\varepsilon)>W_2(H)$. Give $T_0(\varepsilon)>0$ such that $T_0(\varepsilon)>r+2W_2(H)/W_3\left(\frac{1}{2}W_1(\varepsilon)\right)$. Define $T_1(\varepsilon)=r+N(\varepsilon)T_0(\varepsilon)$.

Let $x(t)$ be a solution of (1) defined on $[t_0-r, t_0+T_1(\varepsilon)]$ with $\|x_t\|<H$ and $P(t, x_t)<\delta_1(\varepsilon)$ for $t\in[t_0, t_0+T_1(\varepsilon)]$, and let $W(t)=W(t, x(t))$.

Obviously, $W(t)\leqslant W_2(|x(t)|)\leqslant W_2(H)<\frac{1}{2}W_1(\varepsilon)+N(\varepsilon)\gamma(\varepsilon)$ for $t\in[t_0, t_0+T_1(\varepsilon)]$

Suppose that $W(t)\leqslant\frac{1}{2}W_1(\varepsilon)+[N(\varepsilon)-k]\gamma(\varepsilon)$ for a nonnegative integer $k<N$ and for all $t\in[t_0+kT_0(\varepsilon), t_0+T_1(\varepsilon)]$. Then we can assert that there exists $t_1\in[t_0$

$+kT_0(\varepsilon)+r,\ t_0+(k+1)T_0(\varepsilon)]$ with $W(t_1)<\frac{1}{2}W_1(\varepsilon)+[N(\varepsilon)-(k+1)]\gamma(\varepsilon)$.

If it is not true, then for all $t\in[t_0+kT_0(\varepsilon)+r,\ t_0+(k+1)T_0(\varepsilon)]$ and for $s\in[-r,\ 0]$, we have

$$\frac{1}{2}W_1(\varepsilon)\leqslant W(t)\leqslant W_2(H)$$

and

$$\begin{aligned}W(t+s)&\leqslant\frac{1}{2}W_1(\varepsilon)+[N(\varepsilon)-k]\gamma(\varepsilon)\\&\leqslant\frac{1}{2}W_1(\varepsilon)+[N(\varepsilon)-(k+1)]\gamma(\varepsilon)+\gamma(\varepsilon)\\&\leqslant W(t)+\gamma(\varepsilon).\end{aligned}$$

So, by the choice of $\gamma(\varepsilon)$, we get

$$\begin{aligned}\dot W(t)&\leqslant -W_3(W(t))+W_4(P(t,\ x_t))\\&\leqslant -W_3\Big(\frac{1}{2}W_1(\varepsilon)\Big)+W_4(\delta_1(\varepsilon))\\&<-\frac{1}{2}W_3\Big(\frac{1}{2}W_1(\varepsilon)\Big).\end{aligned}$$

This implies

$$W(t_0+(k+1)T_0(\varepsilon))<W(t_0+kT_0(\varepsilon)+r)-\frac{1}{2}W_3\Big(\frac{1}{2}W_1(\varepsilon)\Big)[T_0(\varepsilon)-r]<0,$$

which is contrary to $W\geqslant 0$.

If there exists the first $t_2\geqslant t_1$ with $W(t_2)=\frac{1}{2}W_1(\varepsilon)+[N(\varepsilon)-(k+1)]\gamma(\varepsilon)$, then by the same argument as above, we get

$$\dot W(t_2)\leqslant -\frac{1}{2}W_3\Big(\frac{1}{2}W_1(\varepsilon)\Big)<0,$$

which is contrary to the definition of t_2. So for all $t\in[t_0+(k+1)T_0(\varepsilon),\ t_0+T_1(\varepsilon)]$ we have

$$W(t)<\frac{1}{2}W_1(\varepsilon)+[N(\varepsilon)-(k+1)]\gamma(\varepsilon).$$

Then by induction principle we get $W(t)<\frac{1}{2}W_1(\varepsilon)$ for $t\in[t_0+T_1(\varepsilon)-r,\ t_0+T_1(\varepsilon)]$ and therefore from

$$W_1(|x(t)|)\leqslant W(t)+P(t,\ x_t)\leqslant W(t)+\delta_1(\varepsilon)\leqslant W(t)+\frac{1}{2}W_1(\varepsilon)$$

we get $\|x_{t_0+T_1(\varepsilon)}\|<\varepsilon$. This completes the proof.

Corollary 3. *Suppose that there exist a continuous function W: $R^+\times R^n_H\to R$, wedges $W_i(i=1,\ \cdots\ 4)$ and a continuous function f: $R^+\to R^+$ with $f(s)>s$ for $s>0$ satisfying* (i) *and* (ii) *of Lemma* 4, *and*

$$\dot W_{(1)}(t,\ \varphi(0))\leqslant -W_3(W(t,\ \varphi(0)))+W_4(P(t,\ \varphi))$$

whenever $(t,\ \varphi)\in R^+\times C^H_n$ and $W(t+s,\ \varphi(s))\leqslant f(W(t,\ \varphi(0)))$ for $s\in[-r,\ 0]$. Then

(iii) *of Theorem* 1 *holds.*

Proof For any $\alpha>\beta>0$, let $\gamma=\inf\{f(s)-s: \beta\leqslant s\leqslant\alpha\}$. Then $\gamma>0$ and for any $(t, \varphi)\in R^+\times C_n^H$ with $\beta\leqslant W(t, \varphi(0))\leqslant\alpha$ and $W(t+s, \varphi(s))\leqslant W(t, \varphi(0))+\gamma$ for $s\in[-r, 0]$, we have

$$W(t+s,\varphi(s))\leqslant W(t, \varphi(0))+\inf\{f(s)-s; \beta\leqslant s\leqslant\alpha\}\leqslant f(W(t, \varphi(0)))$$

and thus

$$\dot{W}_{(1)}(t, \varphi(0))\leqslant -W_3(W(t, \varphi(0)))+W_4(P(t, \varphi)).$$

This completes the proof by Lemma 4.

Corollary 4. *Suppose that there exist a continuous function* W: $R^+\times R_H^n\to R^+$ *and wedges* W_i $(i=1, \cdots, 5)$ *satisfying* (i) *and* (ii) *of Lemma* 4, *and suppose that for any constant* $N\geqslant 0$ *and* $(t, \varphi)\in R^+\times C_n^H$ *with* $W(t+s, \varphi(s))\leqslant N$ *for* $s\in[-r, 0]$, *we have*

$$\dot{W}_{(1)}(t, \varphi(0))\leqslant F(t, W(t, \varphi(0)), N)+W_3(P(t, \varphi)),$$

where

(a) $F(t, W, W)\leqslant -W_4(W)$ *for* $t\geqslant 0$ *and* $W\geqslant 0$,

(b) $|F(t, W, W_1)-F(t, W, W_2)|\leqslant W_5(|W_1-W_2|)$ *for* $t, W, W_1, W_2\geqslant 0$.

Then (iii) *of Theorem* 1 *holds.*

Proof For $\alpha\geqslant\beta>0$ choose $\gamma=W_5^{-1}\left(\frac{1}{2}W_4(\beta)\right)$. If $(t, \varphi)\in R^+\times C_n^H$, $\beta\leqslant W(t, \varphi(0))\leqslant\alpha$ and $W(t+s, \varphi(s))\leqslant W(t, \varphi(0))+\gamma$ for $s\in[-r, 0]$, then

$$\begin{aligned}\dot{W}_{(1)}(t, \varphi(0))&\leqslant F(t,W(t, \varphi(0)), W(t, \varphi(0))+\gamma)\\ &\leqslant F(t, W(t, \varphi(0)), W(t, \varphi(0)))\\ &\quad+|F(t, W(t, \varphi(0)), W(t, \varphi(0))+\gamma)\\ &\quad-F(t, W(t, \varphi(0)), W(t, \varphi(0)))|\\ &\leqslant -W_4(W(t, \varphi(0)))+W_5(\gamma)+W_3(P(t, \varphi))\\ &\leqslant -\frac{1}{2}W_4(W(t, \varphi(0)))+W_3(P(t, \varphi)).\end{aligned}$$

This completes the proof by Lemma 4.

As an application of previous results, let us study the uniform asymptotic satbility of the zero solution of the following high dimension system

$$\begin{cases}\dot{x}(t)=F(t, x_t, y_t),\\ \dot{y}(t)=G(t, x_t, y_t),\end{cases}\tag{4}$$

where $x\in R^n$, $y\in R^m$, F: $R^+\times C_n^H\times C_m^H\to R^n$, G: $R^+\times C_n^H\times C_m^H\to R^m$ are continuous, $F(t, 0, 0)=0$ and $G(t, 0, 0)=0$. By Theorem 1 and using the same argument as those of Lemma 2, Lemma 3 and Lemma 4, we get the following theorem.

Theorem 2. *Suppose that there are a continuous functional* $V: R^+\times C_n^H\times C_m^H\to R^+$, *wedges* W_i $(i=1, 2, 3)$ *and an integrally positive function* λ *such that*

(a) $W_1(|\varphi(0)|+|\psi(0)|)\leqslant V(t, \varphi, \psi)\leqslant W_2(\|\varphi\|+\|\psi\|)$;

(b) *for any* $\sigma>0$ *there exists* $\mu(\sigma)>0$ *such that for any* $(t, \varphi, \psi)\in R^+\times C_n^H\times C_m^H$ *with* $|\psi(0)|\geqslant\sigma$ *we have*

$$\dot{V}_{(4)}(t, \varphi, \psi)\leqslant-\mu(\sigma)|G(t, \varphi, \psi)|-\lambda(t)W_3(|\psi(0)|);$$

(c) *for any* $\varepsilon>0$ *there exists* $\delta_1(\varepsilon)$, $\delta_2(\varepsilon)$ *and* $T_1(\varepsilon)>0$ *such that, for any* $t_0\geqslant0$ *and any solution* $(x(t), y(t))$ *of* (4) *defined on* $[t_0-r, t_0+T_1(\varepsilon)]$ *with*

$$\int_{t_0}^{t_0+T_1(\varepsilon)}\lambda(t)W_3(|y(t)|)dt<\delta_2(\varepsilon)$$

and $\|x_t\|+\|y_t\|<H$, $|y(t)|<\delta_1(\varepsilon)$ *for* $t\in[t_0, t_0+T_1(\varepsilon)]$, *there exists* $t_1\in[t_0, t_0+T_1(\varepsilon)]$ *with* $\|x_{t_1}\|<\varepsilon$ *(or* $V(t_1, x_{t_1}, y_{t_1})<\varepsilon$*).*

Then the zero solution of (4) *is uniformly asymptotically stacle. If there exists a constant* $L>0$ *with* $|G(t, \varphi, \psi)|\leqslant L$ *for* $(t, \varphi, \psi)\in R^+\ C_n^H\times C_m^H$, *then* (b) *can be replaced by*

(d) $\dot{V}_{(4)}(t, \varphi, \psi)\leqslant-W_3(|\psi(0)|)$.

Finally, any one of the following conditions ensures (c).

(e) *there exists a continuous functional* W: $R^+\times C_n^H\times C_m^H\to R$ *and wedges* W_i $(i=4, 5, 6)$ *with*

$$0\leqslant W(t, \varphi, \psi)\leqslant W_4(\|\varphi\|+\|\psi\|),$$
$$\dot{W}_{(4)}(t, \varphi, \psi)\leqslant-W_5(\|\varphi\|)+W_6(\|\psi\|);$$

(f) *there exists a continuous functional* W: $R^+\times C_n^H\times C_m^H\to R$ *and wedges* W_i $(i=7, \cdots, 10)$ *such that*

$$W_7(|\varphi(0)|)\leqslant W(t, \varphi, \psi)\leqslant W_8(\|\varphi\|+\|\psi\|),$$
$$\dot{W}_{(4)}(t, \varphi, \psi)\leqslant-W_9(W(t, \varphi, \psi))+W_{10}(\|\psi\|);$$

(g) *there exists a continuous function* W: $R^+\times R_H^n\times R_H^m\to R^+$ *and wedges* $W_i(i=11, \cdots, 14)$ *with*

$$W_{11}(|x|)\leqslant W(t, x, y)\leqslant W_{12}(|x|+|y|),$$

and for any $\alpha\geqslant\beta>0$ *there exists* $\gamma>0$ *such that*

$$\dot{W}_{(4)}(t, \varphi(0), \psi(0))\leqslant-W_{13}(W(t, \varphi(0), \psi(0)))+W_{14}(\|\psi\|)$$

provided that $\beta\leqslant W(t, \varphi(0), \psi(0))\leqslant\alpha$ *and* $W(t+s, \varphi(s))\leqslant W(t, \varphi(0), \psi(0))+\gamma$ *for all* $s\in[-r, 0]$.

Generally, the Liapunov functional (function) W *in* (e), (f) *and* (g) *are constructed from the lower order system*

$$\dot{x}(t)=F(t, x_t, 0). \tag{5}$$

For example we have the following corollary.

Corollary 5. *If there exists a continuous functional* W: $R^+\times C_n^H\to R$, *a constant* $L>0$ *and wedges* W_i $(i=1, 2, 3)$ *with*

$$0\leqslant W(t, \varphi)\leqslant W_1(\|\varphi\|),$$
$$\dot{W}_{(5)}(t, \varphi)\leqslant-W_2(\|\varphi\|),$$
$$|F(t, \varphi, \psi)-F(t, \varphi, 0)|\leqslant W_3(\|\psi\|), \tag{6}$$

$|W(t, \varphi)-W(t, \tilde{\varphi})|\leqslant L\|\varphi-\tilde{\varphi}\|$ *for* $t\geqslant0$, φ, $\tilde{\varphi}\in C_n^H$ *and* $\psi\in C_m^H$, *then* (c) *of*

Theorem 2 *holds.*

This is an immeadiate consequence of (e), if we notice that

$$\dot{W}_{(4)}(t, \varphi) \leqslant \dot{W}_{(5)}(t, \varphi) + L|F(t, \varphi, \psi) - F(t, \varphi, 0)|.$$

By converse theorem (cf. [1]), if

$$|F(t, \varphi, 0) - F(t, \tilde{\varphi}, 0)| \leqslant N\|\varphi - \tilde{\varphi}\| \tag{7}$$

for $t \geqslant 0$, φ, $\tilde{\varphi} \in C_n^H$ and for a constant $N > 0$, then the uniform asymptotic stability of the zero solution of (5) implies the existence of W. This induces a reducing dimension approach for the uniform asymptotic stability of the zero solution of high dimension system (4), which is formulated as the following theorem.

Theoem 3. *If* (a), (b) *of Theorem* 2 *and* (6), (7) *hold, then the zero solution of* (4) *is uniformly asymptotically stable provided that the zero solution of lower dimension subsystem* (5) *is uniformly asymptotically stable.*

Example 2 Consider now the system

$$\begin{cases} \dot{x}(t) = A_{11}x(t) + A_{12}y(t) + B_{11}(t)x(t-r) + B_{12}(t)y(t-r), \\ \dot{y}(t) = A_{21}x(t) + A_{22}y(t) + B_{21}(t)x(t-r) + B_{22}(t)y(t-r), \end{cases} \tag{8}$$

where $x \in R^n$, $y \in R^m$, $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is an $n+m$ order stable matrix,

$$B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}$$

is bounded continuous.

A is stable, so there exists a positive definite matrix $D^T = D$ with $A^T D + DA = -I$. Construct now a Liapunov functional

$$V(t, x_t, y_t) = (x^T(t), y^T(t))D\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \int_{t-r}^{t} [\alpha(s)x^T(s)x(s) + \beta(s)y^T(s)y(s)]ds,$$

where

$$DB(t) = \begin{pmatrix} E_{11}(t), & E_{12}(t) \\ E_{21}(t) & E_{22}(t) \end{pmatrix},$$

$$\alpha(t) = |E_{11}(t+r)| + |E_{21}(t+r)|,$$

$$\beta(t) = |E_{12}(t+r)| + |E_{22}(t+r)|.$$

Then we get

$$\begin{aligned} \dot{V}_{(8)}(t, x_t, y_t) = &-[1 - |E_{11}(t)| - |E_{12}(t)| - \alpha(t)]|x|^2 \\ &-[1 - |E_{21}(t)| - |E_{22}(t)| - \beta(t)]|y(t)|^2 \\ &+[|E_{11}(t)| + ||E_{21}(t)| - \alpha(t-r)]|x(t-r)|^2 \\ &+[|E_{12}(t)| + |E_{22}(t)| - \beta(t-r)]|y(t-r)|^2. \end{aligned}$$

By Theorem 3, the zero solution of (8) is uniformly asymptotically stable, if the following conditions hold:

(i) the zero solution of $\dot{x}(t) = A_{11}x(t) + B_{11}x(t-r)$ is uniformly asymptotically stable,

(ii) $|E_{11}(t)||E_{12}(t)|+|E_{11}(t+r)|+|E_{21}(t+r)|\leqslant 1$,

(iii) $|E_{21}(t)|+|E_{22}(t)|+|E_{12}(t+r)|+|E_{22}(t+r)|\leqslant\delta<1$, δ is a constant.

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