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THE GLOBAL SMOOTH SOLUTIONS OF SECOND ORDER QUASILINEAR HYPERBOLIC EQUATIONS WITH DISSIPATIVE BOUNDARY CONDITIONS

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Abstract

The paper deals with the following boundary problem of the second order quasilinear hyperbolic equation with a dissipative boundary condition on a part of the boundary:

$$u_{tt}-\sum_{i,j=1}^{n}a_{ij}(Du)u_{x_ix_j}=0, \text{ in } (0, \infty)\times\Omega,$$
$$u|_{\Gamma_0}=0,$$
$$\sum_{i,j=1}^{n}a_{ij}(Du)n_ju_{x_i}+b(Du)u_t|_{\Gamma_1}=0,$$
$$u|_{t=0}=\varphi(x),\ u_t|_{t=0}=\psi(x), \text{ in } \Omega,$$

where $\partial\Omega=\Gamma_0\cup\Gamma_1$, $b(Du)\geqslant b_0>0$. Under some assumptions on the equation and domain, the author proves that there exists a global smooth solution for above problem with small data.

§ 1. Introduction

It is well known that the smooth solutions of boundary value problems for quasilinear hyperbolic equations exist only locally in time in general, that is, their smooth solutions may blow up in a finite time, even if the initial data are smooth. However, an additional dissipative term which implies a damping effect on the equation can guarantee the existence for global smooth solution on $t\geqslant 0$ of the boundary value problem at least for small data (for example, see [1, 2]). Greenberg and Li Ta-tsien[3] discussed the nonlinear vibration problem of a string, instead of adding a dissipative term to the equation, they gave a dissipative boundary condition on one end of the string, which implies the damping effect. They proved that this dissipative boundary condition can guarantee the existence for the global smooth solution of the boundary value problem for small initial data, too.

In this paper, we discuss the initial-boundary value problems of second order quasilinear hyperbolic equations for several dimensions with a nonlinear dissipative boundary condition on a part of the boundary. Under some hypotheses on the

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coefficients of the equation and the domain for the boundary value problem, we prove the existence of global smooth solution and the decay of the solution in time for small data.

In § 2, we shall state our main results. Several preliminary lemmas will be given in § 3. In § 4, we shall study the initial boundary value problems for linear second order hyperbolic equations with the dissipative boundary conditions. The results in § 4 are necessary for discussion on the nonlinear problems. As the first step of discussing the nonlinear problems, we shall study the local solutions of the nonlinear boundary value problems in § 5. The final section is devoted to the proof of the main results.

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§ 2. The Main Results

We discuss the following initial–boundary problem:

$$u_{tt}-\sum_{i,j=1}^{n}a_{ij}(Du)u_{x_ix_j}=0, \text{ in } (0, \infty)\times\Omega, \tag{2.1}$$

$$u|_{\Gamma_0}=0, \tag{2.2}$$

$$\sum_{i,j=1}^{n}a_{ij}(Du)n_ju_{x_i}+b(Du)u_t|_{\Gamma_1}=0, \tag{2.3}$$

$$\begin{aligned} u|_{t=0}&=\varphi(x),\\ u_t|_{t=0}&=\psi(x), \end{aligned} \tag{2.4}$$

where $Du=(u, u_t, u_{x_1}, \cdots, u_{x_n})$, $\Gamma_0\cup\Gamma_1=\partial\Omega$ is the boundary of the domain Ω, $\boldsymbol{n}=(n_1, \cdots, n_n)$ is the outer normal to $\partial\Omega$.

Now, we state the hypothese on the coefficients of the equation (2.1), the boundary condition (2.3) and the domain Ω. We assume that

A$_1$) $a_{ij}(\Lambda)=a_{ji}(\Lambda)$, $a_{ij}(\Lambda)$ are sufficiently smooth on their variables and satisfy

$$\sum_{i,j=1}^{n}a_{ij}(\Lambda)\xi_i\xi_j\geqslant a^2|\xi|^2, \ \forall\xi\in\mathbb{R}^n,$$

for suitably small $|\Lambda|$, where $\Lambda=(\lambda_0, \lambda_1, \cdots, \lambda_{n+1})$ and a is a positive constant.

A$_2$) $\nabla_\Lambda a_{ij}(0)=0$.

B$_1$) $b(\Lambda)$ is sufficiently smooth and

$$b(\Lambda)\geqslant b_0>0,$$

for suitably small $|\Lambda|$, where b_0 is a constant.

B$_2$) $\nabla_\Lambda b(0)=0$.

C$_1$) The domain Ω is bounded with sufficiently smooth boundary $\partial\Omega$ which consists of tow parts Γ_0 and Γ_1 which do not intersect.

C$_2$) Ω satisfies the star–shaped condition in the following sense: there is a point

o (without loss of generality, we may take it as origin) in $\mathbb{R}^n$ such that

$$x\cdot n|_{\Gamma_0}\leqslant 0, \tag{2.5}$$

$$x\cdot n|_{\Gamma_1}\geqslant\gamma_1>0, \tag{2.6}$$

where γ_1 is a constant.

The main results in this paper are the following theorem.

Main theorem. *Assume that* $A_1)-C_2)$ *hold,* $s\geqslant 2\left[\frac{n}{2}\right]+3$, *the initial data* (2.4) *and the boundary conditions* (2.2) (2.3) *satisfy the compatibility conditions up to order* s. *Then there exists* $\varepsilon>0$ *such that the initial-boundary value problem* (2.1)—(2.4) *admits a unique global smooth solution* $u\in C^i([0,\infty); H_{s+1-i}(\Omega))$ $(i=0, 1, \cdots, s+1)$ *and* $\|D^{s+1}u(t)\|_0$ *decays to zero exponentially (in time) as* $t\to+\infty$, *provided*

$$\|\varphi\|_{s+1}+\|\psi\|_s\leqslant\varepsilon.$$

Here, H_s denotes the usual Sobolev's space with the norm $\|\cdot\|_s$. And below, when there is no chance of confusion, H_s and $\|\cdot\|_s$ denote the Sobolev's space and its norm on Ω, respectively.

One of typical examples for the main theorem is the following problem of vibration of a fastened membrane with a boundary friction:

$$u_{tt}-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{u_{x_i}}{\sqrt{1+|\nabla u|^2}}\right)=0, \text{ in } (0,\infty)\times\Omega,$$

$$u|_{\Gamma_0}=0,$$

$$\frac{1}{\sqrt{1+|\nabla u|^2}}\frac{\partial u}{\partial n}+\alpha u_t|_{\Gamma_1}=0,$$

$$u|_{t=0}=\varphi(x),\ u_t|_{t=0}=\psi(x),$$

where $\alpha>0$ is a constant. It is easy to see that the main theorem applies to above problem if we take $b(Du)=\frac{\alpha}{1+|\nabla u|^2}$.

§3. Preliminaries

We consider the following linearized problem:

$$u_{tt}-\sum_{i,j=1}^{n}a_{ij}(Dv)u_{x_ix_j}=0, \text{ in } (0,T)\times\Omega, \tag{3.1}$$

$$u|_{\Gamma_0}=0, \tag{3.2}$$

$$\sum_{i,j=1}^{n}a_{ij}(Dv)n_ju_{x_i}+b(Dv)u_t|_{\Gamma_1}=0, \tag{3.3}$$

$$\begin{aligned}u|_{t=0}&=\varphi(x),\\ u_t|_{t=0}&=\psi(x).\end{aligned} \tag{3.4}$$

For the problem (3.1)—(3.4), we introduce the following energy integral:

$$Q_v^{(s)}(u(t))\triangleq\int_\Omega\left\{\frac{1}{2}(t+\lambda)|D_t^su_t|^2+\frac{1}{2}(t+\lambda)\sum_{i,j=1}^{n}a_{ij}(Dv)D_t^su_{x_i}D_t^su_{x_j}\right.$$

$$+2D_t^s u_t \sum_{i=1}^n x_i D_t^s u_{x_i} + (n-1) D_t^s u \cdot D_t^s u_t \Big\} dx, \tag{3.5}$$

where λ is a constant, which is determined later,

$$D_t^k u = \left\{ \left(\frac{\partial}{\partial t}\right)^{\alpha_0} \right\},\ 0 \leqslant \alpha_0 \leqslant k,$$

$$D_x^k u = \left\{ \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \right\},\ 0 \leqslant \alpha_1 + \cdots + \alpha_n \leqslant k,$$

$$D^k u = \left\{ \left(\frac{\partial}{\partial t}\right)^{\alpha_0} \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \right\},\ 0 \leqslant \alpha_0 + \alpha_1 + \cdots + \alpha_n \leqslant k.$$

Especially when $v = u$, we simply write $Q_v^{(s)}(u(t)) = Q^{(s)}(u(t))$. This energy integral, which is similar to that introduced by W. A. Strauss[4] and G. Chen[5] in studying the decays of solutions for wave equation, plays an important role in the following discussion.

We give some lemmas which are used in the folowing sections.

Lemma 3.1. *Suppose that $a_{ij}(\Lambda)$ and $b(\Lambda)$ are sufficiently smooth on their varaibles, $s \geqslant 2\left[\frac{n}{2}\right]+3$, $\lambda \geqslant 1$,*

$$\sup_{0 \leqslant t \leqslant T} \|D^{s+1} v(t)\|_0^2 \leqslant \eta^2.$$

Then

1) $\displaystyle\sum_{i,j=1}^n |D_t^s(a_{ij}(Dv) u_{x_j}) - a_{ij}(Dv) D_t^s u_{x_j}|$

$$\leqslant c(\eta) \Big(\sum_{j=1}^n |D_t^{s-1} u_{x_j}| + \|D^{s+1} u(t)\|_0 |D_t^s Dv| \Big), \tag{3.6}$$

2) $\displaystyle\sum_{i,j=1}^n |D_t^s(a_{ij}(Dv) u_{x_i x_j}) - a_{ij}(Dv) D_t^s u_{x_i x_j}|$

$$\leqslant c(\eta) \Big(\sum_{i,j=1}^n |D_t^{s-1} u_{x_i x_j}| + \|D^{s+1} u(t)\|_0 |D_t^s Dv| \Big), \tag{3.7}$$

3) $|D_t^s(b(Dv) u_t) - b(Dv) D_t^s u_t| \leqslant c(\eta) (|D_t^{s-1} u_t| + \|D^{s+1} u(t)\|_0 |D_t^s Dv|)$. $\quad$ (3.8)

Moreover, if $a_{ij}(\Lambda)$, $b(\Lambda)$ satisfy A_2), B_2) respectively and

$$\sup_{0 \leqslant t \leqslant T} (t+\lambda) \|D^{s+1} v(t)\|_0^2 \leqslant \eta^2,$$

then

1') $\displaystyle\sum_{i,j=1}^n |D_t^s(a_{ij}(Dv) u_{x_j}) - a_{ij}(Dv) D_t^s u_{x_j}|$

$$\leqslant c(\eta) \eta (t+\lambda)^{-1/2} \Big(\sum_{j=1}^n |D_t^{s-1} u_{x_j}| + \|D^{s+1} u(t)\|_0 |D_t^s Dv| \Big), \tag{3.6$'$}$$

2') $\displaystyle\sum_{i,j=1}^n |D_t^s(a_{ij}(Dv) u_{x_i x_j}) - a_{ij}(Dv) D_t^s u_{x_i x_j}|$

$$\leqslant c(\eta) \eta (t+\lambda)^{-1/2} \Big((t+\lambda)^{-1/2} \sum_{i,j=1}^n |D_t^s u_{x_i x_j}| + \|D^{s+1} u(t)\|_0 |D_t^s Dv| \Big), \tag{3.7$'$}$$

3') $|D_t^s(b(Dv) u_t) - b(Dv) D_t^s u_t| \leqslant C(\eta) \eta (t+\lambda)^{-1/2} (|D_t^{s-1} u_t| + \|D^{s+1} u(t)\|_0 |D_t^s Dv|)$

(3.8)$'$

where $c(\eta)$, which are independent of T and λ, are positive increasing functions and

$c(\eta)=O(1)$ *as* $\eta\to 0$.

By using the Sobolev's imbedding theorem, it is not difficult to prove the above lemma. We omit it.

Lemma 3.2. *Let* $s\geqslant 2\left[\frac{n}{2}\right]+3$. *Assume that* $a_{ij}(\Lambda)$ *satisfy the hypothesis* A₁) *and* $v\in H_{s+1}(\Omega)$ *satisfies*

$$\|v\|_{s+1}\leqslant\eta. \tag{3.9}$$

If u *is a solution of the following elliptical boundary value problem:*

$$\sum_{i,j=1}^{n} a_{ij}(Dv)u_{x_ix_j}=f(x),\ in\ \Omega, \tag{3.10}$$

$$u|_{\Gamma_0}=0, \tag{3.11}$$

$$\sum_{i,j=1}^{n} a_{ij}(Dv)n_ju_{x_i}|_{\Gamma_1}=g(x), \tag{3.12}$$

then

$$\|u\|_2\leqslant c(\eta)(\|f\|_0+\|g\|_1+\|u\|_0). \tag{3.13}$$

Lemma 3.3. *Let* $s\geqslant 2\left[\frac{n}{2}\right]+3$. *Assume that* $a_{ij}(\Lambda)$ *satiafy the hypothesis* A₁) *and that* $v\in C^i([0,T];H_{s+1-i}(\Omega))\ (i=0,\cdots,s+1)$ *satisfies*

$$\sup_{0\leqslant t\leqslant T}\|D^{s+1}v(t)\|_0\leqslant\eta. \tag{3.14}$$

If $u\in C^i([0,T];H_{s+1-i}(\Omega))\ (i=0,\cdots,s+1)$ *is a solution of the boundary value problem* (3.1)—(3.4), *then*

$$\|D^{s+1}u(t)\|_0\leqslant C(\eta)\{\|D_t^{s+1}u\|_0+\|D_t^sD_xu\|_0\}. \tag{3.15}$$

The proofs of Lemmas 3.2 and 3.3 are omitted.

Lemma 3.4. *Let* $s\geqslant 2\left[\frac{n}{2}\right]+3$ *and* $\beta>1$ *be an arbitrary constant. Assume that* $a_{ij}(\Lambda)$ *satisfy the hypothesis* A₁) *and that* $u\in C^i([0,T];H_{s+1-i}(\Omega))\ (i=0,\cdots,s+1)$ *is a solution of boundary value problem* (3.1)—(3.4). *Then there exists* $\eta_0>0$ *and* $\lambda_0>0$ such that

$$\alpha(t+\lambda)\|D^{s+1}u(t)\|_0^2\leqslant Q_v^{(s)}(u(t))$$
$$\leqslant\frac{\beta}{2}(t+\lambda)\int_\Omega\left\{|D_t^su_t|^2+\sum_{i,j=1}^{n}Q_{ij}(Dv)D_t^su_{x_i}D_t^su_{x_j}\right\}dx, \tag{3.16}$$

provided $|Dv|\leqslant\eta_0$. and $\lambda\geqslant\lambda_0$, *where* α *is a positive constant.*

Proof From the definition of $Q_v^{(s)}(u(t))$, the right side of inequality (3.16) is easy to justify.

Taking the boundary condition (3.2) into account and using the generalized Poincares inequality and Lemma 3.3, we obtain the left side of inequality (3.16) immediately.

§ 4. Linear Boundary Value Problem

In this section, we discuss the following linear boundary value problem:

$$Lu \triangleq u_{tt}-\sum_{i,j=1}^{n} a_{ij}(x, t)u_{x_i x_j}+\sum_{i=1}^{n} b_i(x, t)u_{x_i}+c(x, t)u=f(x, t), \text{ in } (0, T)\times\Omega, \tag{4.1}$$

$$u|_{\Gamma_0}=0, \tag{4.2}$$

$$\sum_{i,j=1}^{n} a_{ij}(x, t)n_j u_{x_i}+b(x, t)u_t|_{\Gamma_1}=0, \tag{4.3}$$

$$u|_{t=0}=\varphi(x),\ u_t|_{t=0}=\psi(x), \tag{4.4}$$

where the coefficients of equation (4.1) and $b(x, t)$ are sufficiently smooth, $a_{ij}(x, t)=a_{ji}(x, t)$ and

$$\sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \geqslant a^2|\xi|^2,\ \forall (x, t)\in(0, T)\times\Omega,\ \xi\in\mathbb{R}^n, \tag{4.5}$$

$$b(x, t)\geqslant b_0>0. \tag{4.6}$$

We have the following theorem.

Theorem 4.1. *Suppose that* $\varphi\in H_1$, $\psi\in H_0$ *and* $f\in L^2((0, T)\times\Omega)$. *Then the boundary value problem* (4.1)—(4.4) *admits a unique strong solution* $u\in C^i([0, T]; H_{1-i}(\Omega))$ $(i=0, 1)$ *in the following sence: there exists a sequence* $\{u^{(k)}\}$ *of smooth functions which satisfy the boundary conditions* (4.2) (4.3) *such that*

$$\|u^{(k)}-u\|_{H_1((0,T)\times\Omega)}\to 0, \tag{4.7}$$

$$\|Lu^{(k)}-f\|_{L^2((0,T)\times\Omega)}\to 0, \tag{4.8}$$

$$\|u^{(k)}(0)-\varphi\|_1+\|u_t^{(k)}(0)-\psi\|_0\to 0, \tag{4.9}$$

as $k\to\infty$.

Proof By reducing the problem (4.1)—(4.4) to a boundary value problem for a first order symmetric system and then using the related results for first order symmetric hyperbolic systems (cf., for example, [6]), it is not difficult to obtain the theorem.

The boundary $\partial\Omega$ of Ω is characteristic of the first order system which is reduced from equation (4.1), but $\partial\Omega$ is not characteristic of equation (4.1), so we can obtain the result on regularity for the solution of boundary value problem (4.1)—(4.4) in a similar way.

Theorem 4.2. *Assume that* $\varphi\in H_{s+1}$, $\psi\in H_s$ *and* $f\in H_s((0, T)\times\Omega)$ *satisfy the compatibility conditions up to order* s. *Then the boundary value problem* (4.1)—(4.4) *admits a unique strong solution* $u\in C^i([0, T]; H_{s+1-i}(\Omega))$ $(i=0, \cdots, s+1)$ *in the following sense: there exists a sequence* $\{u^{(k)}\}$ *of smooth functions which satisfy the boundary conditions* (4.2)(4.3) *such that*

$$\|u^{(k)}-u\|_{s+1((0,T)\times\Omega)}\to 0, \tag{4.10}$$

$$\|Lu^{(k)}-f\|_{s((0,T)\times\Omega)}\to 0, \tag{4.11}$$
$$\|u^{(k)}(0)-\varphi\|_{s+1}+\|u_t^{(k)}(0)-\psi\|_s\to 0, \tag{4.12}$$

as $k\to\infty$.

In above two theorems, the hypothesis C_2) on the boundary $\partial\Omega$ of Ω is not necessary. Moreover, if the hypothesis C_2) is satisfied, then we have further the following theorem.

Theorem 4.3. *Assume that the hypotheses* C_1) *and* C_2) *are satified, and that* $\varphi\in H_1$, $\psi\in H_0$ and $f\in L^2((0, T)\times\Omega)$. *Then the strong solution* $u\in C^i([0, T]; H_{1-i}(\Omega))$ $(i=0, 1)$ *of boundary value problem* (4.1)—(4.4) *admits boundary values* u_t, u_{x_i} *on* Γ_1 *in the following sense: there exists a sequence* $\{u^{(k)}\}$ *of smooth functions which satisfy the boundary conditions* (4.2) (4.3) *such that*

$$\|u_t^{(k)}-u_t\|_{L^2((0,T)\times\Gamma_1)}+\sum_{i=1}^n\|u_{x_i}^{(k)}-u_{x_i}\|_{L^2((0,T)\times\Gamma_1)}\to 0 \tag{4.13}$$

holds as $k\to\infty$ *besides* (4.7), (4.8) *and* (4.9) *in Theorem* 4.1.

Moreover, if instead of condition (2.5) *we suppose that*

$$\boldsymbol{x}\cdot\boldsymbol{n}|_{\Gamma_0}\leqslant-\gamma_0<0, \tag{4.14}$$

then the solution u *of problem* (4.1)—(4.4) *admits boundary value* u_{x_i} *on* Γ_0 *in the sense mentioned above.*

Proof Let $\{u^{(k)}\}$ be a sequence for the strong solution of problem (4.1)—(4.4) in Theorem 4.1. From

$$\begin{aligned}\int_\Omega\Big(u_{tt}^{(k)}-\sum_{i,j=1}^n a_{ij}u_{x_ix_j}^{(k)}+\sum_{i=1}^n b_iu_{x_i}^{(k)}+cu^{(k)}\Big)\Big\{(t+\lambda)u_t^{(k)}+2\sum_{i=1}^n x_iu_{x_i}^{(k)}+(n-1)u^{(k)}\Big\}dx\\ =\int_\Omega Lu^{(k)}\Big\{(t+\lambda)u_t^{(k)}+2\sum_{i=1}^n x_iu_{x_i}^{(k)}+(n-1)u^{(k)}\Big\}dx,\end{aligned} \tag{4.15}$$

we can obtain

$$\begin{aligned}\frac{d}{dt}Q(u^{(k)}(t))&=I_0+I_1+I_2\\ &+\int_\Omega\Big(Lu^{(k)}-\sum_{i=1}^n b_iu_{x_i}^{(k)}-cu^{(k)}\Big)\Big\{(t+\lambda)u_t^{(k)}+2\sum_{i=1}^n x_iu_{x_i}^{(k)}+(n-1)u^{(k)}\Big\}dx,\end{aligned} \tag{4.16}$$

where

$$\begin{aligned}Q(u^{(k)}(t))=\int_\Omega\Big\{\frac{1}{2}(t+\lambda)|u_t^{(k)}|^2+\frac{1}{2}(t+\lambda)\sum_{i,j=1}^n a_{ij}u_{x_i}^{(k)}u_{x_j}^{(k)}\\ +2u_t\sum_{i=1}^n x_iu_{x_i}^{(k)}+(n-1)u^{(k)}u_t^{(k)}\Big\}dx,\end{aligned} \tag{4.17}$$

$$I_0=2\int_{\Gamma_0}\sum_{i,j,l=1}^n a_{ij}n_jx_lu_{x_i}^{(k)}u_{x_l}^{(k)}dS-\int_{\Gamma_0}(\boldsymbol{x}\cdot\boldsymbol{n})\sum_{i,j=1}^n a_{ij}u_{x_i}^{(k)}u_{x_j}^{(k)}dS, \tag{4.18}$$

$$\begin{aligned}I_1=-\int_{\Gamma_1}\{b(t+\lambda)-\boldsymbol{x}\cdot\boldsymbol{n}\}|u_t^{(k)}|^2dS-2\int_{\Gamma_1}bu_t^{(k)}\sum_{l=1}^n x_lu_{x_l}^{(k)}dS\\ -\int_{\Gamma_1}(\boldsymbol{x}\cdot\boldsymbol{n})\sum_{i,j=1}^n a_{ij}u_{x_i}^{(k)}u_{x_j}^{(k)}dS-(n-1)\int_{\Gamma_1}bu_t^{(k)}u^{(k)}dS,\end{aligned} \tag{4.19}$$

$$I_2=-\frac{1}{2}\int_\Omega|u_t^{(k)}|^2\,dx-\frac{1}{2}\int_\Omega\sum_{i,j=1}^n a_{ij}u_{x_i}^{(k)}u_{x_j}^{(k)}\,dx+\frac{1}{2}\int_\Omega(t+\lambda)\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial t}u_{x_i}^{(k)}u_{x_j}^{(k)}\,dx$$
$$-\int_\Omega(t+\lambda)\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial x_j}u_{x_i}^{(k)}u_t^{(k)}\,dx-\int_\Omega\sum_{i,j,l=1}^n x_l u_{x_i}^{(k)}\left\{2\frac{\partial a_{ij}}{\partial x_j}u_{x_l}^{(k)}-\frac{\partial a_{ij}}{\partial x_l}u_{x_j}^{(k)}\right\}dx$$
$$-(n-1)\int_\Omega\sum_{i,j=1}^n\frac{\partial a_{ij}}{\partial x_j}u_{x_i}^{(k)}u^{(k)}\,dx. \tag{4.20}$$

Taking the boundary condition (4.2), the hypotheses (2.5) and (4.5) into account, we obtain without difficulty

$$I_0=\int_{\Gamma_0}(\boldsymbol{x}\cdot\boldsymbol{n})\sum_{i,j=1}^n a_{ij}n_in_j|\nabla u^{(k)}|^2\,dS\leqslant 0. \tag{4.21}$$

Moreover, if the hypothesis (4.14) holds, then

$$I_0\leqslant-\gamma_0a^2\int_{\Gamma_0}|\nabla u^{(k)}|^2\,dS. \tag{4.22}$$

Using the hypotheses (2.6), (2.5) and the generalized Poincare's inequality, from (4.19) we have for certain constant C

$$I_1\leqslant-\frac{1}{2}\gamma_1a^2\int_{\Gamma_1}|\nabla u^{(k)}|^2\,dS-\int_{\Gamma_1}\{b_0(t+\lambda)-\boldsymbol{x}\cdot\boldsymbol{n}-C\}|u_t^{(k)}|^2\,dS+C\int_\Omega|\nabla u^{(k)}|^2\,dx. \tag{4.23}$$

Choose λ such that

$$\lambda\geqslant 2b_0^{-1}(\boldsymbol{x}\cdot\boldsymbol{n}+C),\ \forall x\in\Gamma_1. \tag{4.24}$$

Then from (4.23) we have

$$I_1\leqslant-\frac{1}{2}\gamma_1a^2\int_{\Gamma_1}|\nabla u^{(k)}|^2\,dS-\frac{1}{2}b_0(t+\lambda)\int_{\Gamma_1}|u_t^{(k)}|^2\,dS+C\int_\Omega|\nabla u^{(k)}|^2\,dx. \tag{4.25}$$

From (4.20), for certain positive constant C_T (depending on T) we obtain

$$I_2\leqslant C_T\int_\Omega(|u^{(k)}|^2+|u_t^{(k)}|^2+|\nabla u^{(k)}|^2)\,dx, \tag{4.26}$$

and

$$\int_\Omega\Big(Lu^{(k)}-\sum_{i=1}^n b_iu_{x_i}^{(k)}-Cu^{(k)}\Big)\Big\{(t+\lambda)u_t^{(k)}+2\sum_{i=1}^n x_iu_{x_i}^{(k)}+(n-1)u^{(k)}\Big\}dx$$
$$\leqslant C_T\int_\Omega(|u^{(k)}|^2+|u_t^{(k)}|^2+|\nabla u^{(k)}|^2+|Lu^{(k)}|^2)dx. \tag{4.27}$$

Integrating (4.16) from 0 to t, from (4.21), (4.25), (4.26) and (4.27) we obtain

$$Q(u^{(k)}(t))+\gamma_0a^2\int_0^t\int_{\Gamma_0}|\nabla u^{(k)}|^2\,dSd\tau+\frac{1}{2}\gamma_1a^2\int_0^t\int_{\Gamma_1}|\nabla u^{(k)}|^2\,dSd\tau$$
$$+\frac{1}{2}b_0\int_0^t(\tau+\lambda)\int_{\Gamma_1}|u_t^{(k)}|^2\,dSd\tau$$
$$\leqslant Q(u^{(k)}(0))+C_T\int_0^T\int_\Omega(|u^{(k)}|^2+|u_t^{(k)}|^2+|\nabla u^{(k)}|^2+|Lu^{(k)}|^2)dx\,d\tau. \tag{4.28}$$

Take λ such that $Q(u^{(k)}(t))\geqslant 0$ and (4.24) holds. Then from (4.28) and the definition of strong solutions, for certain constant C_T, we have

$$\int_0^t\int_{\Gamma_0}|\nabla u^{(k)}|^2dSd\tau+\int_0^t\int_{\Gamma_1}(|u_t^{(k)}|^2+|\nabla u^{(k)}|^2)dSd\tau\leqslant C_T,\ \forall t\in[0,T]. \tag{4.29}$$

According to Banach-Saks' theorem, there exists a subsequence $\{u^{(k_l)}\}$ of $\{u^{(k)}\}$

such that the derivative sequences $\{w_t^{(m)}\}$ and $\{w_{x_i}^{(m)}\}$ of the arithmetic mean sequence

$$\left\{w^{(m)}=\frac{1}{m}(u^{(k_1)}+\cdots+u^{(k_m)})\right\}$$

converge to certain u_t and u_{x_i} in $L^2((0, T)\times\Gamma_1)$ respectively, It is not difficult to show that $\{w^{(m)}\}$ is still a sequence which defines the strong solution.

From the inequality (4.28), it is easy to see that the boundary values u_t and u_{x_i} are uniquely determined.

In a similar way, we can prove the following theorem.

Theorem 4.4. *Assume that the hypotheses C_1) and C_2) are satisfied, and that $\varphi\in H_{s+1}$, $\psi\in H_s$ and $f\in H_s((0, T)\times\Omega)$ satisfy the compatibility conditions up to order s. Then the strong solution*

$$u\in C^i([0, T]; H_{s+1-i}(\Omega)) \quad (i=0, 1, \cdots, s+1)$$

of the boundary value problem (4.1)—(4.4) *admits boundary values $D_t^{s+1}u$ and $D_t^s u_{x_i}$ on Γ_1 in the following sense: there exists a sequence $\{u^{(k)}\}$ of smooth functions which satisfy the bounday conditions* (4.2), (4.3) *such that*

$$\|D_t^{s+1}u^{(k)}-D_t^{s+1}u\|_{L^2((0,T)\times\Gamma_1)},$$
$$\sum_{i=1}^n\|D_t^s u_{x_i}^{(k)}-D_t^s u_{x_i}\|_{L^2((0,T)\times\Gamma_1)}\to 0 \tag{4.30}$$

holds as $k\to\infty$, besides (4.10)—(4.12) *in Theorem* 4.2.

§ 5. The Existence of Local Solution for the Nonlinear Boundary Value Problem

In this section, under the assumption that the initial data φ and ψ are sufficiently small on the boundary $\partial\Omega$ of Ω, we prove the existence of local solution for the nonlinear boundary value problem (2.1)—(2.4).

Assume that there exists a function $\bar{u}\in C^i([0, T]; \mathring{H}_{s+2-i}(\Omega))$ $(i=0, 1, \cdots, s+2)$ such that

$$\|D^{s+1}u(0)-D^{s+1}\bar{u}(0)\|\leqslant\varepsilon, \tag{5.1}$$

where ε is a small positive constant which will be determined by φ, ψ and equation (2.1).

We introduce the following norms

$$\|u\|_{s+1,T}^2\triangleq\sup_{0\leqslant t\leqslant T}\|D^{s+1}u(t)\|_0^2+\int_0^T\left(\|D_t^s u_t(t)\|_{L^2(\Gamma_1)}^2+\sum_{i=1}^n\|D_t^s u_{x_i}(t)\|_{L^2(\Gamma_1)}^2\right)dt.$$

Let $B_T(\delta)$ be the set of all functions

$$v(x, t)\in C^i([0, T]; H_{s+1-i}(\Omega)) \quad (i=0, 1, \cdots, s+1)$$

which satisfy

$$\|D^{s+1}v(0)\|_0=\|D^{s+1}u(0)\|_0,$$
$$\|v-\bar{u}\|_{s+1,T}\leqslant\delta. \tag{5.2}$$

Here δ is a small positive constant which will be determined later.

Now, we first consider the linearized problem (3.1)—(3.4).

Lemma 5.1. *Assume that hypotheses $A_1)$, $B_1)$, $C_1)$ and $C_2)$ are satisfied, and $\varphi \in H_{s+1}(\Omega)$, $\psi \in H_s(\Omega)\left(s \geq 2\left[\frac{n}{2}\right]+3\right)$ satisfy the compatibility conditions up to order s. Then, for any $v \in B_T(\delta)$, the linear boundary value problem (3.1)—(3.4) admits a unique solution*

$$u \in C^i([0, T]; H_{s+1-i}(\Omega)) \ (i=0, 1, \cdots, s+1).$$

Moreover, if

$$\sup_{0\leq\tau\leq t} \|D^{s+1}v(\tau)\|_0 \leq \eta, \tag{5.3}$$

then there exists a constant λ_1 such that when $\lambda \geq \lambda_1$, the following estimate holds:

$$\|u-\bar{u}\|^2_{s+1,t} \leq C\|D^{s+1}u(0)-D^{s+1}\bar{u}(0)\|_0^2 + C(\eta)(t+\lambda)\int_0^t \|D^{s+1}u(\tau)-D^{s+1}\bar{u}(\tau)\|_0^2 d\tau$$
$$+C(\eta)\delta^2 \sup_{0\leq\tau\leq t}\|D^{s+1}u(\tau)-D^{s+1}\bar{u}(\tau)\|_0^2 + C(\eta)t(1+\sup_{0\leq\tau\leq t}\|D^{s+2}\bar{u}(\tau)\|_0^2), \tag{5.4}$$

where C is a constant.

Proof The existence is obtaided immediately from Theorem 4.2. According to the definition of strong soluion, it is sufficient to prove the estimate (5.4) only for

$$u \in C^i([0, T]; H_{s+2-i}(\Omega)) \ (i=0, \cdots, s+2).$$

Let $w(x, t)=u(x, t)-\bar{u}(x, t)$. Then w satisfies the following

$$w_{tt}-\sum_{i,j=1}^n a_{ij}(Dv)w_{x_ix_j} = -\bar{u}_{tt}+\sum_{i,j=1}^n a_{ij}(Dv)\bar{u}_{x_ix_j}, \tag{5.5}$$

$$w|_{\Gamma_0}=0, \tag{5.6}$$

$$\sum_{i,j=1}^n a_{ij}(Dv)n_j w_{x_i}+b(Dv)w_t|_{\Gamma_1}=0. \tag{5.7}$$

From (5.5), we can obtain

$$\int_0^t\int_\Omega\left(D_t^s w_{tt}-D_t^s\sum_{i,j=1}^n a_{ij}(Dv)w_{x_ix_j}\right)\left\{(\tau+\lambda)D_t^s w_t+2\sum_{k=1}^n x_k D_t^s w_{x_k}+(n-1)D_t^s w\right\}dx\,d\tau$$
$$=\int_0^t\int_\Omega\left(-D_t^s\bar{u}_{tt}+D_t^s\sum_{i,j=1}^n a_{ij}(Dv)\bar{u}_{x_ix_j}\right)$$
$$\times\left\{(\tau+\lambda)D_t^s w_t+2\sum_{k=1}^n x_k D_t^s w_{x_k}+(n-1)D_t^s w\right\}dx\,d\tau. \tag{5.8}$$

As in the proof of Theorem 4.3, from (5.8) we have

$$Q_v^{(s)}(w(t))-Q_v^{(s)}(w(0))=I_0+I_1+I_2+I_3, \tag{5.9}$$

where

$$I_0=2\int_0^t\int_{\Gamma_0}\sum_{i,j,k=1}^n a_{ij}(Dv)n_j x_k D_t^s w_{x_i}\cdot D_t^s w_{x_k}\,dS\,d\tau$$
$$-\int_0^t\int_{\Gamma_0}(x\cdot n)\sum_{i,j=1}^n a_{ij}(Dv)D_t^s w_{x_i}\cdot D_t^s w_{x_j}\,dS\,d\tau$$
$$\leq 0, \tag{5.10}$$

$$I_1=-\int_0^t\int_{\Gamma_1}[(\tau+\lambda)b(Dv)-\boldsymbol{x}\cdot\boldsymbol{n}]\,|D_t^s w_t|^2\,dSd\tau$$
$$-\int_0^t\int_{\Gamma_1}(\boldsymbol{x}\cdot\boldsymbol{n})\sum_{i,j=1}^n a_{ij}(Dv)D_t^s w_{x_i}\cdot D_t^s w_{x_j}\,dSd\tau$$
$$+(n-1)\int_0^t\int_{\Gamma_1}\sum_{i,j=1}^n a_{ij}(Dv)n_i D_t^s w_{x_j}\cdot D_t^s w\,dSd\tau$$
$$+2\int_0^t\int_{\Gamma_1}\sum_{i,j,k=1}^n a_{ij}(Dv)n_i D_t^s w_{x_j}\cdot x_k D_t^s w_{x_k}\,dSd\tau$$
$$+\int_0^t\int_{\Gamma_1}(\tau+\lambda)\sum_{i,j=1}^n\{D_t^s(a_{ij}(Dv)w_{x_j})-a_{ij}(Dv)D_t^s w_{x_j}\}n_i D_t^s w_t\,dSd\tau$$
$$+\int_0^t\int_{\Gamma_1}(\tau+\lambda)[b(Dv)D_t^s w_t-D_t^s(b(Dv)w_t)]\,dSd\tau, \tag{5.11}$$

$$I_2=-\frac{1}{2}\int_0^t\int_\Omega|D_t^s w_t|^2\,dx\,d\tau$$
$$-\frac{1}{2}\int_0^t\int_\Omega\sum_{i,k=1}^n a_{ij}(Dv)D_t^s w_{x_i}\cdot D_t^s w_{x_j}\,dx\,d\tau$$
$$-\frac{1}{2}\int_0^t\int_\Omega\sum_{i,j,k=1}^n\{D_t^s(a_{ij}(Dv)w_{x_ix_j})-a_{ij}(Dv)D_t^s w_{x_ix_j}\}x_k D_t^s w_{x_k}\,dx\,d\tau$$
$$+(n-1)\int_0^t\int_\Omega\sum_{i,j=1}^n\{D_t^s(a_{ij}(Dv)w_{x_ix_j})-a_{ij}(Dv)D_t^s w_{x_ix_j}\}D_t^s w\,dx\,d\tau$$
$$-2\int_0^t\int_\Omega\sum_{i,j,k=1}^n\frac{\partial}{\partial x_i}a_{ij}(Dv)\cdot D_t^s w_{x_i}\cdot x_k D_t^s w_{x_k}\,dx\,d\tau$$
$$+\int_0^t\int_\Omega\sum_{i,j,k=1}^n x_k\frac{\partial}{\partial x_k}a_{ij}(Dv)\cdot D_t^s w_{x_i}\cdot D_t^s w_{x_j}\,dx\,d\tau$$
$$-(n-1)\int_0^t\int_\Omega\sum_{i,j=1}^n a_{ij}(Dv)D_t^s w_{x_j}\cdot D_t^s w\,dx\,d\tau$$
$$+\int_0^t\int_\Omega(\tau+\lambda)\sum_{i,j=1}^n\{D_t^s(a_{ij}(Dv)w_{x_ix_j})-a_{ij}(Dv)D_t^s w_{x_ix_j}\}D_t^s w_t\,dx\,d\tau$$
$$-\int_0^t\int_\Omega(\tau+\lambda)\sum_{i,j=1}^n\frac{\partial}{\partial x_i}a_{ij}(Dv)\cdot D_t^s w_{x_j}\cdot D_t^s w_t\,dx\,d\tau$$
$$+\frac{1}{2}\int_0^t\int_\Omega(\tau+\lambda)\sum_{i,j=1}^n\frac{\partial}{\partial t}a_{ij}(Dv)D_t^s w_{x_i}\cdot D_t^s w_{x_j}\,dx\,d\tau, \tag{5.12}$$

$$I_3=\int_0^t\int_\Omega\Big\{D_t^s\bar{u}_{tt}-D_t^s\sum_{i,j=1}^n a_{ij}(Dv)\bar{u}_{x_ix_j}\Big\}$$
$$\times\Big\{(\tau+\lambda)D_t^s w_t+2\sum_{k=1}^n x_k D_t^s w_{x_k}+(n-1)D_t^s w\Big\}\,dx\,d\tau. \tag{5.13}$$

At first, we discuss I_1. Using the boundary condition (5.7), Lemma 3.1 and the trace theorem, we can estimate that

$$(n-1)\int_0^t\int_{\Gamma_1}\sum_{i,j=1}^n a_{ij}(Dv)n_i D_t^s w_{x_j}D_t^s w\,dSd\tau$$
$$=-(n-1)\int_0^t\int_{\Gamma_1}b(Dv)D_t^s w_t\cdot D_t^s w\,dSd\tau$$
$$+(n-1)\int_0^t\int_{\Gamma_1}\sum_{i,j=1}^n\{a_{ij}(Dv)D_t^s w_{x_j}-D_t^s(a_{ij}(Dv)w_{x_j})\}n_i D_t^s w\,dSd\tau$$

$$+(n-1)\int_0^t\int_{\Gamma_1}\{b(Dv)D_t^s w_t - D_t^s(b(Dv)w_t)\}D_t^s w\,dSd\tau$$
$$\leqslant (n-1)b_1\int_0^t\int_{\Gamma_1}\{|D_t^s w_t|^2+|D_t^s w|^2\}\,dSd\tau$$
$$+C(\eta)\int_0^t\int_{\Gamma_1}\Big\{|D_t^{s-1}w_t|+\sum_{j=1}^n|D_t^{s-1}w_{x_j}|+\|D^{s+1}w(\tau)\|_0|D_t^s Dv|\Big\}|D_t^s w|\,dSd\tau$$
$$\leqslant (n-1)b_1\int_0^t\int_{\Gamma_1}|D_t^s w_t|^2\,dSd\tau + C(\eta)\int_0^t\|D^{s+1}w(\tau)\|_0^2\,d\tau$$
$$+C(\eta)\delta^2\sup_{0\leqslant\tau\leqslant t}\|D^{s+1}w(\tau)\|_0^2. \tag{5.14}$$

Similarly, for any small positive constant ν, we can obtain

$$2\int_0^t\int_{\Gamma_1}\sum_{i,j,k=1}^n a_{ij}(Dv)n_i D_t^s w_{x_j}\cdot x_k D_t^s w_{x_k}\,ds\,d\tau$$
$$\leqslant C_1\nu^{-1}\int_0^t\int_{\Gamma_1}|D_t^s w_t|^2\,ds\,d\tau + C(\eta)\nu\int_0^t\int_{\Gamma_1}\sum_{j=1}^n|D_t^s w_{x_j}|^2\,dSd\tau$$
$$+C(\eta)\nu^{-1}\int_0^t\|D^{s+1}w(\tau)\|_0^2 d\tau + C(\eta)\nu^{-1}\delta^2\sup_{0\leqslant\tau\leqslant t}\|D^{s+1}w(\tau)\|_0^2, \tag{5.15}$$

here and everywhere below C_i denotes constant.

$$-\int_0^t\int_{\Gamma_1}(\tau+\lambda)\sum_{i,j=1}^n\{D_t^s(a_{ij}(Dv)w_{x_j})-a_{ij}(Dv)D_t^s w_{x_j}\}n_i D_t^s w_t\,dSd\tau$$
$$\leqslant C(\eta)\nu\int_0^t\int_{\Gamma_1}(\tau+\lambda)|D_t^s w_t|^2\,dSd\tau + C(\eta)\nu^{-1}\int_0^t(\tau+\lambda)\|D^{s+1}w(\tau)\|_0^2\,d\tau$$
$$+C(\eta)\nu^{-1}\delta^2\sup_{0\leqslant\tau\leqslant t}(\tau+\lambda)\|D^{s+1}w(\tau)\|_0^2. \tag{5.16}$$

For the last term of the rignt side in (5.11), we can obtain an estimate similar to (5.16). Then from(5.11), (5.14), (5.15) and (5.16) etc., it is easy to see that

$$I_1\leqslant -\int_0^t\int_{\Gamma_1}[(b_0-C(\eta)\nu)(\tau+\lambda)-C_2\nu^{-1}]\,|D_t^s w_t|^2\,dSd\tau$$
$$-\int_0^t\int_{\Gamma_1}(\gamma_1 a^2-c(\eta)\nu)\sum_{i=1}^n|D_t^s w_{x_i}|^2 dSd\tau$$
$$+C(\eta)\nu^{-1}\int_0^t(\tau+\lambda)\|D^{s+1}w(\tau)\|_0^2\,d\tau$$
$$+C(\eta)\nu^{-1}\delta^2\sup_{0\leqslant\tau\leqslant t}(\tau+\lambda)\|D^{s+1}w(\tau)\|_0^2. \tag{5.17}$$

Noticing

$$\int_0^t\int_\Omega\Big(D_t^s\bar{u}_{tt}-D_t^s\sum_{i,j=1}^n a_{ij}(Dv)\bar{u}_{x_ix_j}\Big)^2dx\,d\tau$$
$$\leqslant C(\eta)t(\sup_{0\leqslant\tau\leqslant t}\|D^{s+2}\bar{u}(\tau)\|_0^2+\eta^2)$$

and using Lemma 3.1, we can estimate that

$$I_2+I_3\leqslant C(\eta)\int_0^t(\tau+\lambda)^2\|D^{s+1}w(\tau)\|_0^2\,d\tau$$
$$+C(\eta)t(1+\sup_{0\leqslant\tau\leqslant t}\|D^{s+2}\bar{u}(\tau)\|_0^2). \tag{5.18}$$

Taking

$$\nu=\frac{1}{2}C(\eta)^{-1}\min\{b_0,\ \gamma_1 a^2\}, \tag{5.19}$$

$$\lambda_1=(2C_2\nu^{-1}+1)b_0^{-1}, \tag{5.20}$$

from (5.9), (5.10), (5.17) and (5.18), we have

$$\begin{aligned}Q_v^{(s)}(w(t))&+\frac{1}{2}\int_0^t\Big\{\|D_t^s w_t\|^2_{L^2(\Gamma_1)}+\gamma_1 a^2\sum_{i=1}^n\|D_t^s w_{x_i}\|^2_{L^2(\Gamma_1)}\Big\}d\tau\\ &\leqslant Q_v^{(s)}(w(0))+C(\eta)t(1+\sup_{0\leqslant\tau\leqslant t}\|D^{s+2}\bar{u}(\tau)\|_0^2)\\ &+C(\eta)\nu^{-1}\int_0^t(\tau+\lambda)^2\|D^{s+1}w(\tau)\|_0^2d\tau\\ &+C(\eta)\nu^{-1}\delta^2\sup_{0\leqslant\tau\leqslant t}(\tau+\lambda)\|D^{s+1}w(\tau)\|_0^2,\ \forall t\in[0,T].\end{aligned}\tag{5.21}$$

Noticing Lemma 3.4 and (5.19), we see tnat the estimate(5.4) follows immediately from (5.21).

Lemma 5.2. *Under the hypotheses in Lemma 5.1, assume that there exists* $\bar{u}\in C^i([0,T];\ \mathring{H}_{s+2-i}(\Omega))$ $(i=0,\cdots,s+2)$ *such that*(5.1) *holds and take*

$$\varepsilon=\min\Big\{\|D^{s+1}u(0)\|_0,\ \sqrt{\frac{1}{8C(\lambda_1+T)}}\Big\}, \tag{5.22}$$

where C *and* λ_1 *are the constants in* (5.4) *and* (5.20) *respectively. Then there exists a constant* $\delta_1>0$ *and a positive* $T_1(\delta,\lambda)$ *such that when* $\delta\leqslant\delta_1$, $T^*\leqslant T_1(\delta,\lambda)$, *for any* $v\in B_T(\delta)$ *the linearlized boundary value problem* (3.1)—(3.4) *admits a unique solution* $u\in B_{T^*}(\delta)$.

Proof Taking

$$\eta=5\|D^{s+1}u(0)\|_0, \tag{5.23}$$

and

$$\delta_1=\min\Big\{\|D^{s+1}u(0)\|_0,\ \sqrt{\frac{1}{2C(\eta)}}\Big\}, \tag{3.24}$$

where $C(\eta)$ is the constant in (5.4), we temporarily assume that (5.3) holds. From (5.4) it is easily seen that

$$\begin{aligned}\|u-\bar{u}\|^2_{s+1,t}&\leqslant 2C\|D^{s+1}u(0)-D^{s+1}\bar{u}(0)\|_0^2\\ &+2C(\eta)(t+\lambda)\int_0^t\|D^{s+1}u(\tau)-D^{s+1}\bar{u}(\tau)\|_0^2d\tau\\ &+2C(\eta)t(1+\sup_{0\leqslant\tau\leqslant t}\|D^{s+2}\bar{u}(\tau)\|_0^2).\end{aligned}\tag{5.25}$$

Using Gronwall's inequality, from (5.25) we have

$$\begin{aligned}\|u-\bar{u}\|^2_{s+1,t}&\leqslant 2\{C\|D^{s+1}u(0)-D^{s+1}\bar{u}(0)\|_0^2\\ &+C(\eta)T^*(1+\sup_{0\leqslant T\leqslant T^*}\|D^{s+2}\bar{u}(\tau)\|_0^2)e^{C(\eta)(t+2\lambda)t},\ \forall t\in[0,T^*].\end{aligned}\tag{5.26}$$

Take $T_1(\delta,\lambda)$ so small that

$$\|D^{s+1}\bar{u}(t)\|_0\leqslant 2\|D^{s+1}\bar{u}(0)\|_0,\ \forall t\in[0,T_1], \tag{5.27}$$

$$2C(\eta)T_1(1+\sup_{0\leqslant t\leqslant T_1}\|D^{s+2}\bar{u}(t)\|_0^2)\leqslant\frac{\delta^2}{4}, \tag{5.28}$$

$$\exp(C(\eta)(2\lambda+T_1)T_1)\leqslant 2. \tag{5.29}$$

From (5.22), (5.24) and (5.27), it is easy to see that $u\in B_{T^*}(\delta)$ means

$$\|D^{s+1}v(t)\|_0\leqslant\eta.$$

From (5.22), (5.28), (5.29) and (5.26), we can obtain

$$\|u-\bar{u}\|_{S,T^*}^2\leqslant\delta^2,$$

provided $T^*\leqslant T_1(\delta,\lambda)$. The Lemma is proved.

Lemma 5.3. *Under the hypotheses in Lemma 5.2, let $v_1, v_2\in B_{T^*}(\delta)$. If u_1 and u_2 are the solutions for the boundary value problem (3.1)—(3.4) with $v=v_1$ and $v=v_2$ respectively, then there exist δ_2, λ_2 and a positive $T_2(\delta,\lambda)$ such that when $\delta\leqslant\delta_2$, $\lambda\geqslant\lambda_2$ and $T^*\leqslant T_2(\delta,\lambda)$, it holds that*

$$\|u_1-u_2\|_{1,T^*}\leqslant q\|v_1-v_2\|_{1,T^*}, \tag{5.30}$$

where q is a constant and $0<q<1$.

Proof First set $\delta_2\leqslant\delta_1$, $T_2\leqslant T_1$ and so

$$\|D^{s+1}v_i(t)\|_0,\ \|D^{s+1}u_i(t)\|_0\leqslant\eta,\ (i=1,2),\ \forall t\in[0,T^*],\ T^*\leqslant T_2, \tag{5.31}$$

where δ_1, T_1 and η are in Lemma 5.2.

Write $w=u_1-u_2$. Then w satisfies the following

$$w_{tt}-\sum_{i,j=1}^n a_{ij}(Dv_1)w_{x_ix_j}=\sum_{i,j=1}^n(a_{ij}(Dv_1)-a_{ij}(Dv_2))(u_2)_{x_ix_j}, \tag{5.32}$$

$$w|_{\Gamma_0}=0, \tag{5.33}$$

$$\sum_{i,j=1}^n a_{ij}(Dv_1)n_jw_{x_i}+b(Dv_1)w_t|_{\Gamma_1}$$
$$=\sum_{i,j=1}^n[a_{ij}(Dv_2)-a_{ij}(Dv_1)]n_j(u_2)_{x_j}+[b(Dv_2)-b(Dv_1)](u_2)_t, \tag{5.34}$$

$$w|_{t=0}=w_t|_{t=0}=0. \tag{5.35}$$

From (5.32)—(5.35) we can obtain

$$Q_{v_1}^{(0)}(w(t))-Q_{v_1}^{(0)}(w(0))=I_0+I_1+I_2+I_3, \tag{5.36}$$

as we did in the proof of Lemma 5.1, where

$$I_0\leqslant 0. \tag{5.37}$$

Noticing $\bar{u}\in C^i([0,T];\mathring{H}_{s+2-i}(\Omega))$, from Sobolev's imbedding theorem we have

$$\sup_{\Gamma_1}|(u_2)_{x_i}|\leqslant C_2\delta. \tag{5.38}$$

Using (5.38), we can estimate the following

$$\begin{aligned}I_1\leqslant&-\int_0^t\int_{\Gamma_1}[(b_0-C(\eta)\delta)(\tau+\lambda)-C(\eta)\delta^{-1}]w_t^2\,dSd\tau\\&-\int_0^t\int_{\Gamma_1}(\gamma_1a^2-C(\eta)\nu-C(\eta)\delta)\sum_{k=1}^n w_{x_k}^2\,dSd\tau\\&+C(\eta)\delta\int_0^t(\|(v_2-v_1)_t\|_{L^2(\Gamma_1)}^2+\sum_{k=1}^n\|(v_2-v_1)_{x_k}\|_{L^2(\Gamma_1)}^2)d\tau\\&+C(\eta)\delta\int_0^t(\|Dw\|_0^2+\|D(v_2-v_1)\|_0^2)d\tau,\end{aligned} \tag{5.39}$$

for any positive number ν. Moreover, we can obtain

$$I_2+I_3\leqslant C(\eta)\int_0^t(\tau+\lambda)\|Dw(\tau)\|_0^2\,d\tau. \tag{5.40}$$

Take

$$\nu=\frac{1}{4}C(\eta)^{-1}\gamma_1 a^2, \tag{5.41}$$

$$\bar{\lambda}_2=(1+2C(\eta)\nu^{-1})b_0^{-1}, \tag{5.42}$$

$$\bar{\delta}_2=\frac{1}{8}C(\eta)^{-1}\min\{1,\ 2b_0,\ \gamma_1 a^2,\ 2\alpha\}, \tag{5.43}$$

$$A_2=\max\{\lambda_1,\ \bar{\lambda}_2,\ 1\}, \tag{5.44}$$

$$\delta_2=\min\{\delta_1,\ \bar{\delta}_2\}, \tag{5.45}$$

where α is the constant in Lemma 3.4.

From (5.36), (5.37), (5.39) and (5.40), we can obtain the following estimate

$$\|Dw(t)\|_0^2+\int_0^t\Big(\|w_t\|^2_{L^2(\Gamma_1)}+\sum_{k=1}^n\|w_{x_k}\|^2_{L^2(\Gamma_1)}\Big)d\tau$$
$$\leqslant\frac{1}{4}\int_0^t\Big(\|(v_2-v_1)\|^2_{L^2(\Gamma_1)}+\sum_{k=1}^n\|(v_2-v_1)_{x_k}\|^2_{L^2(\Gamma_1)}\Big)d\tau$$
$$+\frac{1}{4}\int_0^t\|D(v_2-v_1)\|_0^2d\tau+C(\eta)\int_0^t(\tau+\lambda)\|Dw(\tau)\|_0^2\,d\tau. \tag{5.46}$$

By use of Gronwall's inequality, (5.41) yields immediately

$$\|w\|^2_{1,T^*}\leqslant\frac{1}{2}(1+T^*)e^{C(\eta)\left(\lambda+\frac{T^*}{2}\right)T^*}\|v_1-v_2\|_{1,T^*}. \tag{5.47}$$

The lemma follows from (5.47) immediately.

Theorem 5.1. *Assume that the hypotheses* A_1), B_1), C_1) *and* C_2) *are satisfied, and that* $\varphi\in H_{s+1}(\Omega)$, $\psi\in H_s(\Omega)$ $\left(s\geqslant 2\left[\frac{n}{2}\right]+3\right)$ *satisfy the compatibility conditions up to order* s. *Then there exist* $\varepsilon>0$ *and* $T^*>0$ *such that if* (5.1) *holds for certain* $\bar{u}\in C^i([0,T^*];\ \overset{\circ}{H}_{s+2-i}(\Omega))$ $(i=0,\cdots,s+2)$, *then nonlinear boundary value problem* (2.1)—(2.4) *admits a unique solution* $u\in C^i([0,T^*];\ H_{s+1-i}(\Omega)$ $((i=0,\cdots,s+1)$.

Proof Take ε as in (5.22), $\delta\leqslant\min\{\delta_1,\ \delta_2\}$, $\lambda\geqslant\max\{\lambda_1,\ \lambda_2\}$ and $T^*\leqslant\min\{T_1(\delta,\lambda),\ T_2(\delta,\lambda)\}$. We denote the closure of $B_{T^*}(\delta)$ under norm $\|\cdot\|_{1,T^*}$ by $\bar{B}_{T^*}(\delta)$, From Lemms 5.2 and5.3, it is easy to see that map

$$\Phi:\ v\mapsto u,$$

which is defined by the boundary value problem (3.1)—(3.4), admits a unique fixed point u in $\bar{B}_{T^*}(\delta)$.

From Banach-Saks' theorem, it is not difficult to show that for any $t\in[0,T^*]$, $D_t^i u(t)\in H_{s+1-i}(\Omega)$ $(i=0,1,\cdots,s+1)$ and

$$\|D^{s+1}u(t)\|_0\leqslant\delta+\|D^{s+1}\bar{u}(t)\|_0. \tag{5.48}$$

Moreover, from studying linear boundary value problem, we can show that $u\in C^i[0,T];\ H_{s+1-i}(\Omega))$ $(i=0,\cdots,s+1)$.

The proof of the theorem is completed.

Remark. For the boundary value problem with small data, we can take $\bar{u}\equiv 0$. For this case, from Lemmas 5.2, 5.3 and their proofs, it is easy to see that the height T^* of the existence domain of smooth solutions for nonlinear boundary value problem

(2.1)—(2.4) depends only on $\|D^{s+1}u(0)\|_0$.

§ 6. Global Smooth Solution

In this section, we shall prove the existence of global smooth solution for the nonlinear boundary value problem (2.1)—(2.4) with small data. From the remark of Theorem 5.1, we can see that the height of the existence domain of local smooth solution depents only on $\|\varphi\|_{s+1}$ and $\|\psi\|_s$. Therefore, in order to show the global existence, it is sufficient to obtain a uniform energy estimate of the solutions in a sense.

Lemma. 6.1, *Assume that the hypotheses* $A_1)-C_2)$ *are satisfied. Let* $u\in C^i([0, T]; H_{s+1-i}(\Omega))$ $(i=0, \cdots, s+1)$ *be the solution for nonlinear boundary value problem* (2.1)—(2.4), $s\geqslant 2\left[\frac{n}{2}\right]+3$. *Then for any* $0<\mu<1$, *there exist* $\bar{\delta}$ *and* $\bar{\lambda}$ *which are independent of* T *such that if* $\lambda\geqslant\bar{\lambda}$, $\delta\leqslant\bar{\delta}$ *and*

$$\sup_{0<t<T}(t+\lambda)\|D^{s+1}u(t)\|_0^2\leqslant\delta^2, \tag{6.1}$$

then

$$Q^{(s)}(u(t))\leqslant Q^{(s)}(u(0))\left(\frac{\lambda}{t+\lambda}\right)^{\mu}. \tag{6.2}$$

Proof Without loss of generality, we assume that $u\in C^i([0, T]; H_{s+2-i}(\Omega))$ $(i=0, 1, \cdots, s+2)$. For small date, we can take $\bar{u}\equiv 0$ in § 5. In the same way, we can obtain the following expression similar to (5.9):

$$Q^{(s)}(u(t))-Q^{(s)}(u(0))=I_0+I_1+I_2, \tag{6.3}$$

where

$$I_0\leqslant 0, \tag{6.4}$$

I_1 and I_2 are similar to (5.11) and (5.12) respectively.

First we estimate I_1. Similar to (5.12) and (5.15), it is easy to show that

$$\begin{aligned}(n-1)&\int_0^t\int_{\Gamma_1}\sum_{i,j=1}^n a_{ij}(Du)n_iD_t^su_{x_j}\cdot D_t^su\,dSd\tau\\
&\leqslant C\delta^{-1}\int_0^t\int_{\Gamma_1}|D_t^su_t|^2dSd\tau+C\delta\int_0^t\int_{\Gamma_1}|D_t^su|^2dSd\tau\\
&\quad+C(\delta)\delta\int_0^t\int_{\Gamma_1}(|D_t^sDu_t|^2+|D_s^tu|^2)dSd\tau\\
&\leqslant C(\delta)(\delta^{-1}+\delta)\int_0^t\int_{\Gamma_1}|D_t^su_t^su_t|^2dXd\tau+C(\delta)\delta\int_0^t\int_{\Gamma_1}\sum_{i=1}^n|D_t^su_{x_i}|^2\,dSd\tau\\
&\quad+C(\delta)\delta\int_0^t\int_{\Omega}|D^{s+1}u|^2\,dx\,d\tau\end{aligned} \tag{6.5}$$

and

$$\begin{aligned}2&\int_0^t\int_{\Gamma_1}\sum_{i,j,k=1}^n a_{ij}(Du)n_iD_t^su_{x_j}\cdot x_kD_t^su_{x_k}\,dS\,d\tau\\
&\leqslant C(\delta)(\delta^{-1}+\delta)\int_0^t\int_{\Gamma_1}|D_t^su_t|^2\,dSd\tau+C(\delta)\delta\int_0^t\int_{\Gamma_1}\sum_{k=1}^n|D_t^su_{x_k}|dSd\tau,\end{aligned} \tag{6.6}$$

respectively.

Using Lemma 3.1, we can obtain

$$-\int_1^t\int_{\Gamma_1}(\tau+\lambda)\sum_{i,j=1}^n\{D_t^s(a_{ij}(Du)u_{x_j})-a_{ij}(Du)D_t^su_{x_j}\}n_iD_t^su_t\,dSd\tau$$
$$\leqslant C(\delta)\delta\int_0^t\int_{\Gamma_1}(\tau+\lambda)\,|D_t^su_t|^2\,dSd\tau+C(\delta)\delta\int_0^t\int_{\Gamma_1}\sum_{i=1}^n|D_t^su_{x_i}|^2dSd\tau. \qquad (6.7)$$

Moreover, we have similar estimates for the other terms in I_1. So from (6.5), (6.6), (6.8), etc., we have

$$I_1\leqslant-\int_0^t[(b_0-C(\delta)\delta)(\tau+\lambda)-\boldsymbol{x}\cdot\boldsymbol{n}-C(\delta)(\delta^{-1}+\delta)]\,\|D_t^su_t\|^2_{L^2(\Gamma_1)}\,d\tau$$
$$-\int_0^t(\gamma_1a^2-C(\delta)\delta)\sum_{i=1}^n\|D_t^su_{x_i}\|^2_{L^2(\Gamma_1)}\,d\tau+C(\delta)\delta\int_0^t\|D^{s+1}u\|_0^2\,d\tau. \qquad (6.8)$$

Now, we estimate I_2. From Lemma 3.1, it is easily seen that

$$\int_0^t\int_\Omega(\tau+\lambda)\sum_{i,j=1}^n\{D_t^s(a_{ij}(Du)u_{x_ix_j})-a_{ij}(Du)D_t^su_{x_ix_j}\}D_t^su_t\,dx\,d\tau$$
$$\leqslant C(\delta)\delta\int_0^t\int_\Omega|D^{s+1}u|^2\,dx\,d\tau. \qquad (6.9)$$

From the assumption A_2) and (6.1), we have

$$(t+\lambda)\left(\left|\frac{\partial a_{ij}(Du)}{\partial x_k}\right|+\left|\frac{\partial a_{ij}(Du)}{\partial t}\right|\right)\leqslant C\delta. \qquad (6.10)$$

By using (6.10), we obtain immediately

$$\left|\int_0^t\int_\Omega(\tau+\lambda)\sum_{i,j=1}^n\frac{\partial}{\partial x_i}a_{ij}(Du)\cdot D_t^su_{x_j}\cdot D_t^su_t\,dxd\tau\right|$$
$$\leqslant D\delta\int_0^t\int_\Omega|D^{s+1}u|^2\,dx\,d\tau. \qquad (6.11)$$

We have the similar estimates for the other terms in I_2. So from (6.9), (6.11), etc., we obtain

$$I_2\leqslant-\frac{1}{2}\int_0^t\int_\Omega\left(|D_t^su_t|^2+\sum_{i,j=1}^na_{ij}(Du)u_{x_ix_j}\right)dx\,d\tau+C(\delta)\delta\int_0^t\|D^{s+1}u\|_0^2\,d\tau. \qquad (6.12)$$

From (6.3), (6.4), (6.8) and (6.12), we have

$$Q^{(s)}(u(t))-Q^{(s)}(u(0))$$
$$\leqslant-\int_0^t[(b_0-C(\delta)\delta)(\tau+\lambda)-\boldsymbol{x}\cdot\boldsymbol{n}-C(\delta)(\delta^{-1}+\delta)]\,\|D_t^su_t\|^2_{L^2(\Gamma_1)}\,d\tau$$
$$-\int_0^t(\gamma_1a^2-C(\delta)\delta)\sum_{i,j=1}^n\|D_t^su_{x_i}\|^2_{L^2(\Gamma_1)}\,d\tau$$
$$-\frac{1}{2}\int_0^t\int_\Omega\left(|D_t^su_t|^2+\sum_{i,j=1}^na_{ij}(Du)u_{x_i}u_{x_j}\right)dx\,d\tau$$
$$+C(\delta)\delta\int_0^t\|D^{s+1}u\|_0^2\,d\tau. \qquad (6.13)$$

From Lemma 3.3 and the generalized Poincare's inequality, it is easily seen that

$$\|D^{s+1}u\|_0^2\leqslant C(\delta)\int_0^t\int_\Omega\left(|D_t^su_t|^2+\sum_{i,j=1}^na_{ij}(Du)u_{x_i}u_{x_j}\right)dx\,d\tau. \qquad (6.14)$$

Moreover, let us suppose that

$$\delta \leqslant 1 \tag{6.15}$$

and

$$\sup_{\Gamma_1} |x \cdot n| \leqslant C. \tag{6.16}$$

From (6.13)—(6.16), we have

$$\begin{aligned} Q^{(s)}(u(t)) - Q^{(s)}(u(0)) \leqslant & -\int_0^t [(b_0 - C(1)\delta)(\tau+\lambda) - C - C(1)\delta] \|D_t^s u_t\|^2_{L^2(\Gamma_1)} d\tau \\ & -\int_0^t (\gamma_1 a^2 - C(1)\delta) \sum_{i=1}^n \|D_t^s u_{x_i}\|^2_{L^2(\Gamma_1)} d\tau \\ & -\left(\frac{1}{2} - C(1)\delta\right) \int_0^t \int_\Omega \left(|D_t^s u_t|^2 + \sum_{i,j=1}^n a_{ij}(Du) u_{x_i} u_{x_j} \right) dx\, d\tau. \end{aligned} \tag{6.17}$$

Then take $\beta = (1+\mu)/2\mu$ in Lemma 3.4,

$$\bar{\delta} = \frac{1}{2} \min\left\{ b_0,\ \gamma_1 a^2,\ \frac{1}{2}(1-\mu),\ 2\eta_0 \right\},$$

and

$$\bar{\lambda} = \max\{\lambda_0,\ 2(C + C(1)\bar{\delta}) b_0^{-1}\},$$

where η_0 and λ_0 are the numbers in Lemma 3.4. From (6.17) and Lemma 3.4 we obtain

$$Q^{(s)}(u(t)) \leqslant Q^{(s)}(u(0)) - \mu \int_0^t (\tau+\lambda)^{-1} Q^{(s)}(u(\tau)) d\tau, \tag{6.18}$$

provided $\delta \leqslant \bar{\delta}$ and $\lambda \geqslant \bar{\lambda}$.

By using Bihari's inequality, the lemma follows immediately from (6.18).

Proof of the main theorem We set $\delta \leqslant \bar{\delta}$ and $\lambda \geqslant \bar{\lambda}$, where $\bar{\delta}$ and $\bar{\lambda}$ are as in Lemma 6.1. By using Lemma 3.4 and (6.2), we know that if (6.1) holds then

$$(t+\lambda) \|D^{s+1} u(t)\|_0^2 \leqslant C_1 \lambda^{1+\mu} (t+\lambda)^{-\mu} \|D^{s+1} u(0)\|_0^2, \tag{6.19}$$

where $C_1 > 1$ is a constant independent of T. Let us suppose that $\lambda \geqslant 1$. We take

$$T_0 = (C_1^{\frac{1}{1+\mu}} - 1)\lambda, \tag{6.20}$$

and ε so small that when

$$\|\varphi\|_{s+1} + \|\psi\|_s \leqslant \varepsilon, \tag{6.21}$$

nonlinear boundary value problem (2.1)—(2.4) admits a solution $u \in C^i([0, T_0]; H_{s+1-i}(\Omega))$ $(i = 0, \cdots, s+1)$ and

$$\|D^{s+i} u(0)\|_0^2 \leqslant C_1^{-1} \lambda^{-1} \delta^2. \tag{6.22}$$

Then we claim that

$$(t+\lambda) \|D^{s+1} u(t)\|_0^2 \leqslant \delta^2,\ \forall t \in [0, T_0] \tag{6.23}$$

and

$$\|D^{s+1} u(T_0)\|_0^2 \leqslant \|D^{s+1} u(0)\|_0^2. \tag{6.24}$$

Indeed, if (6.23) is false, let t_0 be the supremum of such $\overline{T}$ that (6.23) holds on $[0, \overline{T}]$. It is evident that $t_0 > 0$. From (6.19) and (6.22), we have

$$(t_0+\lambda) \|D^{s+1} u(t_0)\|_0^2 \leqslant \left(\frac{\lambda}{t_0+\lambda} \right)^\mu \delta^2. \tag{6.25}$$

Above inequality contradicts the definition of t_0. When once (6.23) holds, we have the estimate (6.19). From (6.19) and (6.20), the inequality (6.24) follows immediately.

Now, we discuss the decay of solutions. Take $\overline{T}>0$ such that

$$C_1\left(\frac{\lambda}{\overline{T}+\lambda}\right)^{1+\mu}\leqslant q^2, \tag{6.26}$$

where q is a constant and $0<q<1$. From (6.19) and (6.26), it is easy to see that

$$\|D^{s+1}u(\overline{T})\|_0\leqslant q\|D^{s+1}u(0)\|_0. \tag{6.27}$$

By using (6.27), it is difficult to show $\|D^{s+1}u(t)\|_0$ decays to zero exponentially as $t\to\infty$.

References

[1] Nishida, T., Global smooth solutions for the second-order quasilinear wave equation with first-order dissipation, preprint.

[2] Matsumura, A., Global existence and atsympotics of the solutions of the second-order quasilinear hyperbolic equations with the first order dissipation, Publ. RIMS, Kyoto, Univ., **13**(1977), 349—379.

[3] Greenberg, J. M. and Li Tatsien, The effect of boundary damping for the quasilinear wave equation, *J. Differential Equations*, **52**(1984), 66—75.

[4] Strauss, W. A., Dispersal of waves vanishing on the boundary of an exterior domain, *Comm. Pure Appl Math.*, **28**(1975),265—278.

[5] Chen, G., A note on the boundary stabilization of the wave equation, SIAM *J. Control and Optimization*, **19**(1981), 106—113.

[6] Rauch, J,. B. and Massey, E. J. III, Differentiability of solutions to hyperbolic initial-boundary problems, *Trans. Amer. Math. Soc.*, **189**(1974), 303—318.