Chin. Ann. of Math.
9B (3) 1988

A NOTE ON A COVERING LEMMA OF A CORDOBA AND R FEFFERMAN**

Long Ruilin (龙瑞麟)* Shen Zhongwei (申仲伟)*

Abstract

This note simplifies Cordoba–Fefferman's proof on the weak boundedness of strong maximal operator M_s (with respect to $d\mu$) on $L(1+\log^{+(n-1)}L)$. Some two-weighted boundedness results on $L(1+\log^{+\alpha}L)$ of M_s are investigated.

Let M_s denote the strong maximal operator in $\mathbb{R}^n$. A. Cordoba and R. Fefferman[1] gave a geometric proof of Jessen–Marcinkiewicz–Zygmund theorem which states

$$|\{M_s(f)>\lambda\}|\leqslant C\int_{\mathbb{R}^n}\frac{|f|}{\lambda}\Big(1+\log^{+(n-1)}\frac{|f|}{\lambda}\Big)dx,\ \forall f,\ \forall\lambda>0. \tag{1}$$

The advantage of this proof is that it can be used in many cases in which the simple iteration method is no longer useful. The key of the proof is a covering lemma. In this note we shall simplify the proof of their covering lemma by extrapolation. A similar idea also appeared in A. Carbery–S. Y. Chang–J. Garnett's paper[2].

We shall treat more general case in which the Lebesgue measure dx is replaced by $d\mu$, where $d\mu$ is an absolutely continuous measures discussed by R. Fefferman[ ], i. e. $d\mu=w(x)dx$ satisfies

$$w(x',x_n)\in A_\infty^{(s)}(\mathbb{R}^{n-1})\text{ uniformly in a. e. } x_n, \tag{2}$$

$$|(R)_d|_\mu\leqslant C|R|_\mu,\ \forall R, \tag{3}$$

where R is a rectangle with sides parallel to the axes. $(R)_d$ denotes the rectangle with the same center and $x_i(i<n)$ side lengths but 3 times x_n side length of R, and $A_\infty^{(s)}$ is the rectangle version of classical A_∞ condition: we say $\mu\in A_\infty^{(s)}$ if $\forall\alpha\in(0,1)$, $\exists\beta\in(0,1)$, s. t.

$$\frac{|E|_\mu}{|R|_\mu}\leqslant\beta\Rightarrow\frac{|E|}{|R|}\leqslant\alpha,\ \forall R,\ \forall\text{ measurable set } E\subset R. \tag{4}$$

Observe that (4) is equivalent to the following:

$$\forall\alpha\in(0,1),\ \exists\beta\in(0,1)\text{ s. t. }\frac{|E|}{|R|}\leqslant 1-\alpha\Rightarrow\frac{|E|_\mu}{|R|_\mu}\leqslant 1-\beta,\ \forall R,\ \forall E\subset R.$$

Particularly, take $\alpha=1/2$, $\mu\in A_\infty^{(s)}$ implies that $\exists\beta\in(0,1)$ s. t.

Manuscript received March 6, 1986.
* Institute of Mathematics, Academia Sinica, Beijing, China.
** Supproted by NSF of China.

$$\frac{|E|}{|R|}\leqslant\frac{1}{2}\Rightarrow\frac{|E|_\mu}{|R|_\mu}\leqslant\beta,\ \forall R,\ \forall E\subset R. \tag{4'}$$

This implies μ satisfies the doubling condition. We also remark that for $1\leqslant p\leqslant\infty$, condition $A_p^{(s)}$ are equivalent to A_p condition in each variable separately and uniformly in the other variables. Since we shall use this fact on $A_\infty^{(s)}$, we prove it in the case $p=\infty$. Suppose that $w\in A_\infty$ (uniformly) in each variable. Then there exists $p<\infty$ such that $w\in A_p$ (uniformly) in each variable. So M_s is bounded on $L^p(w)$. This implies $w\in A_p^{(s)}$. Therefore $w\in A_\infty^{(s)}$. Conversely, let $w\in A_\infty^{(s)}$. We want to prove that as a function of arbitrary j variables (say last j variables), $w\in A_\infty^{(s)}(\mathbb{R}^j)$ uniformly in a. e. points in $\mathbb{R}^{n-j}$. In fact, the Reverse Hölder Inequality for rectangles also holds as in classical case, so there exists $p<\infty$ such that $w\in A_p^{(s)}$. Let J be arbitrary rectangle in $\mathbb{R}^j$ whose center and side lengths are both rational. Let $x'=(x_1,\cdots,x_{n-j})$ be a Lebesgue differentiable point of all functions

$$\int_J w^\varepsilon\,dx_{n-j+1}\cdots dx_n,\ \forall \text{ such } J,\ \varepsilon=1,\ -(p-1)^{-1},$$

and I be a cube in $\mathbb{R}^{n-j}$ containing x'. Then letting $I\to x'$ in the following inequality

$$\frac{1}{|I\times J|}\int_{I\times J} w\,dx\Big(\frac{1}{|I\times J|}\int_{I\times J} w^{-(p-1)^{-1}}dx\Big)^{p-1}\leqslant C,$$

we get

$$\frac{1}{|J|}\int_J w\,dx_{n-j+1}\cdots dx_n\left(\frac{1}{|J|}\int_J w^{-(p-1)^{-1}}\,dx_{n-j+1}\cdots dx_n\right)^{p-1}\leqslant C.$$

Obviously, the above inequality holds for all rectangles in a. e. x', i. e. $w\in A_p^{(s)}(\mathbb{R}^j)$ in a. e. x'. Consequently

$$w\in A_\infty^{(s)}(\mathbb{R}^j) \text{ uniformly in a. e. } x'.$$

In section 1 we simplify the proof of Cordoba–Fefferman's covering lemma in general case of $d\mu$. As a consequence we get (1) for $d\mu$, a result which B. Jawerth and A. Torchinsky[4] have obtained by a different approach. In section 2 we discuss some weighted norm inequalities for M_s. But the results are unsatisfactory.

§1. The Weak Type Estimate of the Strong Maximal Operator with Respect to a Measure

Let $\{R_k\}$ be a sequence of rectangles in $\mathbb{R}^n$. $\{R_k\}$ is said to satisfy property P_1 if

$$|R_k\cap\bigcup_{i<k}R_i|\leqslant|R_k|/2; \tag{5}$$

$\{R_k\}$ is said to satisfy property P_2 if its side lengths in the x_n direction are decreasing and

$$|R_k\cap\bigcup_{i<k}(R_i)_d|\leqslant|R_k|/2. \tag{6}$$

Observe that if $\{R_k\}$ satisfies property P_2, then when we slice $\{R_k\}$ by an arbitrary hyperplane perpendicular to the x_n axis, we obtain a sequence of $n-1$ dimension rectangles which depend on parameter x_n and satisfy property P_1 for all x_n.

As we shall discuss the property of M_s on Zygmund spaces $L(1+\log^{+\frac{1}{\alpha}}L)$, $0<\alpha<\infty$, we need the following fact about some convex function $\Phi(u)$ and its Young's complementary function $\Psi(v)$ formulated in a lemma, the proof of which is elementary and can be omitted.

Lemma 1. *Let* $0<\alpha<\infty$, $\varphi(u)=1+\log^{+\frac{1}{\alpha}}u$, $\Phi(u)=\int_0^u \varphi(t)\,dt$, $\Psi(v)$ *be the Young's complementary function of* Φ. *Then*

$$\frac{u}{2}\left(1+\log^{+\frac{1}{\alpha}}\frac{u}{2}\right)\leqslant\Phi(u)\leqslant u(1+\log^{+\frac{1}{\alpha}}u),\ \forall u>0,$$
$$\Psi(v)\leqslant c_{\alpha,\delta}v^{1-\alpha}(\exp(c_1v)^{\alpha}-1),\ v\leqslant\delta, \tag{7}$$
$$\Psi(v)\leqslant c_{\alpha,\delta}(\exp(c_1v)^{\alpha}-1),\ v>\delta.$$

Remark. When $\alpha\leqslant1$, we have the following unified estimate

$$\Psi(v)\leqslant C(\exp(c_1v)^{\alpha}-1),\ \forall v>0. \tag{7$'$}$$

Let μ be a measure satisfying (2) and (3). Define

$$M_{s,\mu}f(x)=\sup_{R\ni x}|R|_\mu^{-1}\int_R|f(y)|\,d\mu(y),\ \forall f,\ \forall x\in\mathbb{R}^n, \tag{8}$$

$$M_{s,\mu'}f(x')=\sup_{\mathbb{R}^{n-1}\supset R\in x'}|R|_\mu^{-1}\int_R|f(y',x_n)|\,d\mu'(y'),\ \forall f,\ \forall x\in\mathbb{R}_n, \tag{9}$$

where

$$d\mu'=w(x',x_n)dx',\ x'=(x_1,\cdots,x_{n-1}).$$

Lemma 2. *Let* $\{R_k\}$ *be a sequence of rectangles in* $\mathbb{R}^n$ *satisfying property* P_2. *Assume that* $M_{s,\mu'}$ *is of weak type* (p,p) *on* $(\mathbb{R}^{n-1},d\mu')$ *with weak type* (p,p) *coefficient* $O((p-1)^{-r})$, $1<p\leqslant2$, *uniformly in a. e.* x_n. *Then there exists a constant* C *independent of* $\{R_k\}$ *such that*

$$\left(\int_{\cup R_k}|\Sigma\chi_{R_k}|^{p'}\,d\mu\right)^{1/p'}\leqslant C(p')^{r+1}|\cup R_k|_\mu^{1/p'},\ 1<p\leqslant2. \tag{10}$$

Proof Let $\{R_k\}$ be a sequence of rectangles in $\mathbb{R}^n$ satisfying property P_2. After slicing $\{R_k\}$ at x_n, we obtain a sequence of rectangles in $\mathbb{R}^{n-1}$ which depend on x_n and satisfy property P_1 for all x_n, i. e.

$$|S_k^{(x_n)}\cap\bigcup_{i<k}S_i^{(x_n)}|\leqslant|S_k^{(x_n)}|/2,\ \forall x_n.$$

Since $\mu'\in A_\infty^s(\mathbb{R}^{n-1})$ uniformly in a. e. x_n, by (4)$'$ we have

$$|S_k^{(x_n)}\cap\bigcup_{i<k}S_i^{(x_n)}|_{\mu'}\leqslant\beta|S_k^{(x_n)}|_{\mu'}.$$

We want to prove

$$\left(\int_{\cup S_k^{(x_n)}}|\Sigma\chi_{S_k^{(x_n)}}(x')|^{p'}\,d\mu'(x')\right)^{1/p'}\leqslant C(p')^{r+1}|\cup S_k^{(x_n)}|_{\mu'}^{1/p'} \tag{11}$$

uniformly in x_n. Once it is proved, we take the p'-th power on both sides of (11) and integrate in x_n, then we get (10).

Proving (11) is nothing but proving that if $\mu\in A_\infty^{(s)}(\mathbb{R}^l)$ and $M_{s,\mu}$ is of weak type (p, p) on $(\mathbb{R}^l, \mu)$ with coefficient $O((p-1)^{-r})$, $1<p\leqslant 2$, then for arbitrary sequence of rectangles $\{R_k\}$ satisfying property P_1, we have

$$\left(\int_{UR_k}|\Sigma\chi_{R_k}(x)|^{p'}d\mu\right)^{1/p'}\leqslant C(p')^{r+1}|UR_k|_\mu^{1/p'}.\tag{11)'$$

This can be proved by direct estimate or by linearization of operator. We make use of the latter. Assume that $\{R_k\}$ satisfies property P_1 and $\mu\in A_\infty^{(s)}$. Then

$$|E_k|=|R_k-\bigcup_{i<k}R_i|\geqslant\frac{1}{2}|R_k|,\ |E_k|_\mu\geqslant(1-\beta)|R_k|_\mu.$$

Define linear operator T by

$$Tf(x)=\Sigma|R_k|_\mu^{-1}\int_{R_k}f\,d\mu\chi_{E_k}(x).$$

Notice that its adjoint is given by

$$T^*g(y)=\Sigma|R_k|_\mu^{-1}\int_{E_k}g\,d\mu\chi_{R_k}(y).$$

Observing

$$|Tf(x)\leqslant M_{s,\mu}f(x),\ T^*\chi_{UR_k}(y)\geqslant(1-\beta)\Sigma\chi_{R_k}(y),$$

and letting $g=\chi_{UR_k}$, we have

$$\int T^*g(y)f(y)d\mu=\int Tf(x)g(x)d\mu\leqslant p'\|g\|_{L^{p',1}(\mu)}\|Tf\|_{L^{p,\infty}(\mu)}$$
$$\leqslant p'O\left(\frac{1}{p-1}\right)^r\|g\|_{L^{p',1}}\|f\|_p=O(p')^{r+1}|UR_k|_\mu^{1/p'}\|f\|_p,$$

which implies the desired (11)$'$. This completes the proof of Lemma 2.

Remark. (10) also holds for $p>2$, since it holds for $p'=2$, 1, so does for $1<p'<2$, i. e. for $p>2$.

Lemma 3. *The assumptions are given as in Lemma 2. Then there exist constants c_0 and c such that*

$$\int_{UR_k}(\exp((c_0\Sigma\chi_{R_k})^{1/(r+1)}-1)d\mu\leqslant C|UR_k|_\mu.\tag{12}$$

Proof Let c_0 be determined later and $g=(c_0\Sigma\chi_{R_k})^{1/(r+1)}$. It follows from Lemma 2 that

$$\int_{UR_k}\left(e^g-1-\cdots-\frac{g^r}{r!}\right)d\mu=\int_{UR_k}\sum_{j=r+1}^\infty\frac{g^j}{j!}d\mu\leqslant\sum_{j=r+1}^\infty((r+1)^{-1}C_0^{1/(r+1)}C^{1/(r+1)}e)^j|UR_k|_\mu$$
$$\leqslant C|UR_k|_\mu \text{ if } c_0 \text{ is smallenough.}$$

And since $g^{j/r+1}\leqslant C\Sigma\chi_{R_k}$, if $j\leqslant r$, we have proved (12). The proof of the lemma is finished.

A careful examination on the proof of R. Fefferman[3] make us be able to obtain the following slightly precise result.

Lemma 4. *Let μ be a measure satisfying* (2) *and* (3). *Then $M_{s,\mu}$ is of weak type (p, p) on $(\mathbb{R}^n, d\mu)$ with coefficient $O(p-1)^{1-n}$, $1<p\leqslant 2$.*

Proof The proof is by induction on n. For $n=1$, μ satisfies the doubling condition, the result is well-known. Suppose that $n>1$ and the lemma holds for $n-1$. Let μ be a measure satisfying (2) and (3), then so does $\mu'=w(x', x_n)\,dx'$ (for index $n-1$) uniformly in a. e. x_n. By induction, $M_{s,\mu'}$ is of weak type (p, p) with coefficient $O((p-1)^{2-n})$. Let $\{R_k\}$ be arbitrary sequence of rectangles in $\mathbb{R}^n$ satisfying property P_2. By Lemma 2, we have

$$\left(\int_{UR_k}|\Sigma\chi_{R_k}|^{p'}\,d\mu\right)^{1/p'}\leqslant C(p')^{n-1}|UR_k|_\mu^{1/p'}.\ 1<p\leqslant 2. \qquad (13)$$

It is routine that (13) implies $M_{s,\mu}$ is of waek type (p, p) with coefficient $O\,(p')^{n-1}$. For the sake of completeness, we give its proof.

Let $\lambda>0$ and $\{\widetilde{R}_k\}$ be a cover of $\{M_{s,\mu}f>\lambda\}$ such that $|\widetilde{R}_k|_\mu^{-1}\int_{\widetilde{R}_k}|f|d\mu>\lambda$. With no loss of generality, we may assume that $\{\widetilde{R}_k\}$ is a finite sequence and x_n side lengths of $\widetilde{R}_k'$s are decreasing. Now let $R_1=\widetilde{R}_1$ and suppose tnat we have chosen R_2, $\cdots$, R_{k-1}. Then R_k is taken to be the first rectangle in the sequence $\{\widetilde{R}_j\}$ after R_{k-1} with the following property:

$$|\widetilde{R}_j\cap\bigcup_{i\leqslant k-1}(R_i)_d|\leqslant|\widetilde{R}_j|/2.$$

Thus we obtain a sequence of rectangles $\{R_k\}$ satisfying property P_2. So we have (13). Observe that we also have $|U\widetilde{R}_j|_\mu\leqslant C|UR_k|_\mu$. In fact, owing to the choice of $\{R_k\}$ we have

$$|\widetilde{R}_j\cap U'(R_k)_d|>|\widetilde{R}_j|/2,\ \forall j,$$

where U' denotes the union of sets $(R_k)_d$ with R_k being ahead of $\widetilde{R}_j$ in $\{\widetilde{R}_i\}$. Let $\widetilde{S}_j^{(x_n)}$ and $S_{k,d}^{(x_n)}$ denote the slices of $\widetilde{R}_j$ and $(R_k)_d$ at x_n respectively. Then

$$|\widetilde{S}_j^{(x_n)}\cap US_{k,d}^{(x_n)}|>|\widetilde{S}_j^{(x_n)}|/2.$$

Since $\mu\in A_\infty^{(s)}(\mathbb{R}^{n-1})$ uniformly in a. e. x_n, it follows that

$$|\widetilde{S}_j^{(x_n)}\cap US_{k,d}^{(x_n)}|_{\mu'}\geqslant c|\widetilde{S}_j^{(x_n)}|_{\mu'},$$

which implies $M_{s,\mu'}(\chi_{US_{k,d}^{(x_n)}})|_{U\widetilde{S}_j^{(x_n)}}\geqslant c$. By induction, $M_{s,\mu'}$ is of weak type (2.2). Therefore

$$|U\widetilde{S}_j^{(x_n)}|_{\mu'}\leqslant c|US_{k,d}^{(x_n)}|_{\mu'},$$

and

$$|U\widetilde{R}_j|_\mu\leqslant c|U(R_k)_d|_\mu\leqslant c\Sigma|R_k|_\mu\leqslant c|UR_k|_\mu.$$

Consequently

$$|\{M_{s,\mu}f>\lambda\}|_\mu\leqslant|U\widetilde{R}_j|_\mu\leqslant c\Sigma|R_k|_\mu\leqslant c\Sigma\int_{R_k}\frac{|f|}{\lambda}\,d\mu$$
$$\leqslant c\left(\frac{1}{p-1}\right)^{n-1}|UR_k|_\mu^{1/p'}\lambda^{-1}\|f\|_p\leqslant c((p')^{n-1}\|f\|_p/\lambda)^p.$$

This completes the proof of the lemma.

Remark. When $d\mu=dx$, the proof of the lemma is very easy. In fact, as the weak type (p, p) and type (p, p) coefficients of Hardy-Littlewood maximal operator are $O(1)$ and $O((p-1)^{-1})$ respectively, we have

$$|\{M_s f>\lambda\}|=\int_{\mathbb{R}^{n-1}}|\{M_s f>\lambda\}|_{x_n}dx'\leqslant\int_{\mathbb{R}^{n-1}}|\{M^{(n)}(M^{(n-1)}\cdots M^{(1)}f)>\lambda\}|_{x_n}dx'$$

$$\leqslant\frac{c}{\lambda^p}\int_{\mathbb{R}^n}|M^{(n-1)}\cdots M^{(1)}f|^p dx_n\,dx'\leqslant\frac{c}{\lambda^p}((p')^{n-1}\|f\|_p)^p.$$

Now we can obtain the lemma of A. Cordoba and R. Fefferman directly.

Lemma 5. *Let μ be a measure satisfying* (2) *and* (3) *and $\{\tilde{R}_j\}$ be a sequence of rectangles. Then there exists subsequence $\{R_k\}$ such that*

$$|U\tilde{R}_j|_\mu\leqslant c|UR_k|_\mu, \tag{14}$$

$$\int_{UR_k}(\exp((c_0\Sigma\chi_{R_k})^{1/(n-1)}-1)d\mu\leqslant c|UR_k|_\mu. \tag{15}$$

Proof Assume that $\{\tilde{R}_j\}$ has been ordered so that its lengths in the x_n direction are decreasing. We choose a subsequence $\{R_k\}$ satisfying property P_2 by the choice process described in the proof of Lemma 4. Recall that we have proved (14). Since $\mu'=w(x', x_n)dx'$ satisfies the assumption of Lemma 4 uniformly in x_n, $M_{s,\mu'}$ is of weak type (p, p) with uniform coefficient $O(p-1)^{2-n}$. So (15) follows from Lemma 2 and Lemma 3. This completes the proof of the lemma.

Now we establish (1).

Theorem 1. *Let μ be a measure satisfying* (2) *and* (3). *Then*

$$|\{M_{s,\mu}f>\lambda\}|_\mu\leqslant\int_{\mathbb{R}^n}\frac{c|f|}{\lambda}\Big(1+\log^{+(n-1)}\frac{c|f|}{\lambda}\Big)d\mu,\ \forall f,\ \forall\lambda>0. \tag{1'}$$

Proof Let $\lambda>0$ and $\{\tilde{R}_j\}$ be a cover of $\{M_{s,\mu}f>\lambda\}$ such that $|\tilde{R}_j|_\mu^{-1}\int_{\tilde{R}_j}|f|d\mu>\lambda$. By Lemma 5, we have subsequence $\{R_k\}$ satisfying (14) and (15). Let $\Phi(u)=u(1+\log^{+(n-1)}u)$ and $\Psi(v)$ be its complementary function. Then

$$|\{M_{s,\mu}f>\lambda\}|_\mu\leqslant c\Sigma|R_k|_\mu,$$

and

$$\Sigma|R_k|_\mu\leqslant\int\Sigma\chi_{R_k}\frac{|f|}{\lambda}\,d\mu\leqslant\int\Phi\Big(\frac{|f|}{\varepsilon\lambda}\Big)\,d\mu+\int\Psi(\varepsilon\Sigma\chi_{R_k})d\mu.$$

By Lemma 1, we see that

$$\Psi(\varepsilon\Sigma\chi_{R_k})\leqslant c(\exp((c_1\varepsilon\Sigma_{R_k})^{(n-1)^{-1}})-1).$$

Notice that if $\varepsilon\leqslant1$, $e^{\varepsilon u}-1\leqslant\varepsilon(e^u-1)$. Therefore if we choose ε enough small, we obtain

$$\int\Psi(\varepsilon\Sigma\chi_{R_k})d\mu\leqslant c\Big(\frac{c_1\varepsilon}{c_0}\Big)^{1/(n-1)}\int(\exp((c_0\Sigma\chi_{R_k})^{1/(n-1)})-1)d\mu\leqslant c_\varepsilon|UR_k|_\mu\leqslant\frac{1}{2}|UR_s|_\mu.$$

Altogether we then get (1)'. We have finished the proof of the theorem.

Remark. Noting that only the proof of Lemma 4, which becomes trivial when $d\mu=dx$, is a bit complicated, but the proof of Lemmas 2 and 3 is merely simple extrapolation, we indeed simplify the proof of C–F covering Lemma.

§2. Weighted Estimate of Strong Maximal Operator on Zygmund Spaccs

Let $d\mu=w(x)dx$ be a nonnegative Borel measure on $\mathbb{R}^n$ and U, V be weights such that $U\,d\mu$ is bounded locally. Let $M_s=M_{s,\mu}$ be strong maximal operator with respect to measure μ, and $0<\alpha<\infty$. We want to discuss for which U, V we have

$$|\{M_sf>\lambda\}|_{Ud\mu}\leqslant\int\frac{c_2f}{\lambda}\Big(1+\log^{+\alpha}\frac{c|f|}{\lambda}\Big)V\,d\mu,\ \forall f,\ \forall\lambda>0, \tag{16}$$

$$\int_R M_sfU\,d\mu\leqslant c\Big(\int|f|(1+\log^{+\alpha}|f|)V\,d\mu+|R|_U\Big),\ \forall f. \tag{17}$$

Theorem 2. (a) *Suppose* (16) *holds. Then* $\exists\varepsilon>0$, *such that for arbitrary sequence of rectangles* $\{R_j\}$ *and sequence of disjoint sets* $\{E_j\}$ *where* $E_j\subset R_j$, *for arbitrary set* E *and constant* $\delta>0$, *we have*

$$\int_{UR_j\cap\{g>\delta\}}(\exp((\varepsilon g^{1/(\alpha+1)})-1)V\,d\mu\leqslant c|UR_j\cap E|_{Ud\mu}, \tag{18}$$

with constant C *only depending on* U, V, δ, ε, *where*

$$g=\sum_1^\infty\frac{|E_j\cap E|_U}{V(x)|R_j|_\mu}\chi_{R_j}(x). \tag{19}$$

(b) *Suppose that* (18) *holds for some* ε, δ *and arbitrary sequence of rectangles satisfying property* P_2, $E_j=R_j-\bigcup_{i<j}(R_i)_d$, *and* $E=\mathbb{R}^n$. *and* $Ud\mu\in A_\infty^{(s)}$. *Then* (16) *holds for index* $\alpha+1(0<\alpha+1<\infty)$.

Proof (a) we need the following elementary inequality

$$u(1+\log^{+\alpha}u)\leqslant c_{\eta,\alpha}(p-1)^{-\alpha}u^p,\ u\in[\eta,\infty),\ 1<p\leqslant2.$$

Suppose (16) holds. Then for $1<p\leqslant2$, we have

$$|\{M_sf>\lambda\}|_{Ud\mu}\leqslant\int_{\{|f|>\frac{\lambda}{2}\}}\frac{c|f|}{\lambda}\Big(1+\log^{+\alpha}\frac{c|f|}{\lambda}\Big)V\,d\mu$$

$$\leqslant c(p-1)^{-\alpha}\lambda^{-p}\int|f|^pVd\mu,\ \forall f,\ \forall\lambda>0. \tag{20}$$

From (20) and the method used in section 1, we see that for arbitrary $\{R_j\}$, $\{E_j\}$, E and $g=\Sigma\frac{|E_j\cap E|_U}{V(x)|R_j|_\mu}\chi_{R_j}(x)$, we have

$$\Big(\int_{UR_j}|g|^{p'}V\,d\mu\Big)^{1/p'}\leqslant c(p-1)^{-\alpha-1}|UR_j\cap E|_{Ud\mu}^{1/p'},\ 1<p\leqslant2; \tag{21}$$

And for $\delta>0$ and ε enough small, then we have

$$\int_{UR_j\cap\{g>\delta\}}(\exp((\varepsilon g)^{1/(\alpha+1)})-1)V\,d\mu=\sum_{j=1}^{2[\alpha+1]}+\sum_{[\alpha+1]+1}^{2[\alpha+1]+1}+\sum_{2[\alpha+1]+2}^{\infty}$$
$$\leqslant I_1+I_2+c|UR_j\cap E|_{Ud\mu}.$$

To estimate I_1 and I_2, we only have to notice that

$$\int_{UR_j}gV\,d\mu=\int_{UR_j}\Sigma\frac{|E_j\cap E|_{Ud\mu}}{|R_j|_\mu\chi_{R_j}}d\mu\leqslant|UR_j\cap E|_{Ud\mu}.$$

Thus we have proved (a).

(b). Suppose (18) holds for some ε, δ and $E=R^n$, and $\{R_j\}$ satisfying property P_2 and $E_j=R_j-\bigcup_{i<j}(R_i)_d$. And assume $Ud\mu\in A_\infty^{(s)}$. Let $\lambda>0$ and $\{\tilde{R}_j\}$ be a cover of $\{M_sf>\lambda\}$ such that $|\tilde{R}_j|_\mu^{-1}\int_{\tilde{R}_j}|f|\,d\mu>\lambda$. We choose a subsequence $\{R_j\}$ satisfying property P_2. let $E_j=R_j-\bigcup_{i<j}(R_i)_d$. Then

$$|E_j|\geqslant\frac{1}{2}|R_j|,\ |E_j|_U\geqslant\beta|R_j|_U,$$

$$|\tilde{R}_j\cap U(R_i)_d|>\frac{1}{2}|\tilde{R}_j|,\ |\tilde{R}_j\cap U(R_i)_d|_U\geqslant\beta|\tilde{R}_j|_U,\ \forall j,$$

which implies $M_{s,Ud\mu}(\chi_{U(R_i)_d})|_{U\tilde{R}_j}\geqslant\beta$. Since $M_{s,Ud\mu}$ is bounded on $L^2(Ud\mu)$, we get $|U\tilde{R}_j|_U\leqslant c|U(R_j)_d|_U\leqslant c\Sigma|E_j|_U$. Let $\Phi(u)=u(1+\log^{+(\alpha+1)}u)$ and $\Psi(v)$ be its Young complementary function. Then with $g=\Sigma\dfrac{|E_j|_U}{V(x)|R_j|_\mu}\chi_{R_j}$, weget

$$\begin{aligned}\Sigma|E_j|_U&\leqslant\int g\frac{|f|}{\lambda}V\,d\mu=\int_{\{g<\delta\}}+\int_{\{g\geqslant\delta\}}\\&\leqslant\int\Phi\Big(\frac{c|f|}{\lambda}\Big)V\,d\mu+c\int_{UR_j\cap\{g\geqslant\delta\}}(\exp((c_1\varepsilon g)^{1/(\alpha+1)})-1)V\,d\mu\\&\leqslant\int\frac{c|f|}{\lambda}\Big(1+\log^{+(\alpha+1)}\frac{c|f|}{\lambda}\Big)V\,d\mu+\frac{1}{2}\Sigma|E_j|_U\end{aligned}$$

for ε enough small.

So we obtain (b). This completes the proof of the theorem.

Now we discuss (17) as in [2]. We have the following analogous result. Its proof may be borrowed from [2], and omitted.

Theorem 4. *Let μ be a measure as in Theorem 2, $M_s=M_{s,\mu}$. Then the following conditions are equivalent:*

(a) *There exists constant $C=C_{U,V}$ (independent of f and R) such that* (17) *holds.*

(b) $\exists\varepsilon>0$, $\delta_\varepsilon>0$ *and $C_{\varepsilon,U,V}$ such that for any positive linear operator T satisfying $|Tf|\leqslant M_sf$, we have*

$$\int_{\{T^*(U\chi_R\geqslant\delta\varepsilon V\}}(\exp(\varepsilon(T^*(U\chi_R)V^{-1})^{\frac{1}{\alpha}})-1)V\,d\mu\leqslant c_{\varepsilon,U,V}|R|_{Ud\mu},\tag{24}$$

where T^* is the adjoint of T in following sense

$$\int gTf\,d\mu=\int fT^*g\,d\mu.$$

(c) $\exists\varepsilon>0$, $\delta_\varepsilon>0$ *and $C_{\varepsilon,U,V}$ such that for any sequence of rectangles $\{R_j\}$ and sequence of disjoint sets $\{E_j\}$, $E_j\subset R_j$, $g=\Sigma\dfrac{|E_j\cap R|_{Ud\mu}}{V|R_j|_\mu}\chi_{R_j}$, we have*

$$\int_{\{g\geqslant\delta_\varepsilon\}}(\exp(\varepsilon g^{1/\alpha}-1)V\,d\mu\leqslant C_{\varepsilon,U,V}|R|_{Ud\mu}.\tag{25}$$

Finally, we remark that when we consider (17) only for those f whose supports are contained in R, we may replace the domains of the integrals in (24) and (25) by $\{x\in R\colon T^*(U\chi_R)\geqslant\delta_s V\}$ and $\{x\in R\colon g\geqslant\delta_s\}$ respectively. In addition, when $U=V$, the domains of the integrals in (24) and (25) may be taken as whole R, owing to the fact that integrals over $\{x\in R\colon T^*(V\chi_R)\leqslant\delta_s V\}$ or $\{x\in R\colon g\leqslant\delta_s\}$ are $O(|R|_{Vd\mu})$ obviously.

References

[1] Cordoba, A. and Fefferman, R., A geometaic proof of the strong maximal theorem, *Ann, Math.*, **102** (1975), 95–100.

[2] Carbery, A., Chang, S-Y. A. and Carnett, J., Weights and LLogL, *Pacific Jour. of Math.* **120**, (1985), 33–45.

[3] Fefferman, R., Strong differentiation with respect to measures, *Amer. J. of Math.*, **103** (1981), 33–40.

[4] Jawerth, B. and Torchinsky, A., The strong maximal function with respect to measures (preprint).