Chin. Ann. of Math.
9B (3) 1988

SECTIONAL CURVATURE OF KAEHLER SUBMANIFOLDS OF A COMPLEX PROJCETIVE SPACE

Liao Ruijia (廖瑞佳)*

Abstract

Let $M^n (n \geqslant 2)$ be a complex Kaehler submanifold immersed in the complex projective space $CP^m(1)$. Let K be the sectional curvature of M^n. Then $K \geqslant 1/8$ if and only if M^n is an imbedding submanifold congruent to the standard imbedding CP^n (1) or $CP^n\left(\frac{1}{2}\right)$.

§1. Introduction

Let M^n be a complete Kaehler submanifold of complex dimension n, immersed in the complex projective space CP^m (1) endowed with the Fubini-Study metric of constant holomorphic sectional curvature 1. Let K be the sectional curvature of M^n. K. Ogiue[1] conjectured the following facts: (1) If $n \geqslant 2$ and $K > 1/8$, then M^n is totally geodesic in $CP^m(1)$; (2) If $m-n < n(n+1)/2$ and $K > 0$, then M^n is totally geodesic in $CP^m(1)$. Recently, A. Ros and L. Verstraelen[2] resolved the conjecture (1). In this paper, we will prove the following

Theorem. *Let M^n $(n \geqslant 2)$ be a complete Kaehler submanifold immersed in the complex projective space $CP^m(1)$. Let K be the sectional curvature of M^n. Then $K \geqslant 1/8$ if and only if M^n is an imbedding submanifold congruent to the standard imbedding $CP^n(1)$ or $CP^n(1/2)$.*

§2. Preliminaries

Let M^n be an n-dimensional Kaehler submanifold immersed in $CP^m(1)$. The Fubini-Study metric of constant holomorphic sectional curvature 1 on $CP^m(1)$ and the induced metric on M^n will both be denoted by g. The complex structure of $CP^m(1)$ and the induced complex structure on M^n will both be denoted by J. Let $\bar{\nabla}$ and ∇ be respectively the Riemannian connections of $CP^m(1)$ and M^n and let σ be the second fundamental form of the immersion. A and $\nabla^{\perp}$ are the Weingarten

Manuscript received March 18, 1986.

* Department of Mathematics, Zhongshan University, Guangzhou, China.

endomorphism and the normal connection. The second covariant derivative of the normal valued tensor σ is given by

$$(\nabla^2\sigma)(X, Y, Z, W)=\nabla^{\perp}_X((\nabla\sigma)(Y, Z, W))-(\nabla\sigma)(\nabla_X Y, Z, W)$$
$$-(\nabla\sigma)(Y, \nabla_X Z, W)-(\nabla\sigma)(Y, Z, \nabla_X W) \quad (2.1)$$

for any vector fields X, Y, Z and W tangent to M^n. Let $\bar{R}$, R and $R^{\perp}$ denote the curvature tensors of the connections $\bar{\nabla}$, ∇ and $\nabla^{\perp}$. Then we have

$$\bar{R}(\bar{X}, \bar{Y})\bar{Z}=\frac{1}{4}\{g(\bar{Y}, \bar{Z})\bar{X}-g(\bar{X}, \bar{Z})\bar{Y}+g(J\bar{Y}, \bar{Z})J\bar{X}$$
$$-g(J\bar{X}, \bar{Z})J\bar{Y}+2g(\bar{X}, J\bar{Y})J\bar{Z}\}, \quad (2.2)$$
$$R(X, Y)Z=\bar{R}(X, Y)Z+A_{\sigma(Y,Z)}X-A_{\sigma(X,Z)}Y, \quad (2.3)$$
$$g(R^{\perp}(X, Y)\xi, \eta)=g(\bar{R}(X, Y)\xi, \eta)+g([A\xi, A\eta]X, Y) \quad (2.4)$$

for all vector fields $\bar{X}$, $\bar{Y}$, $\bar{Z}$ tangent to $CP^m(1)$, X, Y, Z tangent to M^n and ξ, η normal to M^n in $CP^m(1)$. Moreover σ and $\nabla\sigma$ are symmetric and for $\nabla^2\sigma$ we have

$$(\nabla^2\sigma)(X, Y, Z, W)-(\nabla^2\sigma)(Y, X, Z, W)$$
$$=R^{\perp}(X, Y)\sigma(Z, W)-\sigma(R(X, Y)Z, W)-\sigma(Z, R(X, Y)W). \quad (2.5)$$

We also considier the relations

$$\sigma(JX, Y)=\sigma(X, JY)=J\sigma(X, Y), \quad (2.6)$$
$$A_{J\xi}=JA_{\xi}=-A_{\xi}J, \quad (2.7)$$
$$\nabla^{\perp}_X J\xi=J\nabla^{\perp}_X\xi, \quad (2.8)$$
$$(\nabla\sigma)(JX, Y, Z)=(\nabla\sigma)(X, JY, Z)=(\nabla\sigma)(X, Y, JZ)=J(\nabla\sigma)(X, Y, Z).$$

§ 3. Two Lemmas

Let M^n be an n-dimensional compact Kaehler submanifold immersed in $CP^m(1)$. Let π: $UM\to M$ and UM_p^n be the unit tangent bundle of M^n and its fiber over $p\in M^n$, respectively. Then we consider the function f:$UM\to R$ defined by

$$f(u)=\|\sigma(u, u)\|^2$$

for $u\in UM_p^n$. Since M^n is compact, UM is compact, therefore the function f attains its maximum at some vector $v\in UM_p^n$ for some $p\in M^n$. Then from [2] we have

$$A_{\sigma(v,v)}v=\|\sigma(v, v)\|^2 v. \quad (3.1)$$

Since $n\geqslant 2$, we can always choose a vector $u\in UM_p^n$, such that

$$A_{\sigma(v,v)}u=g(\sigma(v, v), \sigma(u, u))u. \quad (3.2)$$

Let $S=\{(u, w)\,|\,u, w\in UM, g(u, w)=g(u, Jw)=0\}$. From [4] we have

Lemma 1. *Let M^n be a compact Kaehler submanifold immersed in $CP^m(1)$. If f attains its maximum at some vector $v\in UM_p^n$ and some $p\in M^n$, then for any vector $u\in UM_p^n$, such that $(v, u)\in S_p$ and $A_{\sigma(v,v)}u=g(\sigma(v, v), \sigma(u, u))u$, we have*

$$\|\sigma(v, v)\|^2(1-8\|\sigma(u, v)\|^2)-4g(\sigma(v, v), \sigma(u, u))^2+4\|(\nabla\sigma)(u, v, v)\|^2\leqslant 0.$$

Now we consider the function h: $S\to R$ defined by

$$h(u, w)=\|\sigma(u, w)\|^2$$

for $(u, w)\in S$. Since M^n is compact, S is compact, therefore the above function h attains its maximum at some vector $(u, v)\in S_p$ for some $p\in M^n$. Let $\gamma(t)$ be a curve in UM_p^n, such that $\gamma(0)=u$, $\gamma'(0)=v$, $|\gamma'(t)|=1$. As the function $h_0(\gamma, \gamma')$ attains its maximum at $t=0$, we have

$$\frac{d(h_0(\gamma, \gamma'))}{dt}(0)=2g(\sigma(v, v), \sigma(u, v))-2g(\sigma(u, u), \sigma(u, v))=0, \tag{3.3}$$

$$\frac{d^2(h_0(\gamma, \gamma'))}{dt^2}(0)=2\|\sigma(v, v)\|^2+2\|\sigma(u, u)\|^2-8\|\sigma(u, v)\|^2 -4g(\sigma(u, u), \sigma(v, v))\leqslant 0. \tag{3.4}$$

Now, we suppose that the function f also attains its maximum at $v\in UM_p^n$. Fixed u, for any vector $w\in UM_p^n$, $(u, w)\in S_p$, $g(v, w)=0$, we can choose a curve $C_{(u,v)}(t)$ in S_p, such that $C_{(u,v)}(0)=(u, v)$, $C'_{(u,v)}(0)=(u, w)$. As the function $h\circ c$ attains its maximum at $t=0$, we have

$$\frac{d(h\circ c)}{dt}(0)=2g(\sigma(u, v), \sigma(u, w))=0. \tag{3.5}$$

From (3.1), (3.3) and (3.5), it follows that

$$A_{\sigma(u,v)}u=\|\sigma(u, v)\|^2 v. \tag{3.6}$$

Similarly

$$A_{\sigma(u,v)}v=\|\sigma(u, v)\|^2 u. \tag{3.7}$$

Let l_u be the geodesic in M^n determined by the initial conditions $l_u(0)=p$, $l'_u(0)=u$. Parallel translations of u and v along $l_u(t)$ yield vector fields $U_u(t)$ and $V_u(t)$. Let $h_u=h\circ(U_u, V_u)$. By direct computations, we obtain

$$\frac{dh_u}{dt}(t)=2g((\nabla\sigma)(l'_u, U_u, V_u), \sigma(U_u, V_u))(t), \tag{3.8}$$

$$\frac{d^2h_u}{dt^2}(0)=2g((\nabla^2\sigma)(u, u, u, v), \sigma(u, v))+2\|(\nabla\sigma)(u, u, v)\|^2. \tag{3.9}$$

From (3.9), we derive that

$$\frac{dh_{Ju}}{dt^2}(0)=2g((\nabla^2\sigma)(Ju, Ju, u, v), \sigma(u, v))+2\|(\nabla\sigma)(u, u, v)\|^2. \tag{3.10}$$

By similar arguments as in [3] we obtain

$$\begin{aligned} &g((\nabla^2\sigma)(Ju, Ju, u, v), \sigma(u, v)) \\ &=g((\nabla^2\sigma)(Ju, u, Ju, v), \sigma(u, v)) \\ &=g((\nabla^2\sigma)(u, Ju, Ju, v), \sigma(u, v))+g(R^\perp(Ju, u)\sigma(Ju, v), \sigma(u, v)) \\ &\quad -g(R(Ju, u)Ju, A_{\sigma(u,v)}v)-g(R(Ju, v)v, A_{\sigma(u,v)}Ju). \end{aligned} \tag{3.11}$$

By the Ricci equation

$$\begin{aligned} g(R^\perp(Ju, u)\sigma(Ju, v), \sigma(u, v)) &=-\frac{1}{2}\|\sigma(u, v)\|^2-2\|A_{\sigma(u,v)}u\|^2 \\ &=-\frac{1}{2}\|\sigma(u, v)\|^2-2\|\sigma(u, v)^4. \end{aligned} \tag{3.12}$$

By the Guass equation

$$g(R(Ju,\ u)Ju,\ A_{\sigma(u,v)}v) = \|\sigma(u,\ v)\|^2 g(R(Ju,\ u)Ju,\ u)$$
$$= -\|\sigma(u,\ v)\|^2 + 2\|\sigma(u,\ v)\|^2\|\sigma(u,\ u)\|^2, \tag{3.13}$$
$$g(R(Ju,\ u)v,\ A_{\sigma(u,v)}Ju) = -\|\sigma(u,\ v)\|^2 g(R(Ju,\ u)v,\ Jv)$$
$$= -\frac{1}{2}\|\sigma(u,\ v)\|^2 + 2\|\sigma(u,\ v)\|^4. \tag{3.14}$$

From (3.9)—(3.14), it follows that

$$\frac{d^2h_u}{dt^2}(0) + \frac{d^2h_{Ju}}{dt^2}(0) = 2\|\sigma(u,\ v)\|^2(1-4\|\sigma(u,\ v)\|^2 - 2\|\sigma(u,\ u)\|^2) + 4\|(\nabla\sigma)(u,\ u,\ v)\|^2. \tag{3.15}$$

Now we have proved

Lemma 2. *Let M^n be a compact Kaehler submanifold immersed in $CP^m(1)$. If h attains its maximum at some $(u,\ v)\in S_p$ for some $p\in M^n$ and f also attains its maximum at $v\in UM_p^n$, then we have*

$$2\|\sigma(u,\ v)\|^2(1-4\|\sigma(u,\ v)\|^2 - 2\|\sigma(u,\ u)\|^2) + 4\|(\nabla\sigma)(u,\ u,\ v)\|^2 \leqslant 0. \tag{3.16}$$

Remark. Obviously, under the same conditions as in Lemma 2, we also have

$$2\|\sigma(u,\ v)\|^2(1-4\|\sigma(u,\ v)\|^2 - 2\|\sigma(v,\ v)\|^2) + 4\|(\nabla\sigma)(v,\ u,\ v)\|^2 \leqslant 0. \tag{3.17}$$

§ 4. Proof of Theorem

First we note that, by a result of Myers, M^n is compact. Let f attain its maximum at some vector $u\in UM_p^n$ for some $p\in M^n$. By the theorem in [3], we can suppose $\|\sigma(v,\ v)\|\geqslant 1/4$. The assumption $K\geqslant 1/8$ implies that

$$1-8\|\sigma(u,\ v)\|^2 \geqslant 8|g(\sigma(u,\ u),\ \sigma(v,\ v))| \tag{4.1}$$

for any $(u,\ v)\in S_p$. Thus, from $|g(\sigma(u,\ u),\ \sigma(v,\ v)|\leqslant 1/8$ and $\|\sigma(v,\ v)\|^2\geqslant 1/4$, we derive that

$$\|\sigma(v,\ v)\|^2(1-8\|\sigma(u,\ v)\|^2) \geqslant 16g(\sigma(u,\ u),\ \sigma(v,\ v))^2. \tag{4.2}$$

By Lemma 1 for any $u\in UM_p^n$, such that $(u,\ v)\in S_p$ and $A_{\sigma(v,v)}u = g(\sigma(u,u),\ \sigma(v,\ v))u$, we have

$$g(\sigma(u,\ u),\ \sigma(v,\ v)) = 0 \text{ and } \|\sigma(u,\ v)\|^2 = 1/8. \tag{4.3}$$

According to the assumption, we know that h attains its maximum at $(u,\ v)\in S_p$. By Lemma 2, we also have

$$\|\sigma(v,\ v)\|^2 \geqslant 1/4 \text{ and } \|\sigma(u,\ u)\|^2 \geqslant 1/4. \tag{4.4}$$

From (3.4), (4.3) and (4.4), it follows that $\|\sigma(u,\ u)\|^2 = \|\sigma(v,\ v)\|^2 = 1/4$. Therefore the holomorphic sectional curvature $H=\frac{1}{2}$. Then by a theorem of A. Ros [5] and Theorem 5.3 in [1], our theorem is proved.

Acknowledgement. The author wishes to thank Professor Wang Qiming for his help.

Remark. Independently and using another method, A. Ros also obtains the same result.

References

[1] Ogiue, K., Differential geometry of Kaehler submanifolds, *Advances in Math.*, **13** (1974), 73—114.

[2] Ros, A. and Verstraelen, L., On a conjecture of K. Ogiue, *J. Differential Geometry*, **19** (1984), 561—566.

[3] Ros, A., Positively curved Kaehler submanifold, *Proc. Amer. Math. Soc.*, **93** (1985), 329—331.

[4] Liao Ruijia, Scalar curvature of Kaehler submanifolds of a complex projective space, *Kexuetongbao*, **16** (1986), 1210—1213.

[5] Ros, A., A characterization of seven compact Kaehler submanifolds by holomorphic pinching, *Ann. of Math.*, **121** (1985), 377—382.