Chin. Ann. of Math.
9B (3) 1988

THE HOMOTOPY TYPES OF THE DELETED PRODUCTS OF SOME VECTOR BUNDLES

Gu Zhiming (古志鸣)*

Abstract

In this paper it is shown that if W is an m-vector bundle over an n-manifold M with some conditions, then the $W^k-\Delta W$, k-deleted product of W, and $M^k\cup \Delta M\times S^{(k-1)(m+n)-1}$ are homotopy equivalent.

In [1] we have seen that the homotopy groups of the k-deleted products of some manifolds may be expressed by the direct sum of the homotopy groups of M and $M^{k-1}-\tilde{x}$, where M is the manifold in question and $\tilde{x}$ is a point of $M^{k-1}=\underbrace{M\times\cdots\times M}_{k-1}$. In this paper we will further point out that the homotopy types of the k-deleted products of some vector bundles, as manifolds, have analogous property.

§ 1. Major Theorem and Its Applications

Let $k(>1)$ be an integer and ΔM is the diagondal subspace of the product space $M^k=\underbrace{M\times\cdots\times M}_{k}$ of a manifold M. Let m be an integer, $M(m, k)=\Delta M\times S^{(k-1)(m+n)-1}\bigcup_t M^k$, where t: $\Delta M\times v\to M^k$, $t(u, v)=u$, $u\in\Delta M$, v is a fixed point in $S^{(k-1)(m+n)-1}$. The major result of this paper is the following theorem.

Theorem 1. *Let W be an m-vector bundle over a simply connected compact differentiable n-manifold M, W have a defferentiable structure. Let $k>\frac{n+1}{m}+1$, $m\geqslant n\geqslant 2$. If the normal bundle of ΔW in W^k is trivial, then $M(m, k)$ is a deformation retract of the k-dileted product $W^*_{(k)}$ of W.*

The theorem has the following corollaries.

Corollary 1. *Let W be an m-vector bundle over S^n with a differentiable structure, $k>\frac{n+1}{m}+1$, $m\geqslant n\geqslant 2$, $n\equiv 3, 5, 6, 7 \pmod 8$. Then $S^n(m, k)$ is a deformation retract of $W^*_{(k)}$.*

Proof It is sufficient to prove that the normal bundle ν of ΔW in W^k is trivial.

Manuscript received March 3, 1986. Revised June 28, 1986.
* Shenyang Polytechnical University, Shenyang, Liaoning, China.

Because ΔW and S^n are homotopy equivalent, it follows from Bott periodicity theorem that the classifying map of $\nu:\Delta W\to BO$ is null homotopic, namely, ν is trivial.

Corollary 2. *Let W be an m-vector bundle over a simply connected compact differentiable n-manifold M, W have a differentiable structure, and its tangent bundle τ be stable trivial, namely, there is a trivial r-vector bundle ε^r such that $\tau\oplus\varepsilon^r$ is trivial, $k>\frac{n+1}{m}+1$, $m\geqslant n\geqslant 2$. Then $M(m, k)$ is a deformation retract of $W^*_{(k)}$.*

Proof The normal bundle ν of ΔW in W^k is isomorphic to $\underbrace{\tau\oplus\cdots\oplus\tau}_{k-1}$. Because $\tau\oplus\varepsilon^r$ is trivial, $\nu\oplus\underbrace{\varepsilon^r\oplus\cdots\oplus\varepsilon^r}_{k-1}$ is trivial. Because the dimension $(k-1)(m+n)$ of ν is greater than the dimension $(m+n)$ of W, ν is trivial. Thus the corollary is true.

Remark. Theorem 1 and its corollaries are still true if the closed disk bundle takes the place of vector bundle in them because of the Proposition 1.1 in [1].

Now we give an application of Corollary 2. Let E be the 4-vector bundle over S^4 corresponding to the element $S\rho+\sigma$ in $\pi_3(SO_4)=\mathbb{Z}\oplus\mathbb{Z}$, where ρ and σ are explanned by [2], N the colosed disk bundle of E. Because ∂N is homeomorphic to S^7, we can construct a manifold $X=N\bigcup_g D^8$, where $g:\partial N\to\partial D$ is a homeomorphism.

Theorem 2. *Let the manifold X have a differentiable structure and the tangent bundle of X is stable trivial, $k>2$. Then $X^*_{(k)}$ and $S^4(4, k)$ are homotopy equivalent.*

Proof From the structure of X we have

$$X^*_{(k)}\cong N^k\cup(D^8)^k\cup C^1_kN^{k-1}\times(D^8)\cup\cdots\cup C^{k-1}_kN\times(D^8)^{k-1}-\Delta X, \tag{1}$$

where $C^i_kN^{k-i}\times(D^8)^i$ expresses the union of C^i_k sets $N^{k-i}\times(D^8)^i$. Because the inner points of $(D^8)^k-\Delta X$ can be retracted to its bonndary, $(D^8)^k-\Delta X$ and $S^{8k-9}\times D^8$ are hemotopy equivalent. If we retract every subset $x\times(D^8)^i$ in the term with factor D^8 in (1) to one point, $(D^8)^k-\Delta X$ will be deformed to S^{8k-9}. It follows from the below proof of the major theorem that this S^{8k-9} is glued to a "cut" in $\Delta S^4\times S^{8k-9}$ in $S^4(4, k)$ by a homeomorphism. Thus $X^*_{(k)}$ and $N^*_{(k)}$ are homotopy-equivalent. By Corollary 2, $N^*_{(k)}$ and $S^4(4, k)$ are homotopy* equivalent, the theorem follows.

§2. The Proof of Major Theorem

We think of W^k as a km-vector bundle ξ over M^k and consider the bundle $\xi|\Delta M$. Deleting the diagonal in every fibre of $\xi|\Delta M$ we obtain a $(k-1)m-1$-sphere bundle. Since $k>\frac{n+1}{m}+1$, it has a cross-section c. If we think of c as a cross-section of $\xi|\Delta M$, we have $c(\Delta M)\subset W^*_{(k)}$. We can extend c to a cross-section on M^k and denote it by c yet. By [1] $W^*_{(k)}$ is simply connected, then there is an embedding $h_1:M^k\to W^*_{(k)}$

which is homotopic to c (see [3], Theorem 7.5).

Since the normal bundle ν of ΔW in W^k is trivial, we can denote the total space of the closed disk bundle $\bar{\nu}$ of ν by $\Delta W\times D^{(k-1)(m+n)}$. Let $a\in\partial D^{(k-1)(m+n)}$. We define $f\colon \Delta M\times\partial I\to W^*_{(k)}$ as follows

$$f|(\Delta M\times 0)=h_1|\Delta M,$$
$$f(x,1) \text{ is a homeomorphism to } \Delta M\times a.$$

Since h_1 is homotopic to a cross-section, it is easy to see that f can be extended to $\tilde{f}\colon \Delta M\times I\to W^*_{(k)}$ if we choose a suitable radius of the fibers of $\bar{\nu}$. We can suppose that $\tilde{f}$ is an embedding because $k>\frac{n+1}{m}+1>\frac{2n+3}{m+n}$ (see [3], Theorem 4.15). When we glue the inclusion: $\Delta M\times\partial D^{(k-1)(m+n)}\to$the total space of ν and h_1 by $\tilde{f}$, the embeding map $h\colon M(m,k)\to W^*_{(k)}$ is obtained. Let $h(M(m,k))=A$, $i\colon A\to W^*_{(k)}$ is the inclusion. Now we need the following lemma.

Lemma. *There is a map* $s\colon W^k/A\to W^k/W^*_{(k)}$ *such that the following diagram commutes and* $\tilde{s}_*$ *is an isomorphism for all* q.

$$\begin{array}{ccc} H_q(W^k, W^*_{(k)}) & \xrightarrow{p'_*} & \tilde{H}_q(W^k/W^*_{(k)}) \\ \uparrow{\scriptstyle j_*} & & \uparrow{\scriptstyle \tilde{s}_*} \\ H_q(W^k, A) & \xrightarrow{p''_*} & \tilde{H}_q)(W^k/A), \end{array}$$

where j_* *is induced by* $1\colon W^k\to W^k$, p'_* *and* p''_* *are induced by the identification maps* p' *and* p''.

Proof First observe $A=h(M^k\cup\Delta M\times S^{(k-1)(m+n)-1})$. When we retract every point in W^k/A which belongs to the F–int$(\Delta M\times D^{(k-1)(m+n)})$, where F is the fiber of ξ, to the zero cross section of ξ along F, the space obtained and W^k/A are homotopically equivalent. Second, since $h|M^k$ can be homotopic to a cross seccion when we identify A to a point the cone on the zero cross section is obtained. Thus we can retract M^k–ΔM to A. Finally, there is an obvious homotopy equivalent map from int $(\Delta M\times D^{(k-1)(m+n)})$ onto $W^k-W^*_{(k)}=\Delta W$. Taking s to the composition of three maps above we have $p'\circ 1\simeq s\circ p''$.

Look at the homomorphism of the exact sequences

$$\begin{array}{ccccccccccc} \cdots\to H_{q+1}(W^k) & \to & H_{q+1}(W^k, W^*_{(k)}) & \to & H_q(W^*_{(k)}) & \to & H_q(W^k) & \to & H_q(W^k, X^*_{(k)}) & \to\cdots \\ \uparrow{\scriptstyle 1_*} & & \uparrow{\scriptstyle j_*} & & \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle 1_*} & & \uparrow{\scriptstyle j_*} & \\ \cdots\to H_{q+1}(W^k) & \to & H_{q+1}(W^k, A) & \to & H_q(A) & \to & H_q(W^k) & \to & H_q(W^k, A) & \to\cdots \end{array}$$

It follows from the lemma that j_* is an isomorphism. 1_* is an isomorphism. The 5–lemma tell us that i_* is an isomorphism. Since A and $W^*_{(k)}$ are all simply connected, i is a homotopically equivalence.

References

[1] Gu Zhiming, Deleted products of manifolds, *Advances in Mathematics*, **14**(1985), 143—146.
[2] Tamura, I., **8**-Manifolds admitting no differentiable structure, *J. Math. Soc. Japan*, **13**(1961), 377—382.
[3] Tamura, I., Differential Topology, Iwanami Shoten.