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# PERIODIC ORBITS AND *P* STABLE ORBIT CLOSURES OF CONTINUOUS FLOWS ON SURFACES

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**Abstract**

In this paper the author determines the maximum number of simple closed curves with certain propertiees on any manifold $M$ of dimension 2, and obtains some general properties of $M$. Then applying the properties to continuous flows defined on $M$, the author obtains the maximum number of distinct $P$ stable orbit closures and non-zero-homotopic periodic orbits which are not homotopic to each other.

At the beginning of the investigation, we introduce some definitions. Let $M$ be a connected 2-manifold (orientable or nonorientable). We also call $M$ a surface. We say that $M$ is a closed (or compact) surface if it is a compact 2-manifold without boundary. Let $f$ be a continuous flow defined on $M$ with the boundary of $r$ components $L_1, \cdots, L_r$. Then, we define $A_{x_1 \cdots x_r}(M)$ as the compactification of $M$ by means of $r$ points at infinity (namely, gluing a disk to $M$ along each $L_i$). Moreover, we can continue the definition of $f$ to $A_{x_1 \cdots x_r}(M)$ in such a way that the number of periodic and $P$ stable orbits keeps constant. So, without loss of generality, we often suppose the surface $M$ has no boundary, i.e., $M$ is closed.

Let $L_1$, $L_2$ be two simple closed curves on $M$. We write $L_1 \sim L_2$ (resp. $L_1 \not\sim L_2$) if $L_1$ is (is not) homotopic to $L_2$ on $M$. In this mark it is clear that $L_1 \sim 0$ and $L_1 \not\sim 0$. We say that a group of simple closed curves $L_1, \cdots, L_n$ on $M$ satisfy property $A$ if $L_i \not\sim 0$, $L_i \not\sim L_j$ and $L_i \cap L_j = \varnothing$ for all $i \neq j$, $i, j = 1, \cdots, n$.

## § 1. Flows on Orientable Surfaces

As a preliminary, we first investigate some general properties of surfaces.

**Lemma 1.** *Let $M_i$ be an orientable surface of genus $g_i$ with the boundary of $k_i$ components, $i=1, 2$. Let $M$ be the surface obtained by gluing $M_1$ and $M_2$ along $r$ components of each boundary, $r \leqslant \min\{k_1, k_2\}$. Then $M$ has the genus $g = g_1 + g_2 + r - 1$.*

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*Proof* Euler characteristic of $M_i$ is $\chi(M_i)=2-2g_i-k_i$, $i=1, 2$. Notice that $\chi(M)=2-2g-(k_1+k_2-2r)$. It follows that $g=g_1+g_2+r-1$ from the relation $\chi(M_1)+\chi(M_2)=\chi(M)$.

**Corollary 1.** *Let $M$ be an orientable surface without boundary and let $L\subset M$ be a non-zero-homotopic simple closed curve such that $M-L=M_1\cup M_2$, $M_1\neq M_2$. Then $g=g_1+g_2$, where $g, g_1, g_2$ denote the genus of $M, M_1, M_2$ respectively.*

If $M=T^2$ (the torus), it is easy to know that any two non-zero-homotopic simple closed curves on $T^2$ must be homotopic as long as both curves have no points in common. For general surfaces we have the following theorem.

**Theorem 1.** *Let $M$ be a closed orientable surface of genus $g$, $g\geqslant 2$. Then there exist at most $3g-3$ simple closed curves which satisfy the property A. Moreover, this number is always achieved.*

*Proof* To prove the theorem we use induction.

1. First consider the case $g=2$. Suppose there exist 4 simple closed curves satisfying the property A. Denote them by $L_i$, $i=1, 2, 3, 4$. Let $M-L_4=M_1\cup M_2$. There are two cases to consider according as $M_1=M_2$ or not.

Case *A*. $M_1=M_2$. Because the boundary of $M_1$ has 2 components, it has compactification $M_1^*=A_{xy}(M_1)$. If one of $L_i$, $i=1, 2, 3$, say $L_1$, is homotopic to zero on $M_1^*$, then there is a disk $D$ on $M_1^*$ such that $L_1=\partial D$ and $x, y\in D$ (the latter is because $L_1\not\sim L_4$ and $L_1\not\sim 0$ on $M$). It follows that $L_2\not\sim 0$, $L_3\not\sim 0$ and $L_1\not\sim L_3$ on $M_1^*$. On the other hand, $M_1^*$ is homeomorphic to $T^2$, which gives a contradiction. If none of the curves is homotopic to zero on $M_1^*$, then $L_1\sim L_2\sim L_3$ on $M_1^*$ since $M_1^*\approx T^2$ ("$\approx$" means homeomorphism). We may suppose the curves are of type (1, 0) (otherwise, cutting $M_1^*$ along $L_1$ we get a cylinder containing $L_2$, $L_3$ (see Fig. 1a). Then we glue the cylinder with itself along its boundary to get a surface homeomorphic to $T^2$ (see Fig. 1b), on which $L_1$, $L_2$, $L_3$ are of type (1, 0)).


Fig. 1

It is clear that we can find out two curves, say, $L_2$, $L_3$ as shown in Fig. 1b, such that the cylinder bounded by $L_2$ and $L_3$ contains no points at infinity. This means $L_2\sim L_3$ on $M$, a contradiction.

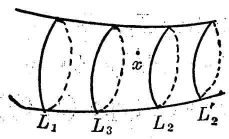
Case B. $M_1\neq M_2$. Obviously, $M_1$ and $M_2$ are all of genus 1 from Corollary 1, and there exist at least two curves on one of both, e.g., $L_1$, $L_2\subset M_1$. Notice that the boundary of $M_1$ has only one component. It is easy to understand that $L_1 \not\sim 0$, $L_2 \not\sim 0$ on its compactification $A_x(M_1)$. On the other hand, $A_x(M_1) \approx T^2$, which gives a contradiction similarly. Thus, we finished the proof for $g=2$.

2. Now suppose the theorem has been proved for surfaces with genus $\leqslant g-1$. Let $M$ be a closed surface with genus $g$. Suppose there exist $3g-3+1$ simple closed curves $L_1, \cdots, L_{3g-2}$ which satisfy the property A. Let $M-L_{3g-2}=M_1 \cup M_2$. Then there are two cases to consider as follows.

Case A. $M_1=M_2$. It is obvious that its compactification $M_1^*=A_{xy}(M_1)$ is of genus $g-1$.

A1. If one of the curves $L_i$, $i=1, \cdots, 3g-3$, say $L_{3g-3}$, is homotopic to zero on $M_1^*$, then there is a disk $D$ on $M_1^*$ such that $L_{3g-3}=\partial D$ and $x$, $y \in D$. It implies that $L_i \not\sim 0$ and $L_i \not\sim L_j$ on $M_1^*$ for all $i \neq j$, $i$, $j=1, \cdots, 3g-4$. On the other hand, $3g-4 > 3(g-1)-3$, which is contrary to the inductive assumption.

A2. If $L_i \not\sim 0$ on $M_1^*$ for all $i=1, \cdots, 3g-3$, we want to prove that among them there are at most two curves which are homotopic to other two ones. Suppose there are two pairs of curves with the property above, say, $L_1 \sim L_2$ and $L_3 \sim L_2'$ on $M_1^*$. Then $L_1$ and $L_2$ bound a cylinder $H$, $L_3$ and $L_2'$ bound an $H'$, and they both contain at least one point at infinity, say $x \in H$. If $y \notin H'$ then $x \in H'$, and the nonempty intersection $H \cap H'$ is a submanifold of $M_1^*$, and so it is also a cylinder since there is no hole on both $H$ and $H'$ (see Fig. 2b). Consequently, one of $L_1$, $L_2$ and one of $L_3$, $L_2'$ must bound a cylinder without points at infinity. Therefore, both of them are homotopic on $M$, a contradiction. Hence, $y \in H'$ (see Fig. 2b).



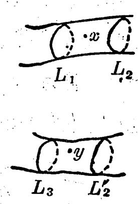




Fig. 2

Notice that if $L_2 \neq L_2'$ we can let $L_2'=L_4$. It follows that there are $3g-5$ curves $L_1, L_3, L_5, \cdots, L_{3g-3}$ (when $L_2 \neq L_2'$) or $L_1, L_4, L_5, \cdots, L_{3g-3}$ (when $L_2=L_2'$) which satisfy the property A on $M_1^*$. On the other hand, $3g-5>3(g-1)-3$, which is contrary to the inductive assumption.

Case B. $M_1 \neq M_2$. We have the compactifications $A_x(M_1)$ and $A_y(M_2)$, and from Corollary 1 $g=g_1+g_2$, $g_i \geqslant 1$, $i=1, 2$. Let $L_{i_1}, \cdots, L_{i_{n_1}} \subset M_1$, $L_{j_1}, \cdots, L_{j_{n_2}} \subset M_2$, $n_1+n_2=3g-3$. It is easy to check that there is at least one $n_i$ such that $n_i > 3g_i-2$. We may suppose $n_1 > 3g_1-2$. Then $L_{i_k} \not\sim 0$ on $A_x(M_1)$ for all $k=1, \cdots, n_1$, and there exist at most two of them which are homotopic to each other. Thus, we obtain $n_1-1$ simple closed curves which satisfy the property A on $A_x(M_1)$. However, $n_1-1\geqslant 3g_1-2=3(g_1-1)+1$, which is contrary to the inductive assumption when $g_1\geqslant 2$. For $g_1=1$, we have $n_1\geqslant 2$ and $A_x(M_1)\approx T^2$, which gives a contradiction, too.

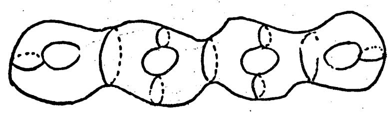


Fig. 3 ($g=4$)

Thus, we finished the proof by the induction. Moreover, Fig. 3 above gives an example which showes that there are $2(g-2)+g+1=3g-3$ simple closed curves satisfying the property A on $T_g$.

**Corollary 2.** *Let $M$ be an orientable surface of genus $g$ with the boundary of $r$ components and let $R$ denote the number of simple closed curves which satisfy the property* A *on $M$. Then*

$$R\leqslant\begin{cases}3g-3+2r & \text{for } g\geqslant 2,\ r\geqslant 0;\\ g+2r-1 & \text{for } g=1,\ r\geqslant 1;\\ 2r-3 & \text{for } g=0,\ r\geqslant 2.\end{cases}$$

*Proof* Suppose there exist $R$ simple closed curves satisfying the property A on $M$. Fix $g\geqslant 2$ (similarly for $g=1$ or 0), and prove by the induction on $r$. Let $r=1$. Then there is at most one curve which becomes homotopic to zero, and at most one which becomes homotopic to another on the compactification $A_x(M)$ of $M$. Hence, there are at least $R-2$ curves which satisfy the property $A$ on $A_x(M)$. Thus, $R-2\leqslant 3g-3$ or $R\leqslant 3g-3+2$ from Theorem 1. Now let us suppose that the proposition has been proved for surfaces with the boundary of $r-1$ components and let $M$ be a surface with the boundary of $r$ components. Let $M_1$ be the surface with the boundary of $r-1$ components, obtained by compactifying one component. We can prove, as before, that there are at least $R-2$ curves satisfying the property $A$ on $M_1$. Then $R-2\leqslant 3g-3+2(r-1)$ by the inductive assumption, and the proof is completed.

From Theorem 1 we have the following theorem.

**Theorem 2.** *Let $M$ be a closed orientablesurface of genus $g$, $g\geqslant 2$, and let $f$ be a continuous flow on $M$. Then $f$ has at most $3g-3$ non-zero-homotopic periodic orbits which are not homotopic to each other. Moreover, this number is always achieved by a flow.*

*Proof* We need only to construct an example which satisfies our requirment. Take a continuous flow on sphere $S^2$ which has three centers and one saddle, and all of its regular orbits are closed, except separatrices (see Fig. 4a). Removing out the three centers we get a cylinder tree as shown in Fig. 4b. Using $2m$ copies of this flow we construct in the obvious way a continuous flow with $3m$ periodic orbits on an orientable surface of genus $m+1$ as in Fig. 5.

Fig. 4

From Corollary 2 we have the following corollary.

**Corollary 3.** *Let $M$ be an orientable surface of genus $g$ with the boundary of $r$ components, and let $f$ be a continuous flow on $M$. Then $f$ has at most $3g-3+2r$ for $g\geqslant 2$ and $r\geqslant 0$; $g+2r-1$ for $g=1$ and $r\geqslant 1$; $2r-3$ for $g=0$ and $r\geqslant 2$ non-zero-homotopic periodic orbits which are not homotopic to each other.*

It is easy to know that the non-zero-homotopic periodic orbits and $P$ stable orbits of a flow on $T^2$ are not compatible. For general surfaces we have the following theorem.

**Theorem 3.** *Let $M$ be a closed surface of genus $g$, $g\geqslant 2$, and let $f$ be a continuous flow on $M$. Denote by $r_1$, $r_2$ the numbers of distinct $P$ stable orbit closures, non-zero-homotopic periodic orbits which are not homotopic to each other respectively. Then*

(1) $r_1+r_2\leqslant 3g-3$;

(2) *there is a flow $f$ with $r_1=g-1$, $r_2=2g-2$.*

*Proof* From Theorem 2 and $g\geqslant 2$ the conclusion is true when $r_1=0$. Now suppose $r_1>0$. By means of the method used in the proof of Lemma 5[2] we may construct a new flow $f'$ on $M$ with the following properties:

(i) $f'$ has $r_1$ periodic orbits $l_1, \cdots, l_{r_1}$ coming from the $r_1$ closures;

(ii) the periodic orbits $L_1, \cdots, L_{r_2}$ of $f$ are also orbits of $f'$.

It is easy to prove from the construction of $f'$ and $l_i$, $i=1, \cdots, r_1$, that the curves $l_i$, $L_j$, $i=1, \cdots, r_1$, $j=1, \cdots, r_2$, satisfy the property A. Hence, $r_1+r_2\leqslant 3g-3$ by applying Theorem 2 to $f'$.

In the rest of the proof we construct a flow with $r_1=g-1$, $r_2=2g-2$. We use a continuous flow on the torus with a nowhere dense recurrent orbit closure. Modify

Fig. 5

the flow so that there is a closed disk in the complement of the closure in which there is a periodic orbit $l$. For example, we can change it in the disk in such a way that its structure of orbits is as in Fig. 5a. Throwing away the open disk with the boundary $l$ and gluing a cylinder tree like that in Fig. 4b along $l$, we get a surface of genus 1 and a flow on it. Using $g-1$ copies of the flow we may construct a flow on a surface of genus $g$ as in Fig. 5b. This flow meets our requirment.

## § 2. Flows on Nonorientable Surfaces

In this chapter we consider flows on a nonorientable surface. We begin with certain topological preliminaries.

**Lemma 2.** (1) *There are at most two simple closed curves which satisfy the property* A *on Moebius band* $B^2$, *of which at most one is one-sided, at most one is two-sided.*

(2) *There is at most one simple closed curve, which satisfies the property* A *on the projective plane* $P^2$, *and if it exists it must be one-sided.*

*Proof* (1) The fundmental group of $B^2$ is $Z$, and there are two simple closed curves $L_1$, $L_2$, which represent the elements $\pm 1$ and $\pm 2$ respectively. Obviously, $L_1$ is one-sided and $L_2$ two-sided. Also, we can prove easily that any other curve, which represents $i(\neq 0, \pm 1, \pm 2)$, must intersect itself.

(2) It is clear that there is at most one one-sided closed curve on $P^2$. Noticing that the fundmental group of $P^2$ is a cyclic group of order 2 we see that the conclusion is evident.

**Lemma 3.** *Let* $M$ *be a nonorientable surface of genus* $g(\geqslant 2)$ *with the boundary of finite components and let* $L$ *be a simple closed curve on* $M$ *such that* $M-L=M_1$ *is connected. By* $g_1$ *denote the genus of* $M_1$.

(1) *If* $L$ *is two-sided, then* $g_1=g/2-1$ *when* $M_1$ *is orientable, otherwise,* $g_1=g-2$;

(2) *If* $L$ *is one-sided, then* $g_1=(g-1)/2$ *when* $M_1$ *is orientable, otherwise,* $g_1=g-1$.

*Proof* It is easy to prove by using the relation $\chi(M_1)=\chi(M)$.

**Lemma 4.** *Let* $M$ *be a closed nonorientable surface of genus* $g$ *and let* $L$ *be a non-zero-homotopic simple closed two-sided curve such that* $M-L=M_1\cup M_2$, $M_1\neq M_2$. *Let* $g_i$ *denote the genus of* $M_i$, $i=1, 2$. *Then*

(1) $g=2g_1+2g_2$ *if both* $M_1$ *and* $M_2$ *are orientable;*

(2) $g=2g_1+g_2$ *if* $M_1$ *is orientable and* $M_2$ *is nonorientable;*

(3) $g=g_1+g_2$ *if both* $M_1$ *and* $M_2$ *are nonorientable.*

*Proof* It can be proved by use of the relation $\chi(M_1)+\chi(M_2)=\chi(M)$.

**Lemma 5.** *There are atmost 3 simple closed curves which satisfy the property A on Klein bottle $K^2$, of which at most one is two-sided, at most two are one-sided.*

*Proof* From Lemma 3 we know that there are at most two one-sided curves which have no points in common. Now let $L_1$, $L_2$ be two two-sided curves such that $L_1 \not\sim 0$, $L_2 \not\sim 0$ and $L_1 \cap L_2 = \varnothing$. If $K^2 - L_1$ is connected, it is a cylinder from Lemma 3. Thus, $L_1 \sim L_2$ on $K^2$. If $K^2 - L_1 = M_1 \cup M_2$, $M_1 \neq M_2$, it is easy to know $M_1$ and $M_2$ are both Moebius band with the boundary $L_1$ from $\chi(K^2) = 0$. Thus, $L_1 \sim L_2$ on $K^2$ from Lemm 2a no matter which $L_2$ belongs to.

Now we can prove the following theorem.

**Theorem 4.** *Let M be a closed nonorientable surface of genus g and let $r_1$ ($r_2$) denote the number of simple closed one (two)-sided curves which satisfy the property A on M. Then*

(1) $r_1 \leqslant g$, $r_2 \leqslant 2g-3$;

(2) *the case that $r_1 = g$ and $r_2 = 2g-3$ can occur on M.*

*Proof* It is clear that $r_1 \leqslant g$ from Lemma 3. Now we prove $r_2 \leqslant 2g-3$ by induction. The proposition is true for $g=1$, 0 by Lemmas 2 and 5. Suppose it has been proved for surfaces of genus $\leqslant g-1$. Let $M$ be a surface of genus $g$ and let us suppose that there are $2g-3+1$ simple closed two-sided curves $L_i$, $i=1, \cdots, 2g-2$, which satisfy the property A. Let $M - L_{2g-2} = M_1 \cup M_2$. Then there are several cases to consider below.

Case A. $M_1 = M_2$.

A1. Assume $M_1$ is orientable. Then $g_1 = g/2 - 1$ from Lemma 3. If one of $L_i$, $i=1, \cdots, 2g-3$, say $L_1$, is homotopic to zero on the compactification $A_{xy}(M_1)$, there must be a disk $D$ in $A_{xy}(M_1)$ such that $L_1 = \partial D$ and $x$, $y \in D$. It follows that the other $2g-4$ curves satisfy the property $A$ on $A_{xy}(M_1)$. On the other hand, we have $2g-4 > 3g_1 - 3$, which is contrary to Theorem 1. If none of the curves is homotopic to zero on $A_{xy}(M_1)$, in a way similar to the proof of Theorem 1 (see Fig. 2), we know there are at least $2g-5$ curves satisfying the property A on $A_{xy}(M_1)$. Then the inequality $2g-5 > 3g_1 - 3$ gives a contradiction as before.

A2. Assume $M_1$ is nonorientable. Then $g_1 = g-2$ from Lemma 3. Similarly, we may prove that there are at least $2g-5$ curves satisfying the property A. However, $2g-5 > 2g_1 - 3$, which is contrary to the inductive assumption.

Case B. $M_1 \neq M_2$.

B1. Suppose both $M_1$ and $M_2$ are orientable. Then $g = 2g_1 + 2g_2$ from Lemma 4. If $n_1$ (resp. $n_2 = 2g-3-n_1$) of the curves $L_i$, $i=1, \cdots, 2g-3$, lie on $M_1$ (resp. $M_2$), then the case that $n_1 \leqslant 3g_1 - 3 + 1$, $n_2 \leqslant 3g_2 - 3 + 1$ cannot occur at the same time. This contradicts Corollary 2.

B2. Suppose that one of $M_1$ and $M_2$, say $M_1$, is orientable, the other ($M_2$) is nonorientable. Then $g=2g_1+g_2$. Let $n_1$ and $n_2$ be as above. The case that $n_1\leqslant 3g_1-3+1$ and $n_2\leqslant 2g_2-3+1$ does not occur. Similarly, the case $n_1>3g_1-3+1$ leads to a contradiction with Corollary 2, and the case $n_2>2g_2-3+1$ with the inductive assumption.

B3. Assume both $M_1$ and $M_2$ are nonorientable. Then $g=g_1+g_2$. Thus, we have either $n_1>2g_1-3+1$ or $n_2>2g_2-3+1$, contradicting the inductive assumption. Hence, we have proved the inequality $r_2\leqslant 2g-3$ by the induction.

The conclusion (2) of the theorem is clearfrom the night picture.

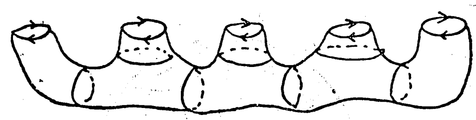


Fig. 6 (g=5)

**Corollary 4.** *Let $M$ be a nonorientable surface of genus $g$ with the boundary of $r$ components $g+r\geqslant 2$. There exist at most $3g-3+2r$ simple closed curves which satisfy the property A, of which at most $g$ are one-sided, at most $2g-3+2r$ are twosided.*

**Theorem 5.** *Let $M$ be a nonorientable surface of genus $g$ ($\geqslant 2$) with the boundary of $r$ components and let $r_1$ ($r_2=r_2'+r_2''$) denote the number of simple closed one (two)-sided curves which satisfy the property A on $M$, where $r_2'$ denotes the number of such curves that all of them do not divide $M$, $r_2''=r_2-r_2'$. Then*

(i) $r_1+2r_2'\leqslant g$;

(ii) $r_2'+r_2''\leqslant 1$ *if* $g=2$ *and* $r=0$, $r_2''+2r_2'\leqslant 2g-3+2r$ *if* $g+r\geqslant 3$. *Hence,* $r_2'\leqslant [(g-r_1)/2]$, $r_1+r_2''+4r_2'\leqslant 3g-3+2r$ *for* $g+r\geqslant 3$.

*Proof* Let $L_1, \cdots, L_{r_2'}$ be the two-sided curves such that $M_0=M-L_1-\cdots-L_{r_2'}$ is still connected. If $M_0$ is nonorientable, then its genus is $g_0=g-2r_2'$ from Lemma 3, and so by Corollary 4 $r_1\leqslant g_0$, $r_2''+2r_2'\leqslant 2g_0-3+2(r+2r_2')$, which give (i) and (ii) for $g+r\geqslant 3$. If $M_0$ is orientable, we write $r_2'=i+j$, $i\geqslant 0$, and suppose without loss of generality that $M_1=M-L_1-\cdots-L_i$ is nonorientable, whereas $M_2=M_1-L_{i+1}$ is orientable. Therefore, $M_3=M_2-L_{i+2}-\cdots-L_{i+j}$ is orientable (in this case $r_1$ must be zero and $g$ even). Then from Lemma 3 the genus of $M_k$, for $k=1$, 2, 3, is the following: $g_1=g-2i$, $g_2=g_1/2-1$, $g_3=g_2-(j-1)=g_1/2-j=g/2-r_2'$. Thus from Corollary 2, $0\leqslant g_3$, and

$$r_2''+2r_2'\leqslant\begin{cases}3g_3-3+2(r+2r_2') \text{ if } g_3\geqslant 2;\\ g_3+2(r+2r_2')-1 \text{ if } g_3=1;\\ 2(r+2r_2')-3 \text{ if } g_3=0,\ r>0.\end{cases}$$

Now it is easy to show the conclusions (i) and (ii) in all the cases. Thus the proof is completed.

In the rest we apply the theorems above to continuous flows. First, by Theorem 4 we have the following theorem.

**Theorem 6.** *Let $M$ be a closed nonorientable surface of genus $g$ ($\geqslant 2$) and let $f$ be a continuous flow on $M$. Then $f$ has at most $3g-3$ periodic orbits satisfying the* *property A on M, of which at most g are one-sided, at most $2g-3$ are two-sided. Moreover, these numbers are always achieved by a flow.*

*Proof* We need only to construct a flow as required. Let $L$ be the equator on sphere $S^2$, $A_i \in L$, $i=1, \cdots, g$. Let $D_i$ be a disk with center at point $A_i$, such that $D_i \cap D_j = \varnothing$ for $i \neq j$, $i, j=1, \cdots, g$. Let $S^2 - L = S_1 \cup S_2$. We define a flow on $S^2 - \bigcup_{i=1}^{g} D_i$ whose structure of orbits is indicated in the following Fig. 7.

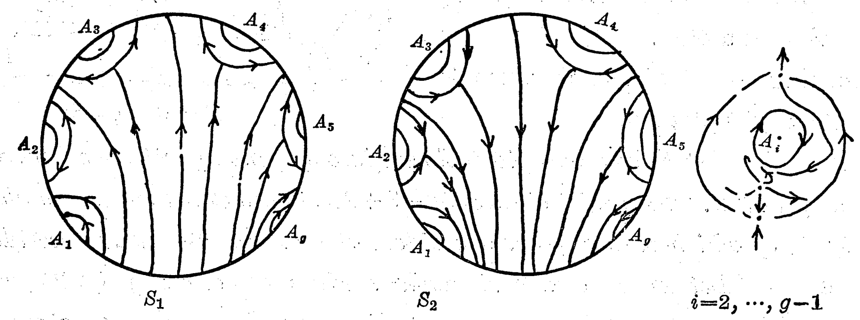



Fig. 7

On considering of $P$ stable orbits and applying Theorem 5 we obtain the following theorem.

**Theorem 7.** *Let $M$ be a nonorientable surface of genus $g$ $(\geqslant 1)$ with the boundary of $r$ components and let $f$ be a continuous flow on $M$. Let $r_0$ (resp. $r_1$, $r_2$) denote the number of distinct $P$ stable orbit closures (resp. one, two-sided periodic orbits which satisfy the property A). Then*

$$2r_0 + r_1 \leqslant g, \quad 2r_0 + r_2 \leqslant 2g - 3 + 2u \ for\ g \geqslant 1,\ r \geqslant 0.$$

*Hence,* $4r_0 + r_1 + r_2 \leqslant 3g - 3 + 2r$ *for* $g + r \geqslant 2$.

*Proof* The conclusion is true from Lemma 2 and Corollary 4 for $g=1$ and 2 since now $r_0 = 0$. For $g \geqslant 3$, we can construct a new flow $f'$ with $r_0$ two-sided periodic orbits all of which do not divide $M$ by means of the method of the proof of Theorem 4[3]. Moreover, the $r_1 + r_2$ periodic orbits of $f$ are still those of $f'$. Now the conclusion is clear from Theorem 5.

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