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THE FIRST EIGENVALUE OF AN IRREDUCIBLE HOMOGENEOUS MANIFOLD

Shen Zhongmin (沈忠民)*

Abstract

Let M be an n-dimensional compact minimal submanifold in the unit sphere. It is shown that the diameter and volume of M satisfy

$$d\geqslant\frac{\pi}{2}+C(n)\frac{d^n}{d^n+V}.$$

An application is that if M is an n-dimensional compact irreducible homogeneous manifold, the first eigenvalue λ_1 of M satisfies

$$\lambda_1\geqslant\frac{n}{d^2}\left(\frac{\pi}{2}+C(n)\frac{d^n}{d^n+V}\right)^2.$$

In the above two cases, $C(n)'$ s are the same constants depending only on n.

§1. Introduction

In 1980, Li, P. [1] established a lower bound for the first eigenvalue of the Laplacian on an n-dimensional compact irreducible homogeneous manifold, that is $\lambda_1\geqslant n\pi^2/4d^2$, where λ_1 is the first eigenvalue, d is the diameter. In this paper, we are going to establish a better lower bound for the first eigenvalue. First of all, we will deal with the problem of the minimally immersed submanifolds in the unit sphere S^m of dimension m.

§2. Minimal Submanifolds in S^m

Let M be an n-dimensional compact manifold which is minimally immersed in a unit sphere S^m of m-dimension $(m\geqslant n)$, i.e. $X=(x^1, \cdots, x^{m+1})$: $M\hookrightarrow S^m\subset E^{m+1}$ is a minimal immersion. Then we have

$$\sum_{k=1}^{m+1}(x^k)^2=1, \tag{1}$$

$$\Delta x^k=-nx^k. \tag{2}$$

Let $d(X_1, X_2)$ be the distance function on S^m, where $X_1, X_2\in S^m$. Let $r(p, q)=d(X(p), X(q))$, where $p, q\in M$. Then we have the following lemma.

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* Institute of Mathematics, Academia Sinica, Beijing, China.

Lemma. *Let $V_n(a)$ be the volume of a geodesic ball with radius a in S^n for $a\leqslant\pi$, and for any fixed $p\in M$, $V(a)$ the volume of the set $\{q\in M,\ r(p,q)\leqslant a\}$, Then*

$$V(a)\geqslant V_n(a)\quad(a\leqslant\pi).$$

Proof See the Corollary 2 of [2].

Theorem 1. *Let M be an n-dimensional compact manifold which is minimally immersed into a unit sphere S^m $(m>n)$. Then*

$$d\geqslant\frac{\pi}{2}+C(n)\frac{d^n}{d^n+V},$$

where $C(n)$ is a positive constant depending only on n, d=diameter of M, V=volume of M.

Proof As is defined above, we obviously have

$$r(p,q)=2\sin^{-1}\frac{\sqrt{\sum_k(x^k(p)-x^k(q))^2}}{2},\quad p,\ q\in M.$$

Thus by (1)

$$\frac{1}{2}-\frac{1}{2}\sum_k x^k(p)x^k(q)=\sin^2\frac{r(p,q)}{2}. \tag{3}$$

Using (2) and integrating (3) for $q\in M$, we obtain

$$\frac{V}{2}=\int_M\sin^2\frac{r(p,q)}{2}dq. \tag{4}$$

It is clear that for any $p,\ q\in M$, $r(p,q)\leqslant d$, the diameter of M, since M is immersed into the unit sphere S^m. In the following, we always assume that $d\leqslant 2\pi/3$, then $r(p,q)\leqslant d\leqslant 2\pi/3$ for any $p,\ q\in M$. From (4), it follows that for $0<\varepsilon<1$,

$$\begin{aligned}\frac{V}{2}&=\int_M\sin^2\frac{r(p,q)}{2}dq\\&=\int_{\{q\in M,r(p,q)\leqslant\varepsilon d\}}\sin^2\frac{r(p,q)}{2}dq+\int_{\{q\in M,r(p,q)\geqslant\varepsilon d\}}\sin^2\frac{r(p,q)}{2}dq\\&\leqslant\sin^2\frac{\varepsilon d}{2}\operatorname{Vol}\{q\in M,\ r(p,q)\leqslant\varepsilon d\}+\sin^2\frac{d}{2}\operatorname{Vol}\{q\in M,\ r(p,q)\geqslant\varepsilon d\},\end{aligned}$$

$$\frac{1}{2}\leqslant\sin^2\frac{d}{2}-\frac{\operatorname{Vol}\{q\in M,\ r(p,q)\leqslant\varepsilon d\}}{V}\sin\left[(1+\varepsilon)\frac{d}{2}\right]\sin\left[(1-\varepsilon)\frac{d}{2}\right]. \tag{5}$$

By use of lemma and an obvious result that $d\geqslant\pi/2$, the following holds:

$$\operatorname{Vol}\{q\in M,\ r(p,q)\leqslant\varepsilon d\}\geqslant V_n(\varepsilon d)\geqslant V_n\left(\frac{\pi}{2}\varepsilon\right), \tag{6}$$

$$\sin\left[(1+\varepsilon)\frac{d}{2}\right]\geqslant\min\left\{\sin\frac{d}{2},\ \sin d\right\}=\sin\frac{d}{2}, \tag{7}$$

$$\sin\left[(1-\varepsilon)\frac{d}{2}\right]\geqslant\sin\left[(1-\varepsilon)\frac{\pi}{4}\right]. \tag{8}$$

Since $\frac{\pi}{2}\leqslant d\leqslant\frac{2}{3}\pi$, it is easy to get (7). Now from the above inequalities (6), (7), (8) and (5), we obtain

$$\frac{1}{2}\leqslant\sin^2\left(\frac{d}{2}\right)-\frac{V_n\left(\frac{1}{2}\pi\varepsilon\right)}{V}\sin\left[(1-\varepsilon)\frac{\pi}{4}\right]\sin\frac{d}{2}.$$

Let $A_n=\sup\limits_{0<\varepsilon<1} V_n\left(\frac{\pi}{2}\varepsilon\right)\sin\left[(1-\varepsilon)\frac{\pi}{4}\right]$, which is a positive constant depending only on n. We have

$$\frac{1}{2}\leqslant\sin^2\left(\frac{d}{2}\right)-\frac{A_n}{V}\sin\frac{d}{2}. \tag{9}$$

It is easy to derive from (9)

$$d\geqslant\frac{\pi}{2}+\frac{B_n}{V}, \tag{10}$$

where B_n is a positive constant depending only on n. Let $C(n)=\left(\frac{3}{2\pi}\right)^n B_n$. Then

$$d\geqslant\frac{\pi}{2}+C(n)\ \frac{d^n}{d^n+V}\left(\text{since } d\leqslant\frac{2}{3}\pi\right). \tag{11}$$

It is obvious that (11) holds with no condition on d, if we substitute $\min\left\{C(n),\frac{\pi}{6}\right\}$ for $C(n)$. Now we complete the proof of this Theorem.

§ 3. The First Eigenvalues of Homogeneous Manifolds

In this section we shall discuss the first eigenvalues of irreducible homogeneous manifolds. Now see the following Theorem 2.

Theorem 2. *Let (M, g) be an n-dimensional compact irreducible homogeneous manifold, λ_1 the first eigenvalue of M, V_1 the eigenspace of λ_1. Then*

(i) *If* $\dim V_1=n+1$,

$$\lambda_1\geqslant n\left(\frac{\pi}{d}\right)^2,$$

(ii) *In general*

$$\lambda_1\geqslant\frac{n}{d^2}\left(\frac{\pi}{2}+C(n)\ \frac{d^n}{d^n+V}\right)^2,$$

where d and V are the diameter and volume of M respectively, $C(n)$ is a positive constant depending only on n.

Proof Let $f_1, \cdots, f_{m+1}$ be the orthonormal basis of V_1. By [3], there exists a constant $c>0$, such that

$$x^k=cf_k\ (1\leqslant k\leqslant m+1),$$
$$X=(x^k):\ M\hookrightarrow E^{m+1}$$

is an isometric immersion, further more X is an isometric immersion of M into $S^m(r)$, where $S^m(r)$ is a sphere of radius r and here $r=\sqrt{\frac{n}{\lambda_1}}$. When $m<n$, this immersion is minimal[3].

Now we construct a new metric on M, that is $\tilde{g}=\frac{1}{r^2}g$. Let $\overline{X}=\frac{1}{r}X$. Then

$$\overline{X}:\ (M, \tilde{g})\hookrightarrow S^m=S^m(1)\subset E^{m+1}$$

is an isometric immersion and further more is minimal when $m>n$. Let $\tilde{d}$, $\tilde{V}$ be the diameter and volume of $(M, \tilde{g})$ respectively, then

$$\tilde{d}=\frac{1}{r}d,\ \tilde{V}=\frac{1}{r^n}V.$$

(i) If $n=m$, i.e. $\dim V_1=n+1$, $\overline{X}$ is onto, since $\overline{X}$ is an isometric immersion of $(M, \tilde{g})$ to S^m. Let X_1, X_2 be a pair of points on S^m with $d(X_1, X_2)=\pi$. Then there exist two points p, q on $(M, \tilde{g})$ such that $\overline{X}(p)=X_1$, and $\overline{X}(q)=X_2$. Let γ be a minimal geodesic on $(M, \tilde{g})$ from p to q. $\overline{X}(\gamma)$ is a geodesic on S^m from X_1 to X_2. Therefore it follows that

$$\tilde{d}\geqslant L(\gamma)=L(\overline{X}(\gamma))$$

i.e.

$$\tilde{d}=\frac{1}{r}d=\sqrt{\frac{\lambda_1}{n}}\,d\geqslant\pi,$$

$$\lambda_1\geqslant\frac{n\pi^2}{d^2}.$$

(ii) If $m>n$, using Theorem 1, we have

$$\tilde{d}=\frac{1}{r}d=\sqrt{\frac{\lambda_1}{n}}\,d\geqslant\frac{\pi}{2}+C(n)\frac{\tilde{d}^n}{\tilde{d}^n+\tilde{V}}=\frac{\pi}{2}+C(n)\frac{d^n}{d^n+V},$$

$$\lambda_1\geqslant\frac{n}{d^2}\left(\frac{\pi}{2}+C(n)\frac{d^n}{d^n+V}\right)^2.$$

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