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ALMOST QUASICONFORMAL MAPPINGS WITH GIVEN BOUNDARY VALUES AND A COMPLEX DILATATION BOUND

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Abstract

In the extremal problems of quasiconformal mappings with given boundary values and a complex dilatation bound which are discussed by Reich, the extremal mapping is required to have no conformal point set of positive measure on the defining set T of the complex dilatation bound $b(w)$. Under the additional assumptions that $\overline{T}\backslash T$ has measure zero and $b(w)$ is continuous a. e. Chen Jixiu proved that the extremal mapping may be relaxed to have a conformal positive measure set and a finite number of singularity points on T. In this paper, the author proves that when the additional assumptions are given up, the same relaxations still hold and the extremal mapping is also allowed to have a countable number of singularity points on T.

§ 1. Introduction

Suppose that $z=F(w)$ is a sense-preserving self homeomorphism of the unit disc U, $\mu(w)$ a complex measurable function in U, $E=\{w_0\in U \mid \operatorname{ess\,sup}_{w\in O(w_0)} |\mu(w)|=1$ for every neighborhood $O(w_0)$ of $w_0\}$ a set of measure zero, and that $\operatorname{ess\,sup}_{w\in U\backslash\Omega}|\mu(w)|<1$ always holds and $F(w)$ is a quasiconformal mapping with the complex dilatation $\mu(w)$ in $U\backslash\overline{\Omega}$ for every open set $\Omega\supset E$. Such F and E will be called an almost quasiconformal self mapping of U and the singularity point set of F respectively.

Let F be an almost quasiconformal self mapping of U, $T\,(\overline{T}\subset U)$ a measurable set and $b(w)$, $0\leqslant b(w)\leqslant 1$, a measurable function on T. Set $k_F=\operatorname{ess\,sup}_{w\in U\backslash T}\left|\frac{F\overline{w}}{Fw}\right|<1$. By continuation, F induces a homeomorphism of the boundary ∂U onto itself. Denote by $A(F, T, b)$ (or simply A) the family of all almost quasiconformal self mappings of U satisfying the following conditions:

i) $G(e^{i\theta})=F(e^{i\theta})$, for $0\leqslant\theta<2\pi$,

ii) $\left|\frac{G\overline{w}}{Gw}\right|\leqslant b(w)$, a.e. on T.

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And further, we assume that F itself belongs to $A(F, T, b)$ and that $A(F, T, b)$ contains another quasiconformal mapping[1], if F is not a quasiconformal mapping.

If $G_0 \in A$ with

$$k_{G_0} = \inf_{G \in A} k_G, \tag{1}$$

where $k_G = \operatorname{ess\,sup}_{w \in U \setminus T} \left| \frac{G_{\bar{w}}}{G_w} \right|$, we shall say that G_0 is an extremal (almost quasiconformal) mapping within A. From now on we write $K_G = \frac{1+k_G}{1-k_G}$.

For the sake of convenience we always regard $b(w)$ as k_F on $U \setminus T$ and assume:

1) $E_1 = \{w_0 \in U \mid \operatorname{ess\,sup}_{w \in O(w_0)} b(w) = 1$ for every neighborhood $O(w_0)$ of $w_0\}$ is a countable set on T.[2]

2) For every $w_0 \in E_1$ and arbitrary $r_2 > r_1 > 0$, if $\{|w - w_0| < r_2\} \subset U$, then the integral

$$I(r_1, r_2) = \int_{r_1}^{r_2} \frac{1}{\frac{1}{2\pi}\int_0^{2\pi} \frac{1+b(w_0+re^{i\theta})}{1-b(w_0+re^{i\theta})} d\theta} \frac{dr}{r} > 0 \tag{2}$$

holds, but $\lim_{r \to 0} I(r_1, r_2) = \infty$.

With the above conditions, we can prove like [1] that the subclass $A_\lambda = \left\{G \in A \,\middle|\, \left|\frac{G_{\bar{w}}}{G_w}\right| \leq \lambda < 1, \text{ a. e. for } w \in U \setminus T\right\}$ of A is a normal family[3], and that the inverse of any uniform convergence sequence also converges uniformly to the inverse of the limit function. Now that A_λ is a normal family, from $\inf_{G \in A} k_G = \inf_{G \in A_k} k_G$ we know that there exists an extremal mapping within $A(F, T, b)$.

Given $A(F, T, b)$, since E_1 is a closed countable set and a continuous function maps a closed set onto a closed one, it is clear that $F(E_1)$ is also a closed countable set. Therefore, $f = F^{-1}$, $\varkappa(z) = f_{\bar{z}}/f_z$ may be defined and we know $k_F = \operatorname{ess\,sup}_{z \in U \setminus F(T)} |\varkappa(z)|$. If $k_F > 0$, we set $T_0 = \{w \in T \mid b(w) = 0\}$ and

$$\tau(z) = \begin{cases} 0, & z \in F(T_0), \\ \dfrac{\varkappa(z)}{b(f(z))}, & z \in U \setminus F(T_0). \end{cases} \tag{3}$$

Denote by $\mathscr{B}(U)$ the Banach space of all analytic L^1 functions in U. When $\varphi \in \mathscr{B}(U)$, $\Omega \subseteq U$, write $\|\varphi\|_\Omega = \iint_\Omega |\varphi(z)| \, dx\, dy$, $\|\varphi\| = \|\varphi\|_U$.

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1) In [1] Chen Jixiu had not made this assumption, but it was used.

2) It is easy to see that E_1 is a closed set.

3) In [1, p. 467], $|\mu_0(w)| \leq \begin{cases} b(w), & w \in T, \\ \sigma < 1, & w \in U \setminus T \end{cases}$ follows from the convergence theorem of quasiconformal mappings since $b(w)$ is continuous a.e.. Now, $b(w)$ is only measurable, in its proof we need to apply the theorem in [2].

In this paper we shall prove the following theorem.

Theorem 1. *Any single one of the following two conditions* (I) *and* (II) *is both necessary and sufficient for F to be an extremal mapping within $A(F, T, b)$.*

Condition (I): *Either $k_F=0$ or*

$$k_F>0 \text{ and } \sup_{\substack{\varphi\in\mathscr{B}(U)\\ \|\varphi\|_{U\setminus F(T_0)}=1}}\left|\iint_U \tau(z)\varphi(z)\,dx\,dy\right|=1. \tag{4}$$

Condition (II): *Either there exists a function $\varphi_0\in\mathscr{B}(U)$ such that*

$$\varkappa(z)=b(f(z))\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|},\ \text{for } z\in U, \tag{5}$$

or there exists a sequence $\varphi_n\in\mathscr{B}(U)$, $\|\varphi_n\|_{U\setminus F(T_0)}=1$, $n=1, 2, 3, \cdots$, such that

$$\lim_{n\to\infty}\varphi_n(z)=0 \text{ locally uniformly in } U, \text{ and } \lim_{n\to\infty}\left|\iint_U \varkappa(z)\varphi_n(z)\,dx\,dy\right|=k_F. \tag{6}$$

A sequence $\{\varphi_n\}$ satisfying (6) is called a degenerating Hamilton sequence.

Our theorem has a course of its development as follows: First, E. Reich in [3] proved the case that $F(w)$ is quasiconformal (i.e., $b_1=\operatorname{ess\,sup}_{w\in T} b(w)<1$) and $F(w)$ has no conformal point set of positive measure on T (i. e., $b_0=\operatorname{ess\,inf}_{w\in T} b(w)>0$). second, under the additional assumptions that T is an open set [4)], $\overline{T}\setminus T$ has measure zero and $b(w)$ is continuous a.e., Chen Jixiu proved the case that $F(w)$ is allowed to have a conformal positive measure set and a finite number of singularity points on T. Now, under the condition that the Chen's additional assumptions are given up, we prove the case that $F(w)$ is allowed to have a conformal positive measure set and a countable number of singularity points on T.

§2. Some Lemmas

We need the following lemmas.

Lemma 1. *If $G\in A(F, T, b)$, $F(w)$, $G(w)$ and $f(z)=F^{-1}(z)$ has its complex dilatation $\hat{\varkappa}(w)$, $\varkappa_1(w)$ and $\varkappa(z)$ respectively, then it holds for all $\varphi\in\mathscr{B}(U)$, $\|\varphi\|_{U\setminus F(T_0)}=1$ that*

$$1\leqslant\iint_{U\setminus F(T_0)}|\varphi|\frac{|1-\varkappa\varphi/|\varphi||^2}{1-|\varkappa|^2}\frac{\left|1+\varkappa\frac{\hat{\varkappa}_1(f(z))}{\varkappa(f(z))}\frac{\varphi}{|\varphi|}\left(\frac{1-\overline{\varkappa\varphi}/|\varphi|}{1-\varkappa\varphi/|\varphi|}\right)\right|^2}{1-|\varkappa_1(f(z))|^2}\,dx\,dy$$

$$\leqslant\iint_{U\setminus F(T_0)}|\varphi|\frac{|1-\varkappa\varphi/|\varphi||^2}{1-|\varkappa|^2}\frac{1+|\varkappa_1(f(z))|}{1-|\varkappa_1(f(z))|}\,dx\,dy. \tag{7}$$

Proof The proof in [4, p. 380] is suitable for the lemma. We can obtain

4) In [1], the assumption that T is open is nonessential.

$$\|\varphi\| \leqslant \iint_U |\varphi| \frac{|1-\varkappa\varphi/|\varphi||^2}{1-|\varkappa|^2} \frac{\left|1-\varkappa_1(f(z))\frac{\bar{p}}{p}\frac{\varphi}{|\varphi|}\left(\frac{1-\bar{\varkappa}\bar{\varphi}/|\varphi|}{1-\varkappa\varphi/|\varphi|}\right)\right|^2}{1-|\varkappa_0(f(z))|^2}\,dx\,dy$$

$$\leqslant \iint_U |\varphi| \frac{|1-\varkappa\varphi/|\varphi||^2}{1-|\varkappa|^2} \frac{1+|\varkappa_1(f(z))|}{1-|\varkappa_1(f(z))|}\,dx\,dy,$$

where $p=f_z$. Observing again $\frac{\bar{p}}{p}=-\frac{\varkappa(z)}{\hat{\varkappa}(f(z))}$ and $\varkappa_1(f(z))=\varkappa(z)=0$ for $z\in F(T_0)$ and removing $\|\varphi\|_{F(T_0)}$ from the right to the left, we obtain (7) at once.

Lemma 2. *Under the mapping $F_0\in A(F, T, b)$, the image area is an absolutely continuous set function.*

Proof The conclusion is clear.

Lemma 3. *Suppose that either $\{F_n\}$ and F_0 are quasiconformal mappings in the domain $D\supset\bar{U}$ or they belong to $A(F, T, b)$. If $\{F_n\}$ converges uniformly on $\bar{U}$ to F_0, then the equality*

$$\lim_{n\to\infty} \operatorname{mes} F_n(e) = \operatorname{mes} F_0(e)$$

holds for any measurable set $e\subseteq\bar{U}$.

Proof In the first case, the conclusion is known, while in the second case, for any given $\varepsilon>0$, since $F_0(E_1)$ is a closed countable set, there exists a finite number of open disks $\Delta_i (i=1, 2, \cdots, m)$ such that the sum $\bigcup_{i=1}^m \Delta_i$ covers $F_0(E_1)$ and mes $\left\{\bigcup_{i=1}^m \Delta_i\right\}<\frac{\varepsilon}{2}$ holds. Since a topological mapping maps open sets onto open sets, there exists an open set $\Omega\left(=F_0^{-1}\left(\bigcup_{i=1}^m \Delta_i\right)\right)$ such that $E_1\subset\Omega$ and mes $F_0(\Omega)<\frac{\varepsilon}{2}$ hold. Furtherfore, since $\{F_n\}$ converges uniformly to F_0 and E_1 is a closed countable set, there exist an open set Ω_1 and n_0 such that $E_1\subset\Omega_1\subset\Omega$, mes $F_0(\Omega_1)<\frac{\varepsilon}{2}$ and for $n\geqslant n_0$, F_n maps Ω_1 into $\bigcup_{i=1}^m \Delta_i$, mes $F_n(\Omega_1)<\frac{\varepsilon}{2}$. Besides, by the symmetry principle there exists a domain $D\supset\bar{U}$ such that the conclusion in the first case can be applied on $D\backslash\Omega_1$. Then there exists n_1 such that

$$|\operatorname{mes} F_n(e\backslash\Omega_1) - \operatorname{mes} F_0(e\backslash\Omega_1)| < \frac{\varepsilon}{2}, \text{ for } n\geqslant n_1.$$

Therefore, if $n\geqslant\max\{n_0, n_1\}$, we have

$$\begin{aligned}|\operatorname{mes} F_n(e) - \operatorname{mes} F_0(e)| &= |(\operatorname{mes} F_n(e\backslash\Omega_1) + \operatorname{mes} F_n(e\cap\Omega_1)) \\ &\quad - (\operatorname{mes} F_0(e\backslash\Omega_1) + \operatorname{mes} F_0(e\cap\Omega_1))| \leqslant |\operatorname{mes} F_n(e\backslash\Omega_1) - \operatorname{mes} F_0(e\backslash\Omega_1)| \\ &\quad + |\operatorname{mes} F_n(e\cap\Omega_1) - \operatorname{mes} F_0(e\cap\Omega_1)| \\ &\leqslant \frac{\varepsilon}{2} + \max\{\operatorname{mes} F_n(\Omega_1), \operatorname{mes} F_0(\Omega_1)\} < \varepsilon,\end{aligned}$$

which completes the proof.

Lemma 4. *Under the hypotheses of Lemma 3, the equalities*

$$\begin{cases} \lim\limits_{n\to\infty} \operatorname{mes}\{F_0(e)\setminus F_n(e)\}=0, \\ \lim\limits_{n\to\infty} \operatorname{mes}\{F_n(e)\setminus F_0(e)\}=0 \end{cases} \tag{8}$$

hold for any measurable set $e\subseteq\overline{U}$.

Proof By the symmetry principle, there exists a domain $D\supset\overline{U}$ such that $\{F_n\}$ and F_0 are quasiconformal mappings in D. Let $\Delta_i(i=1, 2, 3, \cdots)$ be open disks in D such that $\Delta=\bigcup\limits_{i\geq1}\Delta_i$ covers e. We have

$$\begin{aligned} \operatorname{mes}\{F_0(e)\setminus F_n(e)\} &\leqslant \operatorname{mes}\{F_0(\Delta)\setminus F_n(e)\} \\ &\leqslant \operatorname{mes}\{F_0(\Delta)\setminus F_n(\Delta)\}+\operatorname{mes} F_n(\Delta\setminus e) \\ &\leqslant \sum_{i=1}^{m} \operatorname{mes}\{F_0(\Delta_i)\setminus F_n(\Delta)\}+\operatorname{mes} F_0\Big(\Delta\Big\backslash\bigcup_{i=1}^{m}\Delta_i\Big)+\operatorname{mes} F_n(\Delta\setminus e) \\ &\leqslant \sum_{i=1}^{m} \operatorname{mes}\{F_0(\Delta_i)\setminus F_n(\Delta_i)\}+\operatorname{mes} F_0\Big(\Delta\Big\backslash\bigcup_{i=1}^{m}\Delta_i\Big)+\operatorname{mes} F_n(\Delta\setminus e). \end{aligned} \tag{9}$$

According to [5, Theorem II. 2], Lemma 2 and Lemma 3, we take Δ, n_0 such that

$$\operatorname{mes} F_0(\Delta\setminus e)<\frac{3}{16}\varepsilon \tag{10}$$

and

$$|\operatorname{mes} F_n(\Delta\setminus e)-\operatorname{mes} F_0(\Delta\setminus e)|<\frac{1}{16}\varepsilon,\ n\geqslant n_0. \tag{11}$$

Applying Lemma 2, we take m so that

$$\operatorname{mes} F_0\Big(\Delta\Big\backslash\bigcup_{i=1}^{m}\Delta_i\Big)<\frac{\varepsilon}{4}. \tag{12}$$

Since Δ_i is an open disk and $\{F_n\}$ converges uniformly to F_0, we take n_1 such that

$$\operatorname{mes}\{F_0(\Delta_i)\setminus F_n(\Delta_i)\}\leqslant\frac{1}{4m}\varepsilon,\ i=1, 2, \cdots, m,\ n\geqslant n_1. \tag{13}$$

Combining (9), (13), (12), (11) and (10), we have

$$\operatorname{mes}\{F_0(e)\setminus F_n(e)\}<\varepsilon,\ n\geqslant\max\{n_0, n_1\}.$$

Since ε is arbitrary, the first equality in (8) is obtained, while the proof of the second equality is similar. Lemma 4 is proved.

§3. The Proof of Theorem 1

The proof of Theorem 1 will proceed as follows: (I)$\Rightarrow$(II)$\Rightarrow F$ is an extremal mapping$\Rightarrow$(I).

Proof of (I)$\Rightarrow$(II). If $k_F=0$, we have

$$\varkappa(z)=0,\ \text{a.e. for } z\in U\setminus F(T).$$

Choosing

$$\varphi_n(z)=\alpha_n\frac{n+2}{2\pi}z^n,\ n=1, 2, 3, \cdots,$$

we take α_n such that $\|\varphi_n\|_{U\setminus F(T_0)}=1$ and find that (6) holds.

If (4) holds, it follows that either there exists $\varphi_0\in\mathscr{B}(U)$ with $\|\varphi_0\|_{U\setminus F(T_0)}=1$ and

$$\iint_U \tau(z)\varphi_0(z)\,dx\,dy=1, \tag{14}$$

thus from $1\leqslant\iint_U|\tau(z)||\varphi_0(z)|\,dx\,dy\leqslant1$ and $\tau(z)=0$, $z\in F(T_0)$, we obtain $|\tau(z)|=1$, a. e. for $U\setminus F_0(T)$ and then we substitute $\tau(\theta)=e^{i\theta(z)}$ into (14), it induces $\theta(z)=-\arg\varphi_0(z)$, a.e. for $z\in U\setminus F(T_0)$, so the equality (5) holds, or there exists a sequence $\varphi_n\in\mathscr{B}(U)$, $\|\varphi_n\|_{U\setminus F(T_0)}=1$ such that

$$\lim_{n\to\infty}\varphi_n(z)=0 \text{ locally uniformly in } U, \text{ and } \lim_{n\to\infty}\left|\iint_U \tau(z)\varphi_n(z)\,dx\,dy\right|=1,$$

which is equivalent to (6).

Proof of (II)$\Rightarrow F$ is an extremal mapping. If (5) holds, we may assume $\|\varphi_0\|_{U\setminus F(T_0)}=1$. In Lemma 1 we set $\varphi=\varphi_0$ and substitute (5) into (7). If G is an extremal mapping, then $|\varkappa_1(f(z))|\leqslant|\hat{\varkappa}(f(z))|=b(f(z))$ holds. It follows that

$$1\leqslant\iint_{U\setminus F(T^0)}|\varphi_0|\frac{1-b(f(z))}{1+b(f(z))}\frac{\left|1+b(f(z))\dfrac{\varkappa_1(f(z))}{\hat{\varkappa}(f(z))}\right|^2}{1-|\varkappa_1(f(z))|^2}\,dx\,dy$$

$$\leqslant\iint_{U\setminus F(T_0)}|\varphi_0|\frac{1-b(f(z))}{1+b(f(z))}\frac{1+|\varkappa_1(f(z))|}{1-|\varkappa_1(f(z))|}\,dx\,dy\leqslant1.$$

Therefore the equalities

$$\frac{1-b(f(z))}{1+b(f(z))}\frac{1+|\varkappa_1(f(z))|}{1-|\varkappa_1(f(z))|}=1 \tag{15}$$

and

$$\frac{1-b(f(z))}{1+b(f(z))}\frac{\left|1+b(f(z))\dfrac{\varkappa_1(f(z))}{\hat{\varkappa}(f(z))}\right|^2}{1-|\varkappa_1(f(z))|^2}=1 \tag{16}$$

hold a. e. for $z\in U\setminus F(T_0)$.

Solving (15), we have $|\varkappa_1(f(z))|=b(f(z))$, and then we substitute the last into the denominator of (16), it induces $\left|1+b(f(z))\dfrac{\varkappa_1(f(z))}{\hat{\varkappa}(f(z))}\right|=1+b(f(z))$. This is only the case $\varkappa_1(f(z))=\hat{\varkappa}(f(z))$, i. e., $G=F$. Thus F is a unique extremal mapping.

If (6) holds, we can prove $K_F=H$, where H denotes the dilatation of the boundary homeomorphism $F(e^{i\theta})$, i. e., the infimum of maximum dilatations for quasiconformal extensions of $F(e^{i\theta})$ from ∂U into all its inner neighborhoods. In fact, for any given $\varepsilon>0$, there exists a circular ring $D_r=\{0<r\leqslant|w|<1\}\subset U\setminus T$ and a quasiconformal extension h of $F(e^{i\theta})$ from ∂U into D_r with a maximal dilatation $<H+\varepsilon$. Let h^* be a quasiconformal extension of h from D_r to the whole disk

U[6, p.100]. Denote by $K(F)$ and $K(h^*)$ maximal dilatations of F and h^* in U respectively. Applying Lemma 1 in the case of $T=\varnothing$, F, $G=h^*$, $\varphi=\varphi_n$, $\|\varphi_n\|=1$, we have

$$1\leqslant(H+\varepsilon)\iint\limits_{F(D_r)}|\varphi_n|\frac{|1-\varphi_n/|\varphi_n||^2}{1-|\varkappa|^2}dx\,dy+K(F)K(h^*)\iint\limits_{U\setminus F(D_r)}|\varphi_n|dx\,dy. \qquad (17)$$

Because we may assume that φ_n has been modified by multiplicative constant so that the second half of (6) reads

$$\iint\limits_{U}\varkappa\varphi_n\,dx\,dy\to k_F,$$

thus (6) implies

$$\iint\limits_{F(D_r)}\varkappa\varphi_n\,dx\,dy\to k_F.$$

From it we can deduce

$$\iint\limits_{F(D_r)}\frac{1+|\varkappa|^2}{1-|\varkappa|^2}|\varphi_n|\,dx\,dy\to\frac{1+k_F^2}{1-k_F^2}$$

and

$$\iint\limits_{F(D_r)}\frac{\varkappa\varphi_n}{1-|\varkappa|^2}dx\,dy\to\frac{k_F}{1-k_F^2}.$$

Therefore

$$\iint\limits_{F(D_r)}|\varphi_n|\frac{|1-\varkappa\varphi_n/|\varphi_n||^2}{1-|\varkappa|^2}dx\,dy\to\frac{1+k_F^2}{1-k_F^2}-\frac{2k_F}{1-k_F^2}=\frac{1}{K_F}.$$

Setting $n\to\infty$ in (17), we have $K_F\leqslant H+\varepsilon$. Since ε is arbitrary and $K_F\geqslant H$, it follows that $K_F=H$. Hence F is an extremal mapping.

Proof of F is an extremal mapping$\Rightarrow$(I).

α) In the case of $b_0>0$, $b_1<1$. The proof has been completed in [3].

β) In the case of $b_0=0$, $b_1<1$. Take N_0 such that $\frac{1}{n}<k_F$, for $n\geqslant N_0$. Set

$$b_n(w)=\begin{cases}b(w), & b(w)\geqslant\frac{1}{n},\\ \frac{1}{n}, & b(w)<\frac{1}{n}.\end{cases}$$

And let F_n be an extremal mapping within $A(F\ \ T,\ b_n)$, if its complex dilatation on $U\setminus T$ has the essential supremum k_{F_n}. Then k_{F_n} is increasing and $k_{F_n}\leqslant k_F$.

If n_0 is fixed, then $F_n\in A(F,\ T,\ b_{n_0})$ holds for $n\geqslant n_0$. Let $\hat{\varkappa}_n(w)$ be the complex dilatation of F_n, then

$$|\hat{\varkappa}_n(w)|\leqslant\begin{cases}b_{n_0}(w), & w\in T,\\ k_F, & w\in U\setminus T,\end{cases}\quad \text{a. e..}$$

Since $A(F;\ T,\ b_n)$ is a normal family, we can assume without loss of generality that $\{F_n\}$ converges uniformly on $\overline{U}$ to a quasiconformal mapping $F_0\in A(F,\ T,\ b_{n_0})$. Because the last holds for every n_0, it follows that $F_0\in A(F,\ T,\ b)$ and $k_{F_0}\leqslant\lim\limits_{n\to\infty}k_{F_n}$

$\leqslant k_F$. We have $k_{F_0}=k_F$ or $K_{F_0}=K_F$.

In what follows we discuss two possible situations:

i) If $K_F=H$, then we shall prove that $\varkappa(z)$ admits a degenerating Hamilton sequence satisfying (4).

In fact, we define D_r and $K(F)$ as above. Setting $T^*=\{|w|<r\}$, $b^*(w)=k^*>k$, where $k^*<1$, $k=\dfrac{K(F)-1}{K(F)+1}$, we consider the extremal problem within $A(F, T^*, b^*)$. Since $K_F=H$, F is an extremal mapping within $A(F, T^*, b^*)$. According to the case α) the mapping F must satisfy the condition (II), if it is extremal within $A(F, T^*, b^*)$. Because of $|\varkappa(z)|\leqslant k<k^*=b^*(f(z))$ on T^*, there exists a sequence $\varphi_n\in\mathscr{B}(U)$, $\|\varphi_n\|_U=1$ such that

$$\lim_{n\to\infty}\varphi_n(z)=0 \text{ locally uniformly in } U, \text{ and } \lim_{n\to\infty}\left|\iint_U \varkappa\varphi_n\,dx\,dy\right|=k_F.$$

Since $\{\varphi_n\}$ is a degenerating sequence, $\|\varphi_n\|_{U\setminus F(T_0)}\to 1$ and φ_n can be replaced by $\Phi_n=\varphi_n/\|\varphi_n\|_{U\setminus(T_0)}$ with $\Phi_n\in\mathscr{B}(U)$, $\|\Phi_n\|_{U\setminus F(T)}=1$ and

$$\lim_{n\to\infty}\left|\iint_U \tau(z)\Phi_n(z)\,dx\,dy\right|=1.$$

ii) If $K_F>H$, then there exists n_0 such that $K_{F_n}>H$ holds for $n\geqslant n_0$, since $\lim\limits_{n\to\infty}K_{F_n}=K_F$. Set $f_n=F_n^{-1}$. It has been said in § 1 that f_n converges uniformly on $\overline{U}$ to $f_0=F_0^{-1}$. Let $\varkappa_n(z)$ be the complex dilatation of f_n. According to the case α) the mapping F_n must satisfy the condition (II), if it is extremal within $A(F, T, b_n)$. Since, for all such n, $\varkappa_n(z)$ does not admit a degenerating Hamilton sequence (otherwise, it can be proved as above that $K_{F_n}=H$), we see that for every $n\geqslant n_0$ there exists $\varphi_n\in\mathscr{B}(U)$ such that

$$\varkappa_n(z)=b_n(f(z))\frac{\overline{\varphi_n(z)}}{|\varphi_n(z)|}, \text{ a. e. for } z\in U$$

(where we have regarded $b_n(w)$ as k_{F_n} on $U\setminus T$), Because we may assume $\|\varphi_n\|_{U\setminus F(T_0)}=1$, $\{\varphi_n\}$ is locally uniformly bounded. Without loss of generality we suppose that $\{\varphi_n\}$ converges locally uniformly to φ_0, φ_0 cannot vanish identically (otherwise, we have $K_F=H$). Therefore we may define

$$\varkappa_0(z)=b(f(z))\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}, \text{ for } z\in U. \tag{18}$$

For any given positive numbers σ and ε, it is valid that

$$\begin{aligned}\{z\in U\,|\,|\varkappa_n(z)-\varkappa_0(z)|\geqslant\sigma\}&\subset\{z\in F_0(T)\cap F_n(T)\,|\,|\varkappa_n(z)-\varkappa_0(z)|\geqslant\sigma\}\\ &\cup\{z\in F_0(T)\setminus F_n(T)\}\cup\{z\in F_n(T)\setminus F_0(T)\}\cup\{z\,|\,r<|z|<1\}\\ &\cup\{z\in(U\setminus F_0(T))\cap(U\setminus F_n(T))\,|\,|z|\leqslant r,\ |\varkappa_n(z)-\varkappa_0(z)|\geqslant\sigma\}.\end{aligned} \tag{19}$$

By Lemma 4, since F_n converges uniformly on $\overline{U}$ to F_0, there exists n_0 such that

$$\operatorname{mes}\{z\in F_0(T)\setminus F_n(T)\}+\operatorname{mes}\{z\in F_n(T)\setminus F_0(T)\}<\frac{\varepsilon}{4}, \text{ for } n\geqslant n_0. \tag{20}$$

Take r, $0<r<1$ so that

$$\operatorname{mes}\{z \mid r<|z|<1\}<\frac{\varepsilon}{4}. \tag{21}$$

And for almost every $z\in F_0(T)\cap F_n(T)$ it holds that

$$\begin{aligned}|\varkappa_n(z)-\varkappa_0(z)| &\leqslant \left|b_n(f_n(z))\frac{\overline{\varphi_n(z)}}{|\varphi_n(z)|}-b_n(f_n(z))\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}\right| \\ &\quad+\left|b_n(f_n(z))\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}-b(f_0(z))\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}\right| \\ &\leqslant\left|\frac{\varphi_n(z)}{|\varphi_n(z)|}-\frac{\varphi_0(z)}{|\varphi_0(z)|}\right|+|b(f_n(z))-b(f_0(z))|+\frac{1}{n} \\ &\leqslant\frac{1}{n}+\left|\frac{\varphi_n(z)}{|\varphi_n(z)|}-\frac{\varphi_0(z)}{|\varphi_0(z)|}\right|+|b(f_n(z))-b_c(f_n(z))| \\ &\quad+|b_c(f_n(z))-b_c(f_0(z))|+|b_c(f_0(z))-b(f_0(z))|,\end{aligned}$$

where $b_c(w)$ is a continuous function of w.

If we take n_1 such that $\frac{1}{n}\leqslant\frac{\sigma}{3}$, for $n\geqslant n_1$, then

$$\begin{aligned}&\{z\in F_0(T)\cap F_n(T) \mid |\varkappa_n(z)-\varkappa_0(z)|\geqslant\sigma\} \\ &\quad\subset\left\{z\in F_0(T)\cap F_n(T) \;\middle|\; \left|\frac{\varphi_n(z)}{|\varphi_n(z)|}-\frac{\varphi_0(z)}{|\varphi_0(z)|}\right|\geqslant\frac{\sigma}{3}\right\} \\ &\quad\cup\{z\in F_0(T)\cap F_n(T) \mid b(f_n(z))\neq b_c(f_n(z))\} \\ &\quad\cup\left\{z\in F_0(T)\cap F_n(T) \;\middle|\; |b_c(f_n(z))-b_c(f_0(z))|\geqslant\frac{\sigma}{3}\right\} \\ &\quad\cup\{z\in F_0(T)\cap F_n(T) \mid b(f_0(z))\neq b_c(f_0(z))\}.\end{aligned} \tag{22}$$

Take again n_2 so that

$$\operatorname{mes}\left\{z\in F_0(T)\cap F_n(T) \;\middle|\; \left|\frac{\varphi_n(z)}{|\varphi_n(z)|}-\frac{\varphi_0(z)}{|\varphi_0(z)|}\right|\geqslant\frac{\sigma}{3}\right\}<\frac{\varepsilon}{16},\ \text{for } n\geqslant n_2. \tag{23}$$

We note that under a quasiconformal mapping the image area is an absolutely continuous set function. Take $b_c(w)$ by the Lusin's theorem such that mes $F_0(M)<\frac{3\varepsilon}{64}$ holds for $M=\{w\in T \mid b(w)\neq b_c(w)\}$. Hence

$$\operatorname{mes}\{z\in F_0(T)\cap F_n(T) \mid b(f_0(z))\neq b_c(f_0(z))\}\leqslant\operatorname{mes}F_0(M)<\frac{3\varepsilon}{64}. \tag{24}$$

Applying Lemma 3 and mes $F_0(M)<\frac{3\varepsilon}{64}$, we take n_3 such that mes $F_n(M)<\frac{\varepsilon}{16}$ for $n\geqslant n_3$. Therefore

$$\operatorname{mes}\{z\in F_0(T)\cap F_n(T) \mid b(f_n(z))\neq b_c(f_n(z))\}\leqslant\operatorname{mes}F_n(M)<\frac{\varepsilon}{16}. \tag{25}$$

And we take n_4 so that

$$\operatorname{mes}\left\{z\in F_0(T)\cap F_n(T) \;\middle|\; |b_c(f_n(z))-b_c(f_0(z))|\geqslant\frac{\sigma}{3}\right\}<\frac{\varepsilon}{16},\ n\geqslant n_4. \tag{26}$$

On the other hand, for almost every $z\in(U\setminus F_0(T))\cap(U\setminus F_n(T))$, it holds that

$$|\varkappa_n(z)-\varkappa_0(z)|\leqslant\left|b_n(f_n(z))\frac{\overline{\varphi_n(z)}}{|\varphi_n(z)|}-b_n(f_n(z))\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}\right|$$
$$+\left|b_n(f_n(z))\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}-b(f_0(z))\frac{\overline{\varphi_0(z)}}{|\varphi_0(z)|}\right|$$
$$\leqslant\left|\frac{\varphi_n(z)}{|\varphi_n(z)|}-\frac{\varphi_0(z)}{|\varphi_0(z)|}\right|+|k_{F_n}-k_F|.$$

If we take n_5 such that $|k_{F_n}-k_F|\leqslant\frac{\sigma}{2}$ holds for $n\geqslant n_5$, then

$$\{z\in(U\backslash F_0(T))\cap(U\backslash F_n(T))\,|\,|z|\leqslant r,\ |\varkappa_n(z)-\varkappa_0(z)|\geqslant\sigma\}$$
$$\subset\left\{z\in(U\backslash F_0(T))\cap(U\backslash F_n(T))\,|\,|z|\leqslant r,\ \left|\frac{\varphi_n(z)}{|\varphi_n(z)|}-\frac{\varphi_0(z)}{|\varphi_0(z)|}\right|\geqslant\frac{\sigma}{2}\right\}. \tag{27}$$

Take again n_6 so that

$$\operatorname{mes}\left\{z\in(U\backslash F_0(T))\cap(U\backslash F_n(T))\,|\,|z|\leqslant r,\ \left|\frac{\varphi_n(z)}{|\varphi_n(z)|}-\frac{\varphi_0(z)}{|\varphi_0(z)|}\right|\geqslant\frac{\sigma}{2}\right\}$$
$$<\frac{\varepsilon}{4},\ \text{for } n\geqslant n_6. \tag{28}$$

Combining (19) to (28) we get

$$\operatorname{mes}\{z\in U\,|\,|\varkappa_n(z)-\varkappa_0(z)|\geqslant\sigma\}<\varepsilon,\ \text{for } n\geqslant\max\{n_0, n_1, n_2, n_3, n_4, n_5, n_6\}.$$

We have proved that $\{\varkappa_n(z)\}$ converges in measure in U to $\varkappa_0(z)$. Hence there exists a subsequence $\{\varkappa_{n_k}(z)\}$ converging almost everywhere in U to $\varkappa_0(z)$. Since $\{f_{n_k}(z)\}$ converges uniformly on $\overline{U}$ to $f_0(z)$, by a theorem of Bers [6, p.197], $f_0(z)$ is a quasiconformal mapping in U with the complex dilatation $\varkappa_0(z)$. And for $\varkappa_0(z)$ which possesses the representative (18), it has been shown as above that F_0 is a unique extremal mapping within $A(F, T, b)$. Hence $F=F_0$, $\varkappa(z)=\varkappa_0(z)$ and $\tau(z)$ has the representative (3). Therefore, if we set $\varphi=\varphi_0/\|\varphi_0\|_{U\backslash F(T_0)}$, then $\varphi\in\mathscr{B}(U)$ and $\|\varphi\|_{U\backslash F(T_0)}=1$, it holds that

$$\iint_U\tau(z)\varphi(z)\,dx\,dy=1.$$

Thus, for the case of $b_0\geqslant0$, $b_1<1$, Theorem 1 is proved.

γ) In the case of $b_0\geqslant0$, $b_1=1$. We take N_0 such that $1-\frac{1}{n}\geqslant\max\{k_F, \tilde{k}\}$ for $n\geqslant N_0$, where $\tilde{k}$ denotes the essential supremum of the complex dilatation for certain quasiconformal mapping (its existence is guaranteed by the hypothesis) within $A(F, T, b)$. Set

$$b_n(w)=\begin{cases}b(w), & \text{for } b(w)\leqslant1-\dfrac{1}{n},\\[2mm] 1-\dfrac{1}{n}, & \text{for } b(w)>1-\dfrac{1}{n}.\end{cases}$$

Thus $A(F, T, b_n)$ is nonempty for $n\geqslant N_0$. Let F_n be an extremal mapping within $A(F, T, b_n)(\subset A(F, T, b))$, k_{F_n} the essential supremum on $U\backslash T$ of its complex dilatation. Then k_{F_n} is decreasing and $k_{F_n} \geqslant k_F$. If we set $\lim_{n\to\infty} k_{F_n} = k_0$, then $k_0 \geqslant k_F$. Since, for $n \geqslant N_0$, it holds that $F_n \in A^{k_{F_{N_0}}}$, and the last is a normal family, we may assume without loss of generality that $\{F_n\}$ converges uniformly on $\overline{U}$ to F_0. It is obvious that $F_0 \in A(F, T, b)$ and $k_{F_0} \leqslant k_0$.

In what follows we still discuss two possible situations:

i) For $K_F = H$, provided we substitute a quasiconformal extension F^* of F from D_r to the whole disk U for F, we will be able to prove as β) that there exists a sequence $\Phi_n \in \mathscr{B}(U)$, $\|\Phi_n\|_{U\setminus F(T_0)} = 1$ such that

$$\lim_{n\to\infty} \left| \iint_U \tau(z) \Phi_n(z)\, dx\, dy \right| = 1. \tag{29}$$

ii) For $K_F > H$, provided in the representative (18) we substitute the constant k_0 for k_F on $U\setminus T$, we will be able to prove as β) that $f_0 = F_0^{-1}$ is an almost quasiconformal mapping in U with the complex dilatation $\varkappa_0(z)$ (from a quasiconformal mapping in β) to an almost quasiconformal mapping, when we apply the fact that the image area is an absolutely continuous set function, we only need to substitute Lemma 2 for it, and when we apply the Bers' theorem, we only need to apply it in $U\setminus\overline{\Omega}$, where Ω is any open set containing E_1, and then set mes $\Omega \to 0$), and that F_0 is a unique extremal mapping within $A(F, T, b)$. Hence $F = F_0$, $\varkappa(z) = \varkappa_0(z)$ and $\tau(z)$ has the representative (3). Therefore, for $\varphi = \varphi_0 / \|\varphi_0\|_{U\setminus F(T_0)} \in \mathscr{B}(U)$, $\|\varphi\|_{U\setminus F(T_0)} = 1$, it is valid that

$$\iint_U \tau(z) \varphi(z)\, dx\, dy = 1. \tag{30}$$

Theorem 1 is completely proved.

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