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SOME THEOREMS ON CONVEX HYPERSURFACES IN AN AFFINE SPACE

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Abstract

Let M and M^* be hypersurfaces in an affine space A^{n+1} of dimension $n+1$. The main results of this paper are the following:

(1) Two types of integral formulas for M and M^*.

(2) Some conditions for an affine convex hypersurface to be an affine sphere.

(3) Some conditions for M and M^* to be different only by an affine transformation or a translation.

§ 1. Preliminary

Let A^{n+1} be a unimodular affine space of dimension $n+1$, $x=(x^\alpha)$ be the coordinates of a point with respect to a unimodular affine frame $xe_1\cdots e_{n+1}$, i. e., with respect to $n+1$ vectors e_α whose determinant satisfies the condition

$$(e_1, e_2, \cdots, e_{n+1})=1. \tag{1.1}$$

Under the unimodular affine group, points in A^{n+1} are changed according to the equations

$$\begin{gathered}\tilde{x}^\alpha=A^\alpha_\beta x^\beta+A^\alpha,\\ \det(A^\beta_\alpha)=1,\ 1\leqslant\alpha,\ \beta\leqslant n+1,\end{gathered} \tag{1.2}$$

and vectors $V=(v^1, \cdots, v^{n+1})$ are changed by the equations

$$\tilde{x}^\alpha=A^\alpha_\beta x^\beta. \tag{1.3}$$

As a consequence of (1.3) the determinant of $n+1$ vectors $v_1, \cdots, v_{n+1}$

$$(v_1, \cdots, v_{n+1})=\det(v^\alpha_\beta)$$

is an invariant. For a frame $xe_1\cdots e_{n+1}$ we can write

$$dx=\omega^\alpha e_\alpha,\ de^\beta=\omega^\beta_\alpha e_\alpha. \tag{1.4}$$

Differentiating (1.1) and using (1.4) we get

$$\sum_\alpha \omega^\alpha_\alpha=0, \tag{1.5}$$

and the Maurer-Cartan equations

$$d\omega^\alpha=\omega^\beta\wedge\omega^\alpha_\beta,\ d\omega^\alpha_\beta=\omega^\gamma_\beta\wedge\omega^\alpha_\gamma. \tag{1.6}$$

Let M be a hypersurface in A^{n+1}, i. e., let x: $M\to A^{n+1}$ be a hypersurface. We

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choose a frame $xe_1\cdots e_{n+1}$, such that $e_1, \cdots, e_n$ are tangent to M. Then $\omega^{n+1}=0$ on M, and by $d\omega^{n+1}=0$ and the first equation of (1.6), we have

$$\omega_i^{n+1}=h_{ij}\omega^j,\ h_{ij}=h_{ji}. \tag{1.7}$$

Consider the transformation of the frames $xe_1\cdots e_{n+1}$ and $xe_1^*\cdots e_{n+1}^*$, where $e_1^*, \cdots, e_n^*$ are also tangent to M:

$$\begin{aligned} &e_i^*=a_i^je_j,\ A=\det(a_i^j)\neq 0,\\ &e_{n+1}^*=A^{-1}e_{n+1}+a_{n+1}^ie_i. \end{aligned} \tag{1.8}$$

Then we have

$$\begin{aligned} &\omega^i=a_j^i\omega^{*j},\\ &\omega_i^{n+1}=A^{-1}b_i^j\omega_j^{*n+1},\ (b_j^i)=(a_j^i)^{-1},\\ &\omega^i\omega_i^{n+1}=A^{-1}\omega^{*i}\omega_i^{*n+1}. \end{aligned} \tag{1.9}$$

If $H=\det(h_{ij})\neq 0$, then M is called a nondegerate hypersurface. It is easy to see $H^*=HA^{n+2}$. Let $(H^{ij})=(h_{ij})^{-1}$. Then we have

$$\omega^j=H^{ij}\omega_i^{n+1},\ H^{ij}=H^{ji}. \tag{1.10}$$

We call

$$\mathrm{II}=|H|^{-1/(n+2)}\omega^i\omega_i^{n+1}=|H|^{-1/(n+2)}h_{ij}\omega^i\omega^j \tag{1.11}$$

the second fundamental form of M, which is an affine invariant. Thus the volume element

$$dV=|H|^{1/(n+2)}\omega^1\wedge\cdots\wedge\omega^n \tag{1.12}$$

is also an affine invariant. If II is a positive definite form, then M is said to be locally convex.

We can choose e_{n+1} such that

$$(n+2)\omega_{n+1}^{n+1}+d\log|H|=0. \tag{1.13}$$

The line through x in the direction of e_{n+1} is called the affine normal at x. The vector

$$V=|H|^{1/(n+2)}e_{n+1} \tag{1.14}$$

is called the affine normal vector of M, and its differential is

$$dV=|H|^{1/(n+2)}\omega_{n+1}^ie_i. \tag{1.14$'$}$$

Taking the exterior differential of (1.13), we get

$$\begin{aligned} &\omega_{n+1}^i=e^{ij}\omega_j^{n+1},\ e^{ij}=e^{ji},\\ &\omega_{n+1}^i=e_j^i\omega^j,\ e_j^i=e^{ik}h_{kj}, \end{aligned} \tag{1.15}$$

where $\det(e_j^i)\neq 0$. Let $(L_j^i)=(e_j^i)^{-1}$. Then we have

$$\omega^i=L_j^i\omega_{n+1}^j, \tag{1.15$'$}$$

and we can define the third fundamental form

$$\mathrm{III}=-\omega_{n+1}^i\omega_i^{n+1}=-e^{ik}h_{ij}h_{ke}\omega^j\omega^e. \tag{1.16}$$

III is invariant under the change (1.8) of frames.

Since II and III are invariant, the roots and coefficients of the equation in λ

$$\det(h_{ij}+\lambda|H|^{1/(n+2)}e_i^kh_{kj})=0$$

are also invariant. The left-hand side of above equation can be written as

$$\det(\delta^i_j+\lambda|H|^{1/(n+2)}e^i_j)=\sum_{r=0}^{n}\binom{n}{r}L_r\lambda^r. \tag{1.17}$$

We call L_r the rth affine mean curvature of N. In particvlar, we have

$$L_0=1,\ L_1=\frac{1}{n}|H|^{1/(n+2)}e^i_i,\ L_n=|H|^{n/(n+2)}\det(e^i_j). \tag{1.18}$$

Let $S_r(r=0,\ \cdots,\ n)$ be the rth elementary symmetric function of the eigenvalues λ_i of the martix (e^i_j). Then

$$\begin{aligned}L_r&=|H|^{-r/(n+2)}S_r,\\ S_r&=\frac{(n-r)!}{n!}\delta^{i_1\cdots i_r}_{j_1\cdots j_r}e^{j_1}_{i_1}\cdots e^{j_r}_{i_r}.\end{aligned} \tag{1.19}$$

Moreover, we introduce the forms

$$\Omega_r=\delta^{1\cdots n}_{i_1\cdots i_n}\omega^{i_1}_{n+1}\wedge\cdots\wedge\omega^{i_r}_{n+1}\wedge\omega^{i_{r+1}}\wedge\cdots\wedge\omega^{i_n}=n!\,|H|^{1/(n+2)}S_r\,dV, \tag{1.20}$$

$$|H|^{-(r+1)/(n+2)}\Omega_r=n!\,L_r\,dV. \tag{1.21}$$

Suppose that II is positive definite. Then we can define it as a positive riemannian metric on M, and shall compute its riemannian curvature tensor. Choose a frame $e_1\cdots e_{n+1}$, such that e_{n+1} is the direction along the affine normal, and e_i satisfy

$$h_{ij}=\delta_{ij}, \tag{1.22}$$

so that $\det(h_{ij})=H=1$. By (1.14), $V=e_{n+1}$, i. e., e_{n+1} is the affine normal vector. From (1.7) we have

$$\begin{aligned}&\omega^{n+1}_i=\omega^i.\\ &d\omega^{n+1}_i=\omega^j_i\wedge\omega^{n+1}_j=\omega^j\wedge\omega^i_j=d\omega^i,\end{aligned} \tag{1.23}$$

and therefore

$$d\omega^i=\omega^j\wedge\phi^i_j, \tag{1.24}$$

$$\phi^i_j=-\phi^j_i=\frac{1}{2}(\omega^i_j-\omega^j_i). \tag{1.25}$$

So ϕ^j_i is the connection form of II. Denoting the symmetric part of ω^i_j by ψ^i_j, we have

$$\begin{aligned}&\omega^i_j=\phi^i_j+\psi^i_j.\\ &\psi^i_j=\frac{1}{2}(\omega^i_j+\omega^j_i)=K^i_{jl}\omega^l.\end{aligned} \tag{1.26}$$

It is easy to see that K^i_{jl} are symmetric in $i,\ j,\ l$. That is,

$$K^i_{jl}=K^j_{il}=K^i_{lj}. \tag{1.27}$$

Since the curvature form of II is

$$\Omega^i_j=d\phi^i_j-\phi^k_j\wedge\phi^i_k=\frac{1}{4}(\omega^k_j+\omega^j_k)\wedge(\omega^i_k+\omega^k_i)+\frac{1}{2}(\omega^{n+1}_j\wedge\omega^i_{n+1}-\omega^{n+1}_i\wedge\omega^j_{n+1}), \tag{1.27'}$$

the curvature of II is

$$R^i_{ijk}=K^m_{jk}K^i_{ml}-K^m_{jl}K^i_{mk}-\frac{1}{2}(l^{ik}\delta_{jl}-l^{il}\delta_{jk}+l^{jl}\delta_{ik}-l^{jk}\delta_{il}). \tag{1.28}$$

Since $\omega_i^i=0$, from (1.26) we get

$$K^i_{ij}=0. \tag{1.29}$$

The Ricci curvature tensor and the scalar curvature can be derived by (1.28) as follows:

$$R_{il}=K^m_{ij}K^j_{ml}-\frac{1}{2}[l^{jj}\delta_{il}+(n-2)l^{il}], \tag{1.30}$$

$$R=\sigma^2-n(n-1)S_1,$$
$$\sigma^2=\sum_{i,j,k}(K^i_{jk})^2,\ S_1=l^{ii}/n. \tag{1.31}$$

Here we have used (1.27) to get (1.31).

§ 2. Integral Formulas

(1) First type. Let M, $\overline{M}$ be two closed convex orientable hypersurfaces in A^{n+1}. We choose affien frames $xe_1\cdots e_{n+1}$ and $\bar{x}\bar{e}_1\cdots\bar{e}_{n+1}$, respectively, such that e_{n+1} and $\bar{e}_{n+1}$ are in the directions of the respective affine normals of M, $\overline{M}$, and for a map $f: M\to\overline{M}$, we have

$$\bar{e}_i=f_*e_i, \tag{2.1}$$

so that

$$\bar{\omega}^i=\omega^i,\ \bar{\omega}^i_j=\omega^i_j. \tag{2.2}$$

It follows from (1.5) that

$$\bar{\omega}^{n+1}_{n+1}=\omega^{n+1}_{n+1}. \tag{2.3}$$

When f is a unimodular affine transformation, by the last equation of (1.10), the two hypersurfaces M, $\overline{M}$ have equal H, i. e.,

$$\overline{H}=H. \tag{2.4}$$

Consider the forms

$$\Omega_{rs}=\delta^{1\cdots n}_{i_1\cdots i_n}\ \bar{\omega}^{i_1}_{n+1}\wedge\cdots\wedge\bar{\omega}^{i_r}_{n+1}\wedge\omega^{i_{r+1}}_{n+1}\wedge\cdots\wedge\omega^{i_{r+1}}_{n+1}\wedge\omega^{i_{r+s+1}}\wedge\cdots\wedge\omega^{i_n}.$$

By (1.15) and the corresponding forms on $\overline{M}$, we can get the following, which is similar to (1.21):

$$H^{(r+s+1)/(n+2)}\Omega_{rs}=n!\,L_{rs}\,dV,$$
$$L_{rs}=H^{(r+s)/(n+1)}S_{rs}; \tag{2.5}$$

$$S_{rs}=\frac{(n-r-s)!}{n!}\ \delta^{i_1\cdots i_{r+s}}_{j_1\cdots j_{r+s}}\bar{l}^{j_1}_{i_1}\cdots\bar{l}^{j_r}_{i_r}l^{j_{r+1}}_{i_{r+1}}\cdots l^{j_{r+s}}_{i_{r+s}},$$
$$L_{0r}=L_r,\ L_{r0}=\bar{L}_r. \tag{2.6}$$

Consider the variation

$$M_t: X_t=(1+t)X,\ t\in(-\varepsilon,\ \varepsilon),\ X\in M. \tag{2.7}$$

Suppose for the frame $x_te_1,\ \cdots e_{n+1}$ of M_t, we have

$$dx_t=\omega^i(t)e_i,\ de_\alpha=\omega^\beta_\alpha(t)e_\beta, \tag{2.8}$$

where

$$\omega^i(t)=(1+t)\omega^i,$$
$$\omega^\beta_\alpha(t)=\omega^\beta_\alpha,\ H(t)=(1+t)^{-n}H. \tag{2.9}$$

Put

$$\Omega_{rs}(t)=\delta^{1\cdots n}_{i_1\cdots i_n}\overline{\omega}^{i_1}_{n+1}\wedge\cdots\wedge\overline{\omega}^{i_r}_{n+1}\wedge\omega^{i_{r+1}}_{n+1}\wedge\cdots\wedge\omega^{i_{r+s}}_{n+1}\wedge\omega^{i_{r+s+1}}\wedge\cdots\wedge\omega^{i_n}. \tag{2.10}$$

Then

$$\Delta\equiv[H(t)]^{(r+t+1)/(n+2)}\Omega_{rs}(t)$$

is an affine invariant. A direct computation gives

$$\left.\frac{\partial\Delta}{\partial t}\right|_{t=0}=\left[n-r-s-\frac{n(r+s+1)}{n+2}\right]H^{(r+s+1)/(n+2)}\Omega_{rs}. \tag{2.11}$$

Now we use another method to compute $\left.\frac{\partial\Delta}{\partial t}\right|_{t=0}$. By $\tilde{\omega}^{\alpha}(t)$, $\tilde{\omega}^{\alpha}_{\beta}(t)$ we denote the corresponding forms on $MX(-t,\ t)$. Then

$$d=dM+dt\cdot\partial/\partial t,$$
$$\tilde{\omega}^i(t)=\omega^i(t)+x^i dt,\ \tilde{\omega}^{n+1}(t)=x^{n+1}dt. \tag{2.12}$$

Writing

$$\tilde{\Omega}_{rs}(t)=\delta^{1\cdots n}_{i_1\cdots i_n}\omega^{i_1}_{n+1}\wedge\cdots\wedge\tilde{\omega}^{i_r}_{n+1}\wedge\omega^{i_{r+1}}_{n+1}\wedge\cdots\wedge\omega^{i_{r+s}}_{n+s}\wedge\omega^{i_{r+s+1}}\wedge\cdots\wedge\omega^{i}{}_{n}, \tag{2.13}$$

taking the exterior differential of (2.13), and using (1.6), (1.13), (2.2), (2.3), we get

$$d\tilde{\Omega}_{rs}(t)=(r+s+1)\omega^{n+1}_{n+1}\wedge\tilde{\Omega}_{rs}(t)+(n-r-s)\tilde{\omega}^{n+1}\wedge\tilde{\Omega}_{r+s+1}(t).$$

Hence we have

$$d(\tilde{E}^{(r+s+1)/(n+2)}\tilde{\Omega}_{rs})=-\frac{n(r+s+1)}{n+2}\tilde{H}^{(r+s+1)/(n+2)}dt\wedge\tilde{\Omega}_{rs}$$
$$+(n-r-s)\tilde{H}^{(r+s+1)/(n+2)}\tilde{\omega}^{n+2}\wedge\Omega_{r,s+1}. \tag{2.14}$$

Noticing (2.12) we have also

$$\tilde{\Omega}_{rs}=\Omega_{rs}(t)+dt\wedge\phi_{rs}, \tag{2.15}$$

where ϕ_{rs} does not include dt. Taking the exterior differential of (2.15), we get

$$d\tilde{\Omega}_{rs}=d\Omega_{rs}(t)-dt\wedge d\phi_{rs}. \tag{2.16}$$

Substituting (2.15) in both sides of (2.14), we have

$$d(\tilde{H}^{(r+s+1)/(n+2)}\tilde{\Omega}_{rs})=\tilde{H}^{(r+s+1)/(n+2)}dt\wedge\left[-\frac{n(r+s+1)}{n+2}\tilde{\Omega}_{rs}+(n-r-s)x^{n+1}\tilde{\Omega}_{r,s+1}\right],$$

$$d(\tilde{H}^{(r+s+1)/(n+2)}\Omega_{rs})=d_M(H^{(r+s+1)/(n+2)}\Omega_{rs})$$
$$+dt\wedge\left[\frac{\partial(H^{(r+s+1)/(n+2)}\Omega_{rs})}{\partial t}-d_M(H^{(r+s+1)/(n+2)}\phi_{rs})\right].$$

Comparing the terms including dt in the last two equations, we get

$$\left.\frac{\partial\Delta}{\partial t}\right|_{t=0}=d_M(H^{(r+s+1)/(n+2)}\phi_{rs})$$
$$+H^{(r+s+1)/(n+2)}\left[\frac{n(r+s+1)}{n+2}\Omega_{rs}+(n-r-s)x^{n+1}\Omega_{r,s+1}\right]. \tag{2.17}$$

A comparison of (2.11), (2.17) gives

$$d_M(H^{(r+s+1)/(n+2)}\phi_{rs})=(n-r-s)H^{(r+s+1)/(n+2)}(\Omega_{rs}-x^{n+1}\Omega_{r,s+1}).$$

If M is compact, then integrating the above equation and using (2.5), we get

$$\int_M L_{rs}\,dV=\int_M pL_{r,s+1}\,dV, \tag{2.18}$$

where

$$p = H^{-1/(n+2)} x^{n+1} \tag{2.19}$$

is an affine supporting function on M.

If $\overline{M}$ coincides with M, then (2.18) becomes[4]

$$\int_M L_r\, dV = \int_M p L_{r+1}\, dV_i. \tag{2.20}$$

(2) Second type. Let f: $M \to \overline{M}$ be a map such that the tangent spaces of M, $\overline{M}$ at the corresponding points are parallel. Choose their affine frames $xe_1 \cdots e_{n+1}$ and $\bar{x}\bar{e}_1 \cdots \bar{e}_{n+1}$ such that $e_i = \bar{e}_i$, and e_{n+1} and $\bar{e}_{n+1}$ are normal vectors of M and $\overline{M}$ respectively. In general, the normal direction of M need not be that of $\overline{M}$. Consider the following forms:

$$\begin{aligned} A_{rs} &= (x,\ v,\ \underbrace{dv,\ \cdots,\ dv}_{n-1-r-s},\ \underbrace{dx,\ \cdots,\ dx}_{r},\ \underbrace{d\bar{x},\ \cdots,\ d\bar{x}}_{s}), \\ B_{rs} &= (x,\ \bar{x},\ \underbrace{dv,\ \cdots,\ dv}_{n-1-r-s},\ \underbrace{dx,\ \cdots,\ dx}_{r},\ \underbrace{d\bar{x},\ \cdots,\ d\bar{x}}_{s}), \\ D_{rs} &= (v,\ \underbrace{dv,\ \cdots,\ dv}_{n-r-s},\ \underbrace{dx,\ \cdots,\ dx}_{r},\ \underbrace{d\bar{x},\ \cdots,\ d\bar{x}}_{s}). \end{aligned} \tag{2.21}$$

Taking the exterior differential of (2.21), from (1.4) and (1.14) we have

$$dA_{rs} = pD_{rs} - D_{r+1,s},$$

$$dB_{rs} = pD_{r,s+1} - \bar{p}D_{r+1,s}.$$

Intergrating the above, we get

$$\int_M (pD_{rs} - D_{r+1,s}) = 0,\ \int_M (\bar{p}D_{rs} - D_{r,s+1}) = 0,\ \int_M (pD_{r,s+1} - \bar{p}D_{r+1,s}) = 0. \tag{2.22}$$

Using(1.4), (1.14), (1.14)$'$, (1.25)$'$, we can derive

$$D_{rs} = Q_{rs} dW,\ dW = H^{(n-1-r-s)/(n+1)} \omega^1_{n+1} \wedge \cdots \wedge \omega^n_{n+1}, \tag{2.23}$$

$$Q_{sr} = \frac{(n-r-s)!}{n!} \delta^{i_1 \cdots i_{r+s}}_{j_1 \cdots j_{r+s}} L^{j_1}_{i_1} \cdots L^{j_r}_{i_r} \overline{L}^{j_{r+1}}_{i_{r+1}} \cdots \overline{L}^{i_{r+s}}_{j_{r+s}}, \tag{2.24}$$

$$Q_{0s} = \overline{Q}_s,\ Q_{r0} = Q_r,$$

where Q_r, $\overline{Q}_r$ are the rth elementary symmetric functions of the eigenvalues λ_i of (L^i_j), $(\overline{L}^i_j)$, and dW is the volume element of M with respect to its third fundamental form III. It should be remarked that Q_r is taken with respect to the affine normal vector v of M, while $\overline{Q}_r$ for $\overline{M}$ is taken with respect also to the affine normal vector v of M, but not to the affine normal vector $\bar{v}$ of $\overline{M}$ itself. Since $(L^v_j) = (l^i_j)^{-1}$ (cf. (1.15)), its eigenvalue A_i is the affine curvature diameter of M. Comparing (2.6)with (2.24), one sees that S_{sr} and Q_{rs} have similar forms.

Substituting (2.23) in (2.22), we have another kind of integral formulas

$$\begin{aligned}\int_M Q_{rs}\,dW&=\int_M pQ_{r-1,s}\,dW,\\ \int_M Q_{rs}\,dW&=\int_M \bar{p}Q_{r,s-1}\,dW,\\ \int_M pQ_{r,s+1}\,dW&=\int_M \bar{p}Q_{r+1,s}\,dW.\end{aligned}\tag{2.25}$$

When $M=\overline{M}$, (2.25) becomes[4]

$$\int_M Q_r\,dW=\int_M pQ_{r+1}\,dW.\tag{2.26}$$

§ 3. Some Theorems

Using the integral formulas in the last section, we can obtain some properties of hypersurfaces in A^{n+1}, e. g., conditions for a hypersurface to be an affine sphere, and for two affine hypersurfaces to be different only by an affine transformation in A^{n+1}.

Let M be a closed convex hypersurface; if it is necessary, we can take $e^*_{n+1}=-e_{n+1}$ instead of e_{n+1} such that (h_{ij}) is negative definite. In fact, by taking $A=-I$ in (1.9), it follows from the last formula of (1.10) that $(h^*_{ij})=-(h_{ij})$.

Now we let (h_{ij}) be negative definite. Denoting $A=(I^i_j)$, $B=(I^{ij})$, $C=(h_{ij})$, by $I^i_j=I^{ik}h_{kj}$ we get $A=BC$. Since C is negative definite, there is a nonsingular matrix T such that $C=-T'T$. Therefore $\det(I-\lambda A)=\det(I+\lambda BT'T)=\det(I+\lambda TBT')$. It is clear that the eigenvalues of (I^i_j) are real, since the matrices A, B are symmetric, and TBT' is also.

Lemma 1. *Let M be a closed convex hypersurface in A^{n+1}. Then the following statements are equivalent:*

(1) *Either $A=(I^i_j)$ is positive definite everywhere, or* III *is negative definite everywhere.*

(2) *The nth affine mean curvature $L_n>0$ holds everywhere.*

Proof It is clear that (1) implies (2) by (1.17). To show the converse we consider the affine distance function from the origin O to a point $x\in M$:

$$f(x)=H^{-1/(n+2)}(x,e_1,\cdots,e_n).\tag{3.1}$$

$f(x)$ is a continuous function on a compact manifold so that it reaches the maximum value at some points, e. g., at some x_0. Taking the differential of (3.1), and using (1.5), (1.13), we have

$$df=f_{.i}\omega^i=H^{-1/(n+2)}[(x,\omega^{n+1}_n e_{n+1},e_2,\cdots,e_n)+\cdots+(x,e_1,\cdots,e_{n-1},\omega^{n+1}_n e_{n+1})],\tag{3.2}$$

$$H^{1/(n+2)}f_{.i}=(x,h_{1i}e_{n+1},e_2,\cdots,e_n)+\cdots+(x,e_1,\cdots,e_{n-1},h_{ni}e_{n+1}).\tag{3.3}$$

1. We choose an affine frame $xe_1\cdots e_{n+1}$ such that the matrix (h_{ij}) is a diagonal

matrix $(\lambda_i\,\delta_{ij})$ at x_0. Then

$$H^{1/(n+2)}df=\lambda_1\omega^1(x,\ e_{n+1},\ e_2,\ \cdots,\ e_n)+\cdots+\lambda_n\omega^n(x,\ e_1,\ \cdots,\ e_{n-1},\ e_{n+1}).$$

Since $df=0$ at x_0, we have

$$\begin{aligned}&\lambda_1(x,\ e_{n+1},\ e_2,\ \cdots,\ e_n)=0,\\&\lambda_2(x,\ e_1,\ e_{n+1},\ \cdots,\ e_n)=0,\\&\cdots\quad\cdots\quad\cdots\quad\cdots\\&\lambda_n(x,\ e_1,\ e_2,\ \cdots,\ e_{n+1})=0.\end{aligned}\tag{3.4}$$

The solutions of the above system of equations are $x=\alpha e_{n+1}$ with $\alpha>0$. Since $\lambda_i\neq 0$ and the origin O is inside M, e_{n+1} points outward and is the same as it at point x_0.

2. Choose an affine frame $xe_1\cdots e_{n+1}$ such that the matrix (l^i_j) is a diagonal matrix $(\mu_i\delta_{ij})$ at x_0. Now we show that at the point x_0, $\mu_i>0$, $i=1,\ \cdots,\ n$, e. g., $\mu_1>0$, by (3.3) we have

$$\begin{aligned}f_{,1}=&H^{-1/(n+2)}[h_{11}(x,\ e_{n+1},\ e_2,\ \cdots,\ e_n)+\cdots+h_{n1}(x,\ e_1,\ \cdots,\ e_{n-1},\ e_{n+1})],\\df_{,1}=&-H^{-1/(n+2)}\left(\frac{1}{n+2}\,d\log H+\omega^{n+1}_{n+1}\right)[h_{11}(x,\ e_{n+1},\ e_2,\ \cdots,\ e_n)+\cdots\\&+h_{n1}(x,\ e_1\cdots,\ e_{n+1},\ e_{n-1})]+H^{-1/(n+2)}[h_{11}(\omega^1e_1,\ e_{n+1},\ e_2,\ \cdots,\ e_n)\\&+h_{11}(x,\ \omega^1_{n+1}e_1,\ e_2,\ \cdots,\ e_n)+\cdots+h_{n1}(\omega^ne_n,\ e_1,\ \cdots,\ e_{n-1},\ e_{n+1})\\&+h_{n1}(x,\ e_1,\ \cdots,\ e_{n-1},\ \omega^n_{n+1}e_n)].\end{aligned}\tag{3.5}$$

Noticing (3.4) and $x=\alpha e_{n+1}$, we can get

$$\begin{aligned}f_{,11}&=H^{-1/(n+2)}[h_{11}(e_1,\ e_{n+1},\ e_2,\ \cdots,\ e_n)+h_{11}\mu_1(x,\ e_1,\ \cdots,\ e_n)]\\&=H^{-1/(n+2)}(-h_{11}+\alpha\mu_1h_{11}).\end{aligned}$$

Since f reaches the maximum value at x_0, $f_{,11}<0$. By the above equation we thus have

$$-h_{11}(1-\alpha\mu_1)<0.$$

The condition $h_{11}<0$ gives

$$\alpha\mu_1>1,\ \mu_1>1/\alpha>0.$$

Hence the eigenvalues of (I^i_j) at x_0 are all positive.

3. The eigenvalues of (I^i_j) at any point $x\in M$ are positive; otherwise there exists some point such that $\mu_i=0$, so that it will lead to $L=0$ which contradicts $L_n>0$.

Theorem 1. *Let M be a closed convex hypersurface of A^3 with affine Gauss curvature $S_2>0$ (or $L_2>0$). Then the volume V of M with respect to* III *is boundead above by 4π, i. e., $V\leqslant 4\pi$, where the equality holds if and only if M is an ellipsoid.*

Proof Choose a frame $xe_1\cdots e_{n+1}$ such that $h_{ij}=\delta_{ij}$, $I^{ii}=\lambda_1+\lambda_2$. Then it follows from (1.31) that

$$R=\sigma^2+\lambda_1+\lambda_2\geqslant\lambda_1+\lambda_2\geqslant 2\sqrt{\lambda_1\lambda_2}.$$

Integrating the above inequality we have

$$\int_M R\,dV\geqslant 2\int_M\sqrt{\lambda_1\lambda_2}\,dV,\tag{3.6}$$

where $dv=\omega^1\wedge\omega^2$ is the volume element of II, and, $\int_M \sqrt{\lambda_1\lambda_2}\,dV$ is the volume V of M with respcet to III. By Gauss-Bonnet formula,

$$\int_M R\,dV=2\int_M K\,dV=4\pi\chi(M),$$

where K is the Gauss curvature of II, and χ is Euler-characteristic of M. Since M is closed and convex, it is homeomorphic to the sphere, so that $\chi(M)=2$, and therefore $\int_M R\,dV=8\pi$. Thus it follows from (3.6) that

$$V\leqslant 4\pi, \tag{3.7}$$

where the equality holds if and only if $\lambda_2=\lambda_1$. In the case where M is an affine sphere, M is an ellipsoid, since a two-dimensional affine sphere is an ellipsoid.

Theorem 2. *Let M be a closed convex hypersurface in A^{n+1}. Then the scalar curvature R of M with respect to* II *satisfies*

$$R\geqslant(n-1)S_1, \tag{3.8}$$

where the equality holds if and only if M is an affine sphere.

Proof It follows immediately from (1.31) that when the equality holds in (3.8) we have $\sigma^2=0$, i. e., $K^i_{jk}=0$, so that

$$\omega^j_i+\omega^i_j=0.$$

Differentiating the above equation we get

$$\omega^k_i\wedge\omega^j_k+\omega^{n+1}_i\wedge\omega^j_{n+1}+\omega^k_j\wedge\omega^i_k+\omega^{n+1}_j\wedge\omega^i_{n+1}=0,$$

i. e.,

$$\omega^{n+1}_i\wedge\omega^j_{n+1}+\omega^{n+1}_j\wedge\omega^i_{n+1}=0,$$

or

$$\delta_{ik}l^{lj}-\delta_{il}l^{kj}=\delta_{jl}l^{ki}-\delta_{jk}l^{li}.$$

By contraction for j, k, we have

$$l^{il}=\lambda\delta_{il},\ \lambda=\sum_j l^{jj}.$$

Thus $\lambda_1=\cdots=\lambda_n$, and M is an affine sphere.

Theorem 3. *Let a hypersurface $M\subsetneq A^{n+1}$ be closed convex with constant first affine mean curvature L_1. Then the hypersurface M is an affine sphere.*

Proof Suppose the the origin O of A^{n+1} to be inside M, such that the affine supporting function preserves the sign. Since L_1 is constant, from (2.20) for $r=0, 1$ we have

$$\int_M p(L_1^2-L_2)dV=0.$$

But

$$H^{2/(n+2)}(L_1^2-L_2)=\left(\frac{\lambda_1+\cdots+\lambda_n}{n}\right)^2-\frac{2(\lambda_1\lambda_2+\cdots+\lambda_{n+1}\lambda_n)}{n(n-1)}$$
$$=\frac{1}{n^2(n-1)}\sum_{i<j}(\lambda_i-\lambda_j)^2\geqslant 0,$$

where the equality holds only if $\lambda_1=\cdots=\lambda_n=\lambda$, so that

$$\omega_{n+1}^i=\lambda H^{-1/(n+2)}\omega^i,\ \lambda=\text{const.},$$

It follows from (1.14)′ that

$$dV=\lambda\omega^i e_i=\lambda\, dx,\ V-\lambda x=a=\text{const.},$$

It is easy to see that M is an affine sphere.

Let $M\subsetneq A^{n+1}$ be a closed convex hypersurface, the origin O of A^{n+1} be chosen inside M, and e_{n+1} be along the affine normal direction and point outward. Then the affine supporting function $p>0$, and (h_{ij}) is negative definite. By the same method we can obtain the following Theorems 4 and 5.

Theorem 4. *Let M be a closed convex hypersurface in A^{n+1}, and L_n be always positive. If there is an r such that the rth affine mean curvature L_r is constant, then M is an affine sphere.*

Theorem 5. *Let M be a closed convex hypersurface in A^{n+1}, and L_n be positive everywhere. If there are two indices τ, r with $1\leqslant\tau<r\leqslant n$ such that the ratio of the two affine mean curvatures L_r and L_τ is constant, i. e., $L_r/L_\tau=a$ where a is constant, then M is an affine sphere.*

Theorem 6. *Let M be a closed convex hypersurface in A^{n+1}, and L_n be positive everywhere. If there is an r, $1\leqslant r\leqslant n$, such that*

$$L_{r-1}^{\alpha}L_{\alpha}^{\beta}=const.,\ \alpha+\beta>0,\ \alpha\geqslant 0,\ \beta\geqslant 0, \tag{3.9}$$

then M is an affine sphere.

Proof Let $L_{\gamma-1}^{\alpha}=L_r^{\beta}=C^{\alpha(r-1)+\beta r}$. Then we have

$$L_r=C^{\alpha(\beta-1)/\beta+r}L_{r-1}^{-\alpha/\beta}. \tag{3.10}$$

By $L_{r-1}^{1/(r-1)}\geqslant L_r^{1/r}$, we get

$$L_{r-1}\geqslant L_r^{(r-1)/r}=C^{(r-1)[\alpha(1-1/r)/\beta+1]}L_{r-1}^{-\alpha(1-1/r)/\beta},$$

or $L_{r-1}\leqslant C^{r-1}$. It follows from (3.10) that $L_r\leqslant C^r$. Moreover, $L_1\geqslant C$ holds (see Lemma 8.3 in [4]). As a consequence of the last two inequalities and integral formulas, we have the following inequality

$$\begin{aligned}\int_M C^r p\, dV&=\int_M C^{r-1}pL_1\, dV=\int_M C^{r-1}L_0\, dV\\ &\leqslant\int_M L_{r-1}\, dV=\int_M pL_r\, dV\leqslant\int_M C^r p\, dV.\end{aligned}$$

So

$$\int_M p(L_1-C)dV=0$$

gives $L_1=C$. Then by Theorem 1 we see that M is an affine sphere.

Similarly we can also get the following theorem.

Theorem 7. *Let M be closed convex hypersurface in A^{n+1}, and L_n be positive everywhere. If there is an r such that*

$$L_{r-2}/L_{r-1} \leqslant C \leqslant L_{r-1}/L_r, \quad C = const., \quad 1 < r < n, \tag{3.11}$$

then N is an affine sphere,

Using integral formulas, we can also discuss when two hypersurfaces are the same under an affine transformation. This is a problem similar to Cohn–Vossen's theorem, which determines when two hyperssurfaces in a Euclidean space E^{n+1} are the same under a rigid motion.

Let A, B be two matrices whose eigenvalues are all positive. If there exists o nonsingular matrix T such that $T^{-1}AT$ and $T^{-1}BT$ are symmetric, then we call A, B an S–pair.

Lemma 2. *Let the eigenvalues of A and B be positive. Then the following conditions are equivalent:*

(1) *A, B form an S–pair.*

(2) *There exists a positive definite matrix C such that AC and BC are symmetric matrices.*

Proof To show that (1) implies (2) suppose there is a nonsingular matrix T such that

$$A_1 = T^{-1}AT, \quad B_1 = T^{-1}BT$$

are symmetric matrices. Then

$$A = TA_1T^{-1}, \quad B = TB_1T^{-1},$$

and therefore we have two symmetric matrices

$$AC = TA_1T', \quad BC = TB_1T',$$

where $C = TT'$.

To show that (2) imples (1), we assume that there is a positive definite matrix C such that

$$AC = A_1, \quad BC = B_1$$

are symmetric matrices. Putting $C = TT'$, we then get two symmetric matrices

$$T^{-1}AT = T^{-1}AT(T'T^{-1'}) = T^{-1}A(TT')T^{-1'} = T^{-1}A_1T^{-1},$$
$$T^{-1}BT = T^{-1}BT(T'T^{-1}) = T^{-1}B(TT')T^{-1} = T^{-1}B_1T^{-1}.$$

For matrices A and B, we write

$$\det(\lambda A + \mu B + I) = \sum_{s+r+0}^{n} \frac{n!}{r!s!(n-r-s)!} \lambda^r \mu^s P_{rs}. \tag{3.12}$$

Lemma 3. *If the matrices A and B form an S–pair, then the P_{rs} determined by (3.12) satisfy*

$$P_{i-1,1} > P_{i0}^{1-1/i} P_{0i}^{1/i}. \tag{3.13}$$

Proof By assumption, there is a nonsingular matrix T such that $T^{-1}AT$ and $T^{-1}BT$ are positive symmetric matrices. Since

$$\det(\lambda A + \mu B + I) = \det(\lambda T^{-1}AT + \mu T^{-1}BT + I), \tag{3.14}$$

the P_{rs} for the symmetric matrices $T^{-1}AT$, $T^{-1}BT$ and those for A, B are the same.

Hence the Gårding's inequality gives (3.13) immediately.

Theorem 8. *Suppose that M and M' are closed convex hypersurfaces in A^{n+1} and that f: $M\to M'$ is a diffeomorphism if under the mapping f the two hypersurfaces have the same affine normal vector at each pair of the corresponding points, i e.,*

$$V'_{f(x)}=V_x,\ for\ x\in M.$$

Then M and M' are different only by a translation.

Proof By the assumption and (1.14)', we have

$$H'^{1/(n+2)}\omega'^{i}_{n+i}e'_i=H^{1/(n+2)}\omega^{i}_{n+1}e_i,$$

so that M and M' have parallel tangent space at each pair of the corresponding points. Take a Euclidean metric on A^{n+1} such that $e_1, \cdots, e_n$, is a set of orthogonal unit vectors in the tangent space of M, and e_{n+1} is the affine normal direction of M. Then $e_1, \cdots, e_n$ are also tangent to M, and e_{n+1} is the affine normal direction of M'. Since $V'=V$, from

$$V_{,}=H'^{1/(n+2)}e_{n+1},\ V=H^{1/(n+2)}e_{n+1},$$

we have $H'=H$. It is easy to see that with respect to this frame, H and H' are the Gauss curvature of M and M' respectively. Hence by Minkowski's theorem, M' and M are different only by a translation.

Theorem 9. *Suppose that M and M' are two closed convex hypersurfaces in A^{n+1}, and their respective nth affine mean curvatures L_n, L'_n are positive everywhere. If f: $M\to M'$ is an affine map, $A=(L^i_j)$ and $A'=(L'^i_i)$ form an S-pair, and there is an r such that $L_r=L'_r$, then M and M' are different only by an affine transformation on A^{n+1}.*

Proof Choose the affine frames of M and M' such that

$$\omega'^i_j=\omega^i_j \tag{3.15}$$

holds By integral formulas (2.18) we have

$$\int_M L_{0,r-1}\,dV=\int_M pL_{0,r}\,dV,\ \int_M L_{r-1,0}\,dV=\int_M pL_{r-1,1}\,dV,$$

and therefore

$$\int_M (L_{0,r-1}-L_{r-1,0})dV=\int_M p(L_{0,r}-L_{r-1,1})dV. \tag{3.16}$$

It follows from (3.13) and $L_r=L'_r$, i. e., $L_{0r}=L_{0r}'$, that

$$L_{r-1,1}\geqslant L_{r0}^{(r-1)/r}L_{0r}^{1/r}=L_{0r},$$

where the equality holds if and only if A and A' are in proportion. Choose the direction of vector e_{n+1} such that $p<0$. By (3.16) we have

$$\int_M L_{0,r-1}\,dV\geqslant\int_M L_{r-1,0}\,dV.$$

Interchanging the roles of M and M' in the above process, we have

$$\int_M L_{0,r-1}\,dV\leqslant\int_M L_{r-1,0}\,dV.$$

Thus

$$\int_M L_{0,r-1}\,dV=\int_M L_{r-1,0}\,dV, \tag{3.17}$$

and A and A' are in proportion. By $L_r=L'_r$ we have $A=A'$, i. e.,

$$\omega^i_{n+1}=\omega'^i_{n+1}, \tag{3.18}$$

which together with (3.15) implies

$$\omega_i^{n+1}\wedge\omega^j_{n+1}=\omega_i'^{n+1}\wedge\omega^j_{n+1},$$

i. e.,

$$h_{ik}l^j_i-h_{il}l^j_k=h'_{ik}l^j_l-h'_{il}l^j_k.$$

It is easy to get $h_{ij}=h'_{ij}$, by choosing the frames of M and M' such that $l^j_k=\mu_k\delta^j_k$. Thus with respect to these frames,

$$\omega_j^{n+1}=\omega_j'^{n+1},$$

and therefore $\omega'^\alpha=\omega^\alpha$, $\omega^\beta_\alpha=\omega'^\beta_\alpha$. Hence f is an affine transformation of A^{n+1}, restricted on M.

If we substitute a Euclidean space E^{n+1} for the affine space A^{n+1} and use the same method, we can get a theorem on convex hypersurfaces, which is stronger than the Cohn-Vossen's theorem.

Theorem 10. *Let M and M' be two closed convex hypersurfaces in E^{n+1}. If f: $M\to M'$ is an affine map, $A=(I^i_j)$ and $B=(I'^i_i)$ form an S-pair, and there exists an r such that the rth mean curvatures S_r, S'_r of M, M' are equal, then M and M' are the same except a rigid motion E^{n+1}.*

Theorem 11. *Suppose that M and M' are two closed convex hypersurfaces in A^{n+1}, $L_n>0$, and F: $M\to M'$ is a diffeomorphism such that M and M' have parallel tangent spaces at each pair of the corresponding points. Let* III *and* III′ *be the third fundamental forms of M and M' with respect to the affine normal vector v of M, and Q_i and Q'_i be the ith elementary symmetric functions of* III *and* III′ *respectiely (see* (2.24)). *If* III′ *is positive definite and there exists an r, $2\leqslant r\leqslant n$, such that*

$$Q_{r-1}\leqslant Q'_{r-1},\ Q_r\geqslant Q'_r, \tag{3.19}$$

then M and M' are different only by a translation.

This theorem is a generalization to an affine space A^{n+1} of the theorem on a Euclidean space E^n given in [3], and the method of proof here is essentially the same as that in [3].

Proof From the integral formula (2.25), we have

$$\begin{aligned}\int_M Q_{r0}\,dV&=\int_M p\,Q_{r-1,0}\,dV,\\ \int_M Q_{1,r-1}\,dV&=\int_M p\,Q_{0,r-1}\,dV.\end{aligned} \tag{3.20}$$

By the Gårding's inequality

$$Q_{1,r-1}\geqslant Q_{r0}^{1/r}\,Q_{r0}^{1-1/r}\geqslant Q_{r0}\quad (Q_{0r}\geqslant Q_{r0}),$$

we have $\int Q_{1,r-1}\,dV \geqslant \int Q_{r0}\,dV$. On the other hand, $Q_{r-1,0} \geqslant Q_{0,r-1}$, $p>0$. Thus

$$Q_{1,r-1}=Q_{r0},\ Q_{r-1,0}=Q_{0,r-1},$$

and M and M' are different by a translation.

Example. In A^3 ellipsoids have constant affine principle curvature or principle curvature diameter, If we take two different ellipsoids, for example, one is a sphere M, and the other is an ellipsoid M' different from the sphere, such that M and M' have equal affine principle curvature diameters, then the elementary symmetric functions of these diameters are equal, i, e., $Q_r=Q'_r$, and M' cannot be obtained from M by a translation, However, these facts do not contradict Theorem II, since the Q'_r in the theorem is taken with respect to the affine normal vector V of M, but not to that V' of M'.

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